Análise Matemática II E

Exame de Época 2

14/07/2020

Nota: Esta é apenas uma proposta de resolução de entre muitas outras possibilidades.

1

a) $y(t)$ → quantidade (em gramas) do elemento químico no organismo do paciente no instante t

$$\begin{cases} y(0) = 27 \\ y' = ky \end{cases}$$

b) $y' = ky \Leftrightarrow y' - ky = 0$

Sendo esta uma EDO linear, começamos por determinar um factor integrante para a mesma

$$\mu(t) = e^{\int -k\,dt} = e^{-kt}$$

Assim vem

$$e^{-kt} y' - k e^{-kt} y = 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{d}{dt}\left(e^{-kt} y\right) = 0 \Leftrightarrow e^{-kt} y = c, \; c \in \mathbb{R}$$

$$\Leftrightarrow y = c e^{kt}, \; c \in \mathbb{R}$$

②

Como $y(0) = 27$ temos

$$27 = c\,e^{0} \Leftrightarrow c = 27$$

$$\therefore\ y = 27\,e^{kt}$$

c) $y(1) = 26.9 \Leftrightarrow 26.9 = 27\,e^{k} \Leftrightarrow$

$$\Leftrightarrow \frac{26.9}{27} = e^{k} \Leftrightarrow$$

$$\Leftrightarrow k = \log\left(\frac{26.9}{27}\right)$$

$$\therefore\ y = 27\,e^{\log\left(\frac{26.9}{27}\right)t}$$

$y(t) = 13.5 \Leftrightarrow 13.5 = 27\,e^{\log\left(\frac{26.9}{27}\right)t}$

$$\Leftrightarrow \frac{13.5}{27} = e^{\log\left(\frac{26.9}{27}\right)t}$$

$$\Leftrightarrow \log\left(\frac{13.5}{27}\right) = \log\left(\frac{26.9}{27}\right)t$$

$$\Leftrightarrow t = \frac{\log\left(\frac{13.5}{27}\right)}{\log\left(\frac{26.9}{27}\right)} \text{ horas}$$

2 $x^3+x^3y^4=yy'e^{-x^2} \Leftrightarrow$

$\Leftrightarrow x^3(1+y^4)=yy'e^{-x^2} \Leftrightarrow$

$\Leftrightarrow x^3e^{x^2} = \frac{y}{1+y^4}y' \Leftrightarrow$

$\Leftrightarrow \int x^3e^{x^2}dx = \int \frac{y}{1+y^4}y'dx \Leftrightarrow$

$\Leftrightarrow \int x^3e^{x^2}dx = \int \frac{y}{1+y^4}dy$

Determinemos $\int x^3e^{x^2}dx$ utilizando primitivação por partes

$$\int x^3e^{x^2}dx = \int x^2xe^{x^2}dx = x^2\frac{e^{x^2}}{2} - \int xe^{x^2}dx =$$

$$= x^2\frac{e^{x^2}}{2} - \frac{e^{x^2}}{2} + c, \ c\in\mathbb{R}$$

Assim vem

$$x^2\frac{e^{x^2}}{2} - \frac{e^{x^2}}{2} + c_1 = \frac{1}{2}\operatorname{arctg}(y^2) + c_2, \ c_1, c_2 \in\mathbb{R} \Leftrightarrow$$

$$\Leftrightarrow \operatorname{arctg}(y^2) = x^2e^{x^2} - e^{x^2} + c_3, \ c_3\in\mathbb{R} \Leftrightarrow$$

$$\Leftrightarrow y^2 = \operatorname{tg}(x^2e^{x^2} - e^{x^2} + c_3), \ c_3\in\mathbb{R} \Leftrightarrow$$

$$\Leftrightarrow y = \pm\sqrt{\operatorname{tg}(x^2e^{x^2} - e^{x^2} + c_3)}, \ c_3\in\mathbb{R}$$

3

a) É possível prolongar f por continuidade a $(0,-2)$ sse $\lim\limits_{(x,y)\to(0,-2)} f(x,y)$ existe.

$$\lim_{(x,y)\to(0,-2)} f(x,y) = \lim_{(x,y)\to(0,-2)} x(y+2)\log(x^2+(y+2)^2)+x$$

$$= \lim_{\rho\to 0} \rho^2 \cos\theta \operatorname{sen}\theta \log(\rho^2) + \rho\cos\theta$$

$$\begin{cases} x = \rho\cos\theta & \rho > 0 \\ y = -2+\rho\operatorname{sen}\theta & \theta\in[0,2\pi[\end{cases}$$

Como

$$\lim_{\rho\to 0} \rho^2\log(\rho^2) = \lim_{\rho\to 0} \frac{-\log\left(\frac{1}{\rho^2}\right)}{\frac{1}{\rho^2}} = -0 = 0$$

$$\lim_{\rho\to 0} \rho = 0$$

e $\cos\theta\operatorname{sen}\theta$ e $\cos\theta$ são limitadas entre -1 e 1,

podemos concluir que

$$\lim_{\rho\to 0} \rho^2\log(\rho^2)\cos\theta\operatorname{sen}\theta + \rho\cos\theta = 0+0 = 0$$

Assim, é possível prolongar f por continuidade a $(0,-2)$ sendo a função prolongamento definida hor

$$\bar{f}(x,y) = \begin{cases} x(y+2)\log(x^2+(y+2)^2)+x, & \text{se } (x,y)\neq(0,-2) \\ 0, & \text{se } (x,y)=(0,-2) \end{cases}$$

b)

$$\frac{d\bar{f}}{dx}(0,-2) = \lim_{h\to 0} \frac{\bar{f}(h,-2)-\bar{f}(0,-2)}{h} =$$

$$= \lim_{h\to 0} \frac{h \times 0 \times \log(h^2)+h-0}{h} =$$

$$= \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1$$

$$\frac{d\bar{f}}{dy}(0,-2) = \lim_{h\to 0} \frac{\bar{f}(0,-2+h)-\bar{f}(0,-2)}{h} =$$

$$= \lim_{h\to 0} \frac{0 \times h \times \log(h^2)+0}{h} = \lim_{h\to 0} 0 = 0$$

Se $\bar{f}$ for diferenciável em $(0,-2)$ então

$$d\bar{f}(0,-2)(h_1,h_2) = h_1$$

$\bar{f}$ é diferenciável em $(0,-2)$ sse

$$\bar{f}(0+h_1, -2+h_2) = \bar{f}(0,-2) + d\bar{f}(0,-2)(h_1,h_2) +$$

$$+ \|(h_1,h_2)\| \, \varepsilon(h_1,h_2) \, , \text{com}$$

$$\lim_{(h_1,h_2)\to(0,0)} \varepsilon(h_1,h_2) = 0$$

$$\lim_{(h_1,h_2)\to(0,0)} \varepsilon(h_1,h_2) = \lim_{(h_1,h_2)\to(0,0)} \frac{h_1 h_2 \log(h_1^2+h_2^2) + \cancel{h_1} - \cancel{h_1}}{\sqrt{h_1^2+h_2^2}}$$

$$= \lim_{(h_1,h_2)\to(0,0)} \frac{h_1 h_2 \log(h_1^2+h_2^2)}{\sqrt{h_1^2+h_2^2}}$$

$$\begin{cases} h_1 = \rho\cos\theta \\ h_2 = \rho\,\text{sen}\,\theta \end{cases} \quad = \lim_{\rho\to 0} \frac{\rho^2 \cos\theta\,\text{sen}\,\theta \log(\rho^2)}{\rho} =$$

$$\rho > 0$$
$$\theta \in [0, 2\pi[\quad = \lim_{\rho\to 0} 2\rho\cos\theta\,\text{sen}\,\theta \log(\rho)$$

Como

$$\lim_{\rho\to 0} \rho\log(\rho) = \lim_{\rho\to 0} \frac{-\log\left(\frac{1}{\rho}\right)}{\frac{1}{\rho}} = -0 = 0$$

e $\cos\theta\,\text{sen}\,\theta$ é limitada entre -1 e 1 temos

$$\lim_{\rho\to 0} 2\rho\log(\rho)\cos\theta\,\text{sen}\,\theta = 2\times 0 = 0$$

Assim, $\bar{f}$ é diferenciável em $(0,-2)$

e) Sendo $\bar{f}$ diferenciável em $(0,-2)$ faz sentido usar o seu diferencial para determinar uma aproximação de primeira ordem para

$$\bar{f}(0.02,-1.999) = \bar{f}((0,-2)+(0.02,0.001))$$

$$\simeq \bar{f}(0,-2) + d\bar{f}(0,-2)(0.02,0.001) =$$

$$= 0 + 0.02 = 0.02$$

4 a) Comecemos por determinar $Jac\, f(x,y)$.

$$Jac f(x,y) = \begin{bmatrix} e^x \operatorname{arctg}(x+y) + \frac{e^x}{1+(x+y)^2} & \frac{e^x}{1+(x+y)^2} \\ \\ \\ \frac{2x}{x^2+y^2}\cos(x^3y) - \log(x^2+y^2)\operatorname{sen}(x^3y)3x^2y & * \end{bmatrix}$$

$$* \ \frac{2y}{x^2+y^2}\cos(x^3y) - \log(x^2+y^2)\operatorname{sen}(x^3y)x^3$$

Atendendo a que todas as derivadas parciais da função são funções contínuas em $\mathbb{R}^2 \setminus \{(0,0)\}$, pois resultam de operações algébricas e compostas de funções contínuas (exponencial, seno, cosseno, arcotangente, logaritmo e polinómios), podemos concluir que $f \in C^1(\mathbb{R}^2 \setminus \{(0,0)\})$ logo f é diferenciável em $\mathbb{R}^2 \setminus \{(0,0)\}$.

b) $(g \circ f)(1,0) = g(f(1,0)) = g\left(\frac{e\pi}{4}, 0\right)$

$g \in C^1(\mathbb{R}^2)$ logo g é diferenciável em $\left(\frac{e\pi}{4}, 0\right)$

$f \in C^1(\mathbb{R}^2 \setminus \{(0,0)\})$ logo f é diferenciável em $(1,0)$

A composta de funções diferenciáveis é diferenciável, pelo que $g \circ f$ é diferenciável em $(1,0)$. Sabe-se ainda que:

$$\text{Jac}\, g\circ f\,(1,0) = \text{Jac}\, g\left(\frac{e\pi}{4}, 0\right) \times \text{Jac}\, f(1,0) =$$

$$= \begin{bmatrix} -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{e\pi}{4} + \frac{e}{2} & \frac{e}{2} \\ 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{e\pi}{4} - \frac{e}{2} + 6 & -\frac{e}{2} \end{bmatrix} = \nabla g\circ f\,(1,0)^T$$

5

$$g(u,v) = f(x,y) = f(u+v,\ u-v)$$

$$\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$$

$$\frac{\partial g}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}$$

$$\frac{\partial^2 g}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right) =$$

$$= \frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y \partial x}\frac{\partial y}{\partial u} + \frac{\partial^2 f}{\partial x \partial y}\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial u} =$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y \partial x} + \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial^2 g}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) =$$

$$= \frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial v} + \frac{\partial^2 f}{\partial y \partial x}\frac{\partial y}{\partial v} - \frac{\partial^2 f}{\partial x \partial y}\frac{\partial x}{\partial v} - \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial v} =$$

$$= \frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y \partial x} - \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} = \frac{\partial^2 f}{\partial x^2} + \cancel{\frac{\partial^2 f}{\partial y \partial x}} + \cancel{\frac{\partial^2 f}{\partial x \partial y}} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial x^2} - \cancel{\frac{\partial^2 f}{\partial y \partial x}}$$

$$- \cancel{\frac{\partial^2 f}{\partial x \partial y}} + \frac{\partial^2 f}{\partial y^2} = 2\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) = 2 \times 0 = 0$$

6

a) Seja

$$f(x,y,z) = \operatorname{sen}(e^{x^3y} + 3z - 1) + \operatorname{arctg}(xz + y^2) + 1$$

- $f(0,0,\frac{\pi}{2}) = \operatorname{sen}(e^0 + \frac{3\pi}{2} - 1) + \operatorname{arctg}(0) + 1 =$

$$= \operatorname{sen}(\cancel{1} + \frac{3\pi}{2} - \cancel{1}) + 1 = -1 + 1 = 0$$

- $\frac{\partial f}{\partial x}(x,y,z) = \cos(e^{x^3y} + 3z - 1)\, e^{x^3y}\, 3x^2y +$

$$+ \frac{z}{1 + (xz + y^2)^2}$$

$\frac{\partial f}{\partial y}(x,y,z) = \cos(e^{x^3y} + 3z - 1)\, e^{x^3y}\, x^3 +$

$$+ \frac{2y}{1 + (xz + y^2)^2}$$

$\frac{\partial f}{\partial z}(x,y,z) = 3\cos(e^{x^3y} + 3z - 1) + \frac{x}{1 + (xz + y^2)^2}$

Como todas as derivadas parciais são funções contínuas em $\mathbb{R}^3$, por resultarem de operações algébricas e compostas de funções contínuas (exponencial, cosseno, polinómios), podemos concluir que $f \in \mathcal{C}^1(\mathbb{R}^3)$ pelo que f é de classe $\mathcal{C}^1$ numa vizinhança de $(0,0,\pi/2)$

- $\frac{\partial f}{\partial x}(0,0,\pi/2) = \frac{\pi/2}{1} = \frac{\pi}{2} \neq 0$

Logo, pelo Teorema da função implícita, existem U vizinhança de $(0,\pi/2)$ e V vizinhança de 0 e $\phi : U \to V$ tal que $\phi \in \mathcal{C}^1(U)$ e

$$\forall (y,z) \in U, \forall x \in V, \quad f(x,y,z) = 0 \Leftrightarrow x = \phi(y,z)$$

b) Como $\phi \in \mathcal{C}^1(U)$ e $(0,\pi/2) \in U$ sabemos que ϕ é diferenciável em $(0,\pi/2)$ logo

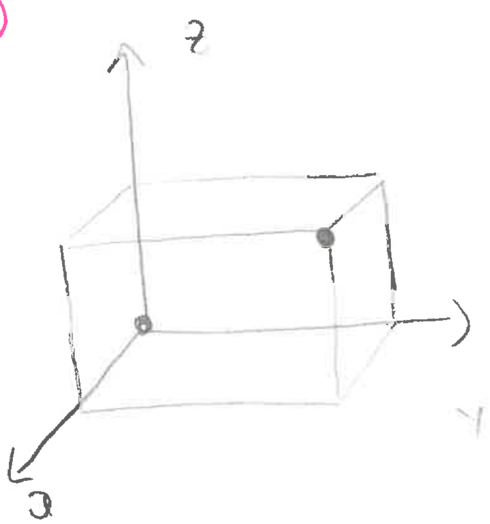
$$\phi'_{(3,-2)}(0,\pi/2) = \nabla\phi(0,\pi/2)^T \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$\frac{\partial \phi}{\partial y}(0,\pi/2) = -\frac{\frac{\partial f}{\partial y}(0,0,\pi/2)}{\frac{\partial f}{\partial x}(0,0,\pi/2)} = -\frac{\cos(\frac{3\pi}{2}) \times 0}{\pi/2} = 0$$

$$\frac{\partial \phi}{\partial z}(0,\pi/2) = -\frac{\frac{\partial f}{\partial z}(0,0,\pi/2)}{\frac{\partial f}{\partial x}(0,0,\pi/2)} = -\frac{3\cos(\frac{3\pi}{2})}{\pi/2} = 0$$

Logo $\phi'_{(3,-2)}(0,\pi/2) = [0 \;\; 0] \begin{bmatrix} 3 \\ -2 \end{bmatrix} = 0$

7



max xyz

s.a $3x+2y+z=1$

$\mathcal{L}(x,y,z,\lambda) = xyz - \lambda(3x+2y+z-1)$, com $\lambda \in \mathbb{R}$

$$\nabla\mathcal{L}(x,y,z,\lambda)=0 \Leftrightarrow \begin{cases} yz - 3\lambda = 0 \\ xz - 2\lambda = 0 \\ xy - \lambda = 0 \\ -(3x+2y+z-1) = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} xyz - 3\lambda x = 0 \\ xyz - 2\lambda y = 0 \\ xyz - \lambda z = 0 \\ 3x+2y+z = 1 \end{cases} \Leftrightarrow \begin{cases} -3\lambda x + 2\lambda y = 0 \\ -2\lambda y + \lambda z = 0 \\ \equiv \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \lambda(-3x+2y) = 0 \\ \equiv \end{cases} \Leftrightarrow \begin{cases} \lambda = 0 \vee 3x = 2y \\ \equiv \end{cases}$$

Se $\lambda = 0$ então

$xyz = 0 \Leftrightarrow x = 0 \vee y = 0 \vee z = 0$ não estando

a caixa bem definida.

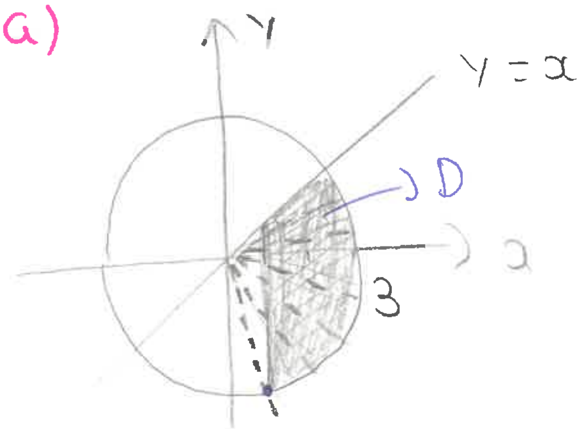
Assim,

$$\begin{cases} x = \frac{2y}{3} \\ \wedge\ (-2y+z) = 0 \\ = \end{cases} \Leftrightarrow \begin{cases} x = \frac{2y}{3} \\ z = 2y \\ - \\ 2y + 2y + 2y = 1 \end{cases} \Leftrightarrow \begin{cases} x = 1/9 \\ z = 1/3 \\ - \\ y = 1/6 \end{cases}$$

$\therefore$ A posição do vértice deverá ser $(1/9, 1/6, 1/3)$

8

a)



$\|x\| = \rho$

$$\begin{cases} x = \rho\cos\theta & -\arccos(1/3) \leq \theta \leq \pi/4 \\ y = \rho\sin\theta & \frac{1}{\cos\theta} \leq \rho \leq 3 \end{cases}$$

$1 = 3\cos\theta \Leftrightarrow \cos\theta = \frac{1}{3} \Leftrightarrow$

$\Leftrightarrow \theta = \arccos(1/3)$

$x = 1 \Leftrightarrow \rho\cos\theta = 1 \Leftrightarrow \rho = \frac{1}{\cos\theta}$

$$\text{Área} = \iint_D 1\, dx\,dy =$$

$$= \int_{-\arccos(1/3)}^{\pi/4} \int_{\frac{1}{\cos\theta}}^{3} \rho\, d\rho\, d\theta$$

b)

$$\text{Área} = \int_{-\arccos(1/3)}^{\pi/4} \left[\frac{\rho^2}{2} \right]_{\frac{1}{\cos\theta}}^{3} d\theta =$$

$$= \int_{-\arccos(1/3)}^{\pi/4} \frac{9}{2} - \frac{1}{2} \frac{1}{\cos^2\theta} \, d\theta =$$

$$= \left[\frac{9}{2}\theta - \frac{1}{2} \operatorname{tg}\theta \right]_{-\arccos(1/3)}^{\pi/4} =$$

$$= \frac{9\pi}{8} - \frac{1}{2} + \frac{9}{2}\arccos(1/3) + \frac{1}{2}\operatorname{tg}(-\arccos(1/3))$$

$$= \frac{9\pi}{8} - \frac{1}{2} + \frac{9}{2}\arccos(1/3) + \frac{1}{2} \frac{-\frac{2\sqrt{2}}{3}}{1/3} =$$

$$x^2 + y^2 = 9 \Leftrightarrow 1 + y^2 = 9 \Leftrightarrow$$

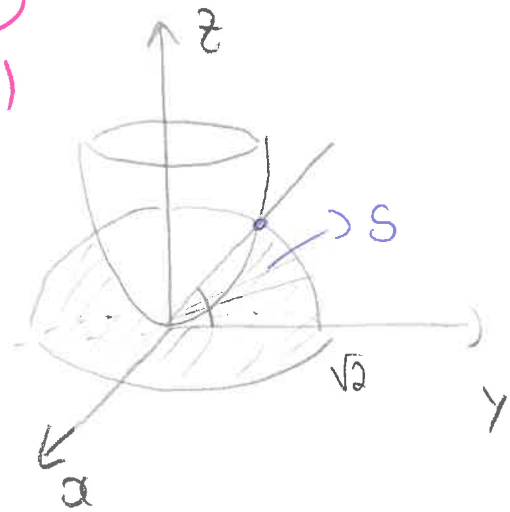
$$\Leftrightarrow y^2 = 8 \Leftrightarrow y = \pm 2\sqrt{2}$$

$$-2\sqrt{2} = 3 \operatorname{sen}\theta \Leftrightarrow \operatorname{sen}\theta = -\frac{2\sqrt{2}}{3}$$

$$= \frac{9\pi}{8} - \frac{1}{2} + \frac{9}{2}\arccos(1/3) - \sqrt{2}$$

9

a)



$$\begin{cases} x = \rho \cos\theta \sin\varphi \\ y = \rho \sin\theta \sin\varphi \\ z = \rho \cos\varphi \end{cases}$$

$\frac{\pi}{4} \leq \varphi \leq \pi/2$

$0 \leq \theta \leq 2\pi$

$\frac{\cos\varphi}{\sin^2\varphi} \leq \rho \leq \sqrt{2}$

$|Jac| = \rho^2 \sin\varphi$

$z = x^2+y^2 \Leftrightarrow$

$\Leftrightarrow \rho\cos\varphi = \rho^2 \sin^2\varphi \Leftrightarrow$

$\Leftrightarrow \rho\cos\varphi - \rho^2\sin^2\varphi = 0 \Leftrightarrow$

$\Leftrightarrow \rho(\cos\varphi - \rho\sin^2\varphi) = 0 \Leftrightarrow$

$\Leftrightarrow \rho = 0 \vee \rho = \frac{\cos\varphi}{\sin^2\varphi}$

$$\begin{cases} z = x^2+y^2 \\ x^2+y^2+z^2 = 2 \end{cases} \Leftrightarrow \begin{cases} — \\ z^2+z-2=0 \end{cases} \Leftrightarrow \begin{cases} — \\ z = \frac{-1\pm\sqrt{1+8}}{2} \end{cases}$$

$$\Leftrightarrow \begin{cases} — \\ z = \frac{-1\pm 3}{2} \end{cases} \Leftrightarrow \begin{cases} — \\ \cancel{z = -2} \vee z = 1 \end{cases}$$

$1 = \sqrt{2}\cos\varphi \Leftrightarrow \cos\varphi = \frac{1}{\sqrt{2}} \Leftrightarrow \varphi = \arccos\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

$$\text{Volume} = \iiint_S 1\, dx\,dy\,dz = \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_{\frac{\cos\varphi}{\sin^2\varphi}}^{\sqrt{2}} \rho^2\sin\varphi \, d\rho\, d\varphi\, d\theta$$

b)

$$\text{volume} = 2\pi \int_{\pi/4}^{\pi/2} \left[\frac{\rho^3}{3} \operatorname{sen}\varphi \right]_{\frac{\cos\varphi}{\operatorname{sin}^2\varphi}}^{\sqrt{2}} d\varphi =$$

$$= \frac{2\pi}{3} \int_{\pi/4}^{\pi/2} 2\sqrt{2}\operatorname{sen}\varphi - \frac{\cos^3\varphi}{\operatorname{sen}^6\varphi} \operatorname{sen}\varphi \, d\varphi$$

$$= \frac{2\pi}{3} \int_{\pi/4}^{\pi/2} 2\sqrt{2}\operatorname{sin}\varphi - \frac{\cos^3\varphi}{\operatorname{sin}^5\varphi} \, d\varphi =$$

$$= \frac{2\pi}{3} \int_{\pi/4}^{\pi/2} 2\sqrt{2}\operatorname{sin}\varphi - \frac{1}{\operatorname{sin}^2\varphi} \operatorname{cotg}^3\varphi \, d\varphi =$$

$$= \frac{2\pi}{3} \left[-2\sqrt{2}\cos\varphi + \frac{\operatorname{cotg}^4\varphi}{4} \right]_{\pi/4}^{\pi/2} =$$

$$= \frac{2\pi}{3} \left(2\sqrt{2}\,\frac{\sqrt{2}}{2} - \frac{1}{4} \right) = \frac{2\pi}{3} \left(2 - \frac{1}{4} \right)$$

$$= \frac{2\pi}{3} \left(\frac{7}{4} \right) = \frac{7\pi}{6}$$

10 x_* é um ponto estacionário de f sse $\nabla f(x_*) = 0$

Consideremos $-\nabla f(x_*)$. Sabemos que existem $\alpha_1, \dots, \alpha_m \geq 0$ tais que

$$-\nabla f(x_*) = \sum_{i=1}^{m} \alpha_i v_i$$

$$-\nabla f(x_*)^T \nabla f(x_*) = \left(\sum_{i=1}^{m} \alpha_i v_i \right)^T \nabla f(x_*) =$$

$$= \sum_{i=1}^{m} \underbrace{\alpha_i}_{\geq 0} \underbrace{v_i^T \nabla f(x_*)}_{\geq 0} \geq 0$$

Mas

$$-\nabla f(x_*)^T \nabla f(x_*) = -\|\nabla f(x_*)\|^2 \leq 0$$

Logo

$$-\|\nabla f(x_*)\|^2 = 0 \Leftrightarrow \nabla f(x_*) = 0$$

pelo que x_* é um ponto estacionário de f.