CHAPTER 16
Topics in Vector Calculus

EXERCISE SET 16.1

1. **(a)** III because the vector field is independent of y and the direction is that of the negative x-axis for negative x, and positive for positive

(b) IV, because the y-component is constant, and the x-component varies priodically with x

2. **(a)** I, since the vector field is constant

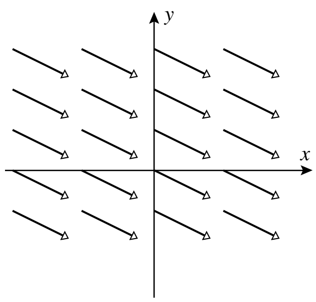
(b) II, since the vector field points away from the origin

3. **(a)** true **(b)** true **(c)** true

4. **(a)** false, the lengths are equal to 1 **(b)** false, the y-component is then zero

(c) false, the x-component is then zero

5.



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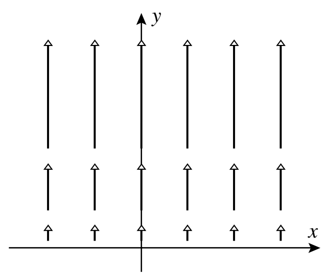

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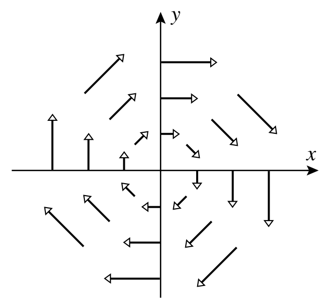

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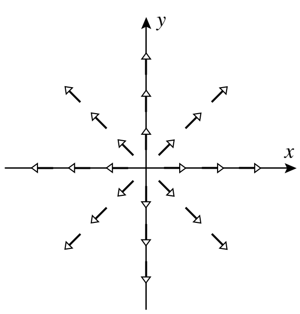

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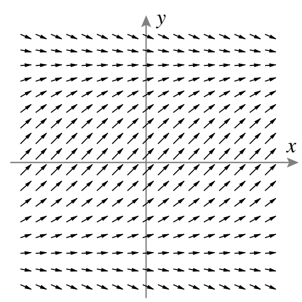

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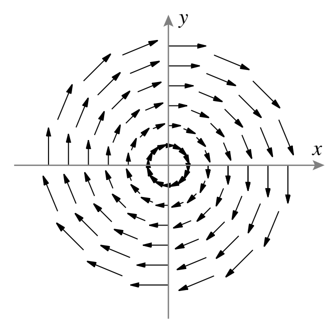


11. **(a)** $\nabla\phi = \phi_x\mathbf{i} + \phi_y\mathbf{j} = \dfrac{y}{1+x^2y^2}\mathbf{i} + \dfrac{x}{1+x^2y^2}\mathbf{j} = \mathbf{F}$, so $\mathbf{F}$ is conservative for all x, y

(b) $\nabla\phi = \phi_x\mathbf{i} + \phi_y\mathbf{j} = 2x\mathbf{i} - 6y\mathbf{j} + 8z\mathbf{k} = \mathbf{F}$ so $\mathbf{F}$ is conservative for all x, y

12. **(a)** $\nabla\phi = \phi_x\mathbf{i} + \phi_y\mathbf{j} = (6xy - y^3)\mathbf{i} + (4y + 3x^2 - 3xy^2)\mathbf{j} = \mathbf{F}$, so $\mathbf{F}$ is conservative for all x, y

(b) $\nabla\phi = \phi_x\mathbf{i} + \phi_y\mathbf{j} + \phi_z\mathbf{k} = (\sin z + y\cos x)\mathbf{i} + (\sin x + z\cos y)\mathbf{j} + (x\cos z + \sin y)\mathbf{k} = \mathbf{F}$, so $\mathbf{F}$ is conservative for all x, y

13. div $\mathbf{F} = 2x + y$, curl $\mathbf{F} = z\mathbf{i}$

14. div $\mathbf{F} = z^3 + 8y^3x^2 + 10zy$, curl $\mathbf{F} = 5z^2\mathbf{i} + 3xz^2\mathbf{j} + 4xy^4\mathbf{k}$

15. div $\mathbf{F} = 0$, curl $\mathbf{F} = (40x^2z^4 - 12xy^3)\mathbf{i} + (14y^3z + 3y^4)\mathbf{j} - (16xz^5 + 21y^2z^2)\mathbf{k}$

16. div $\mathbf{F} = ye^{xy} + \sin y + 2\sin z\cos z$, curl $\mathbf{F} = -xe^{xy}\mathbf{k}$

17. div $\mathbf{F} = \dfrac{2}{\sqrt{x^2+y^2+z^2}}$, curl $\mathbf{F} = \mathbf{0}$

18. div $\mathbf{F} = \dfrac{1}{x} + xze^{xyz} + \dfrac{x}{x^2+z^2}$, curl $\mathbf{F} = -xye^{xyz}\mathbf{i} + \dfrac{z}{x^2+z^2}\mathbf{j} + yze^{xyz}\mathbf{k}$

19. $\nabla\cdot(\mathbf{F}\times\mathbf{G}) = \nabla\cdot(-(z+4y^2)\mathbf{i} + (4xy+2xz)\mathbf{j} + (2xy-x)\mathbf{k}) = 4x$

20. $\nabla\cdot(\mathbf{F}\times\mathbf{G}) = \nabla\cdot((x^2yz^2 - x^2y^2)\mathbf{i} - xy^2z^2\mathbf{j} + xy^2z\mathbf{k}) = -xy^2$

21. $\nabla\cdot(\nabla\times\mathbf{F}) = \nabla\cdot(-\sin(x-y)\mathbf{k}) = 0$

22. $\nabla\cdot(\nabla\times\mathbf{F}) = \nabla\cdot(-ze^{yz}\mathbf{i} + xe^{xz}\mathbf{j} + 3e^{y}\mathbf{k}) = 0$

23. $\nabla\times(\nabla\times\mathbf{F}) = \nabla\times(xz\mathbf{i} - yz\mathbf{j} + y\mathbf{k}) = (1+y)\mathbf{i} + x\mathbf{j}$

24. $\nabla\times(\nabla\times\mathbf{F}) = \nabla\times((x+3y)\mathbf{i} - y\mathbf{j} - 2xy\mathbf{k}) = -2x\mathbf{i} + 2y\mathbf{j} - 3\mathbf{k}$

27. Let $\mathbf{F} = f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$; div $(k\mathbf{F}) = k\dfrac{\partial f}{\partial x} + k\dfrac{\partial g}{\partial y} + k\dfrac{\partial h}{\partial z} = k$ div $\mathbf{F}$

28. Let $\mathbf{F} = f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$; curl $(k\mathbf{F}) = k\left(\dfrac{\partial h}{\partial y} - \dfrac{\partial g}{\partial z}\right)\mathbf{i} + k\left(\dfrac{\partial f}{\partial z} - \dfrac{\partial h}{\partial x}\right)\mathbf{j} + k\left(\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y}\right)\mathbf{k} = k$ curl $\mathbf{F}$

29. Let $\mathbf{F} = f(x,y,z)\mathbf{i} + g(x,y,z)\mathbf{j} + h(x,y,z)\mathbf{k}$ and $\mathbf{G} = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$, then

$$\begin{aligned}\text{div }(\mathbf{F}+\mathbf{G}) &= \left(\frac{\partial f}{\partial x} + \frac{\partial P}{\partial x}\right) + \left(\frac{\partial g}{\partial y} + \frac{\partial Q}{\partial y}\right) + \left(\frac{\partial h}{\partial z} + \frac{\partial R}{\partial z}\right)\\ &= \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) + \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) = \text{div }\mathbf{F} + \text{div }\mathbf{G}\end{aligned}$$

30. Let $\mathbf{F} = f(x,y,z)\mathbf{i} + g(x,y,z)\mathbf{j} + h(x,y,z)\mathbf{k}$ and $\mathbf{G} = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$, then

$$\begin{aligned}\text{curl }(\mathbf{F}+\mathbf{G}) = &\left[\frac{\partial}{\partial y}(h+R) - \frac{\partial}{\partial z}(g+Q)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(f+P) - \frac{\partial}{\partial x}(h+R)\right]\mathbf{j}\\ &+ \left[\frac{\partial}{\partial x}(g+Q) - \frac{\partial}{\partial y}(f+P)\right]\mathbf{k};\end{aligned}$$

expand and rearrange terms to get curl $\mathbf{F}$ + curl $\mathbf{G}$.

31. Let $\mathbf{F} = f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$;

$$\begin{aligned}\text{div }(\phi\mathbf{F}) &= \left(\phi\frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial x}f\right) + \left(\phi\frac{\partial g}{\partial y} + \frac{\partial \phi}{\partial y}g\right) + \left(\phi\frac{\partial h}{\partial z} + \frac{\partial \phi}{\partial z}h\right)\\ &= \phi\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) + \left(\frac{\partial \phi}{\partial x}f + \frac{\partial \phi}{\partial y}g + \frac{\partial \phi}{\partial z}h\right)\\ &= \phi\text{ div }\mathbf{F} + \nabla\phi\cdot\mathbf{F}\end{aligned}$$

32. Let $\mathbf{F} = f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$;
curl $(\phi\mathbf{F}) = \left[\frac{\partial}{\partial y}(\phi h) - \frac{\partial}{\partial z}(\phi g)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(\phi f) - \frac{\partial}{\partial x}(\phi h)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(\phi g) - \frac{\partial}{\partial y}(\phi f)\right]\mathbf{k}$; use the product rule to expand each of the partial derivatives, rearrange to get ϕ curl $\mathbf{F} + \nabla\phi \times \mathbf{F}$

33. Let $\mathbf{F} = f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$;

$$\text{div(curl }\mathbf{F}) = \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)$$
$$= \frac{\partial^2 h}{\partial x\partial y} - \frac{\partial^2 g}{\partial x\partial z} + \frac{\partial^2 f}{\partial y\partial z} - \frac{\partial^2 h}{\partial y\partial x} + \frac{\partial^2 g}{\partial z\partial x} - \frac{\partial^2 f}{\partial z\partial y} = 0,$$

assuming equality of mixed second partial derivatives

34. curl $(\nabla\phi) = \left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right)\mathbf{i} + \left(\frac{\partial^2\phi}{\partial z\partial x} - \frac{\partial^2\phi}{\partial x\partial z}\right)\mathbf{j} + \left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right)\mathbf{k} = \mathbf{0}$, assuming equality of mixed second partial derivatives

35. $\nabla\cdot(k\mathbf{F}) = k\nabla\cdot\mathbf{F},\ \nabla\cdot(\mathbf{F}+\mathbf{G}) = \nabla\cdot\mathbf{F} + \nabla\cdot\mathbf{G},\ \nabla\cdot(\phi\mathbf{F}) = \phi\nabla\cdot\mathbf{F} + \nabla\phi\cdot\mathbf{F},\ \nabla\cdot(\nabla\times\mathbf{F}) = 0$

36. $\nabla\times(k\mathbf{F}) = k\nabla\times\mathbf{F},\ \nabla\times(\mathbf{F}+\mathbf{G}) = \nabla\times\mathbf{F} + \nabla\times\mathbf{G},\ \nabla\times(\phi\mathbf{F}) = \phi\nabla\times\mathbf{F} + \nabla\phi\times\mathbf{F},\ \nabla\times(\nabla\phi) = \mathbf{0}$

37. **(a)** curl $\mathbf{r} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

(b) $\nabla\|\mathbf{r}\| = \nabla\sqrt{x^2+y^2+z^2} = \frac{x}{\sqrt{x^2+y^2+z^2}}\mathbf{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\mathbf{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\mathbf{k} = \frac{\mathbf{r}}{\|\mathbf{r}\|}$

38. **(a)** div $\mathbf{r} = 1 + 1 + 1 = 3$

(b) $\nabla\frac{1}{\|\mathbf{r}\|} = \nabla(x^2+y^2+z^2)^{-1/2} = -\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}} = -\frac{\mathbf{r}}{\|\mathbf{r}\|^3}$

39. **(a)** $\nabla f(r) = f'(r)\frac{\partial r}{\partial x}\mathbf{i} + f'(r)\frac{\partial r}{\partial y}\mathbf{j} + f'(r)\frac{\partial r}{\partial z}\mathbf{k} = f'(r)\nabla r = \frac{f'(r)}{r}\mathbf{r}$

(b) $\text{div}[f(r)\mathbf{r}] = f(r)\text{div }\mathbf{r} + \nabla f(r)\cdot\mathbf{r} = 3f(r) + \frac{f'(r)}{r}\mathbf{r}\cdot\mathbf{r} = 3f(r) + rf'(r)$

40. **(a)** $\text{curl}[f(r)\mathbf{r}] = f(r)\text{curl }\mathbf{r} + \nabla f(r)\times\mathbf{r} = f(r)\mathbf{0} + \frac{f'(r)}{r}\mathbf{r}\times\mathbf{r} = \mathbf{0}+\mathbf{0} = \mathbf{0}$

(b)
$$\nabla^2 f(r) = \text{div}[\nabla f(r)] = \text{div}\left[\frac{f'(r)}{r}\mathbf{r}\right] = \frac{f'(r)}{r}\text{div }\mathbf{r} + \nabla\frac{f'(r)}{r}\cdot\mathbf{r}$$
$$= 3\frac{f'(r)}{r} + \frac{rf''(r) - f'(r)}{r^3}\mathbf{r}\cdot\mathbf{r} = 2\frac{f'(r)}{r} + f''(r)$$

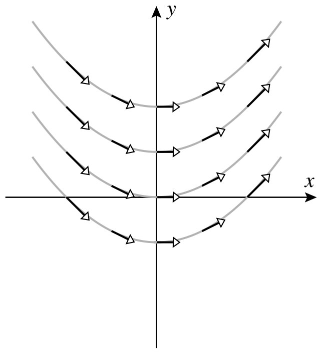
41. $f(r) = 1/r^3, f'(r) = -3/r^4$, $\text{div}(\mathbf{r}/r^3) = 3(1/r^3) + r(-3/r^4) = 0$

42. Multiply $3f(r) + rf'(r) = 0$ through by r^2 to obtain $3r^2f(r) + r^3f'(r) = 0$, $d[r^3f(r)]/dr = 0$, $r^3f(r) = C, f(r) = C/r^3$, so $\mathbf{F} = C\mathbf{r}/r^3$ (an inverse-square field).

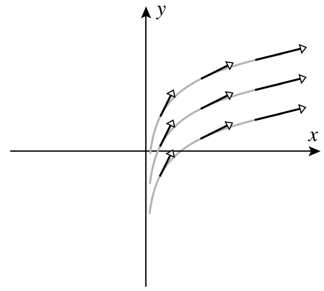
43. **(a)** At the point (x, y) the slope of the line along which the vector $-y\mathbf{i} + x\mathbf{j}$ lies is $-x/y$; the slope of the tangent line to C at (x, y) is dy/dx, so $dy/dx = -x/y$.

(b) $ydy = -xdx,\ y^2/2 = -x^2/2 + K_1,\ x^2 + y^2 = K$

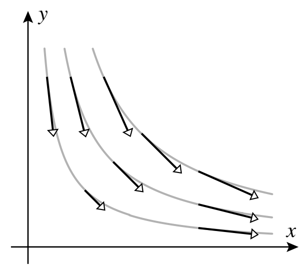
44. $dy/dx = x, y = x^2/2 + K$



45. $dy/dx = 1/x, y = \ln x + K$



46. $dy/dx = -y/x,\ (1/y)dy = (-1/x)dx,\ \ln y = -\ln x + K_1,$
$y = e^{K_1}e^{-\ln x} = K/x$



EXERCISE SET 16.2

1. **(a)** $\int_0^1 dy = 1$ because $s = y$ is arclength measured from $(0,0)$

(b) 0, because $\sin xy = 0$ along C

2. **(a)** $\int_C ds =$ length of line segment $= 2$ **(b)** 0, because x is constant and $dx = 0$

3. **(a)** $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt$, so $\int_0^1 (2t-3t^2)\sqrt{4+36t^2}\,dt = -\frac{11}{108}\sqrt{10} - \frac{1}{36}\ln(\sqrt{10}-3) - \frac{4}{27}$

(b) $\int_0^1 (2t - 3t^2)2\,dt = 0$ **(c)** $\int_0^1 (2t - 3t^2)6t\,dt = -\frac{1}{2}$

4. **(a)** $\int_0^1 t(3t^2)(6t^3)^2\sqrt{1+36t^2+324t^4}\,dt = \frac{864}{5}$ **(b)** $\int_0^1 t(3t^2)(6t^3)^2\,dt = \frac{54}{5}$

(c) $\int_0^1 t(3t^2)(6t^3)^2 6t\,dt = \frac{648}{11}$ **(d)** $\int_0^1 t(3t^2)(6t^3)^2 18t^2\,dt = 162$

5. **(a)** $C: x = t,\ y = t,\ 0 \le t \le 1;\ \int_0^1 6t\,dt = 3$

(b) $C: x = t,\ y = t^2,\ 0 \le t \le 1;\ \int_0^1 (3t + 6t^2 - 2t^3)dt = 3$

(c) $C: x = t,\ y = \sin(\pi t/2),\ 0 \le t \le 1;$
$$\int_0^1 [3t + 2\sin(\pi t/2) + \pi t\cos(\pi t/2) - (\pi/2)\sin(\pi t/2)\cos(\pi t/2)]dt = 3$$

(d) $C: x = t^3,\ y = t,\ 0 \le t \le 1;\ \int_0^1 (9t^5 + 8t^3 - t)dt = 3$

6. **(a)** $C: x = t, y = t, z = t, 0 \le t \le 1; \int_0^1 (t + t - t)\,dt = \frac{1}{2}$

(b) $C: x = t, y = t^2, z = t^3, 0 \le t \le 1; \int_0^1 (t^2 + t^3(2t) - t(3t^2))\,dt = -\frac{1}{60}$

(c) $C: x = \cos \pi t, y = \sin \pi t, z = t, 0 \le t \le 1; \int_0^1 (-\pi \sin^2 \pi t + \pi t \cos \pi t - \cos \pi t)\,dt = -\frac{\pi}{2} - \frac{2}{\pi}$

7. $\int_0^3 \frac{\sqrt{1+t}}{1+t}\ dt = \int_0^3 (1+t)^{-1/2}\ dt = 2$

8. $\sqrt{5}\int_0^1 \frac{1+2t}{1+t^2}\ dt = \sqrt{5}(\pi/4 + \ln 2)$

9. $\int_0^1 3(t^2)(t^2)(2t^3/3)(1+2t^2)\ dt = 2\int_0^1 t^7(1+2t^2)\ dt = 13/20$

10. $\frac{\sqrt{5}}{4}\int_0^{2\pi} e^{-t}\ dt = \sqrt{5}(1 - e^{-2\pi})/4$

11. $\int_0^{\pi/4} (8\cos^2 t - 16\sin^2 t - 20\sin t\cos t)dt = 1 - \pi$

12. $\int_{-1}^1 \left(\frac{2}{3}t - \frac{2}{3}t^{5/3} + t^{2/3}\right) dt = 6/5$

13. $C: x = (3-t)^2/3,\ y = 3 - t,\ 0 \le t \le 3;\ \int_0^3 \frac{1}{3}(3-t)^2 dt = 3$

14. $C: x = t^{2/3},\ y = t,\ -1 \le t \le 1;\ \int_{-1}^1 \left(\frac{2}{3}t^{2/3} - \frac{2}{3}t^{1/3} + t^{7/3}\right) dt = 4/5$

15. $C: x = \cos t,\ y = \sin t,\ 0 \le t \le \pi/2;\ \int_0^{\pi/2} (-\sin t - \cos^2 t)dt = -1 - \pi/4$

16. $C: x = 3 - t,\ y = 4 - 3t,\ 0 \le t \le 1;\ \int_0^1 (-37 + 41t - 9t^2)dt = -39/2$

17. $\int_0^1 (-3)e^{3t}dt = 1 - e^3$

18. $\int_0^{\pi/2} (\sin^2 t\cos t - \sin^2 t\cos t + t^4(2t))\,dt = \frac{\pi^6}{192}$

19. **(a)** $\int_0^{\ln 2} \left(e^{3t} + e^{-3t}\right)\sqrt{e^{2t} + e^{-2t}}\,dt$
$$= \frac{63}{64}\sqrt{17} + \frac{1}{4}\ln(4 + \sqrt{17}) - \frac{1}{8}\ln\frac{\sqrt{17}+1}{\sqrt{17}-1} - \frac{1}{4}\ln(\sqrt{2}+1) + \frac{1}{8}\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}$$

(b) $\int_0^{\pi/2} [\sin t\cos t\,dt - \sin^2 t\,dt] = \frac{1}{2} - \frac{\pi}{4}$

20. **(a)** $\displaystyle\int_0^{\pi/2} \cos^{21} t \sin^9 t \sqrt{(-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2}\, dt$

$$= 3\int_0^{\pi/2} \cos^{22} t \sin^{10} t\, dt = \frac{61{,}047}{4{,}294{,}967{,}296}\pi$$

(b) $\displaystyle\int_1^e \left(t^5 \ln t + 7t^2(2t) + t^4(\ln t)\frac{1}{t}\right) dt = \frac{5}{36}e^6 + \frac{59}{16}e^4 - \frac{491}{144}$

21. **(a)** $C_1 : (0,0)$ to $(1,0); x = t, y = 0, 0 \le t \le 1$
$C_2 : (1,0)$ to $(0,1); x = 1-t, y = t, 0 \le t \le 1$
$C_3 : (0,1)$ to $(0,0); x = 0, y = 1-t, 0 \le t \le 1$

$$\int_0^1 (0)dt + \int_0^1 (-1)dt + \int_0^1 (0)dt = -1$$

(b) $C_1 : (0,0)$ to $(1,0); x = t, y = 0, 0 \le t \le 1$
$C_2 : (1,0)$ to $(1,1); x = 1, y = t, 0 \le t \le 1$
$C_3 : (1,1)$ to $(0,1); x = 1-t, y = 1, 0 \le t \le 1$
$C_4 : (0,1)$ to $(0,0); x = 0, y = 1-t, 0 \le t \le 1$

$$\int_0^1 (0)dt + \int_0^1 (-1)dt + \int_0^1 (-1)dt + \int_0^1 (0)dt = -2$$

22. **(a)** $C_1 : (0,0)$ to $(1,1); x = t, y = t, 0 \le t \le 1$
$C_2 : (1,1)$ to $(2,0); x = 1+t, y = 1-t, 0 \le t \le 1$
$C_3 : (2,0)$ to $(0,0); x = 2-2t, y = 0, 0 \le t \le 1$

$$\int_0^1 (0)dt + \int_0^1 2dt + \int_0^1 (0)dt = 2$$

(b) $C_1 : (-5,0)$ to $(5,0); x = t, y = 0, -5 \le t \le 5$
$C_2 : x = 5\cos t, y = 5\sin t, 0 \le t \le \pi$

$$\int_{-5}^5 (0)dt + \int_0^\pi (-25)dt = -25\pi$$

23. $C_1 : x = t, y = z = 0, 0 \le t \le 1,\ \displaystyle\int_0^1 0\, dt = 0;\ \ C_2 : x = 1, y = t, z = 0, 0 \le t \le 1,\ \displaystyle\int_0^1 (-t)\, dt = -\frac{1}{2}$

$C_3 : x = 1, y = 1, z = t, 0 \le t \le 1,\ \displaystyle\int_0^1 3\, dt = 3;\ \ \displaystyle\int_C x^2 z\, dx - yx^2\, dy + 3\, dz = 0 - \frac{1}{2} + 3 = \frac{5}{2}$

24. $C_1 : (0,0,0)$ to $(1,1,0); x = t, y = t, z = 0, 0 \le t \le 1$
$C_2 : (1,1,0)$ to $(1,1,1); x = 1, y = 1, z = t, 0 \le t \le 1$
$C_3 : (1,1,1)$ to $(0,0,0); x = 1-t, y = 1-t, z = 1-t, 0 \le t \le 1$

$$\int_0^1 (-t^3)dt + \int_0^1 3\, dt + \int_0^1 -3dt = -1/4$$

25. $\displaystyle\int_0^\pi (0)dt = 0$

26. $\displaystyle\int_0^1 (e^{2t} - 4e^{-t})dt = e^2/2 + 4e^{-1} - 9/2$

27. $\displaystyle\int_0^1 e^{-t} dt = 1 - e^{-1}$

28. $\displaystyle\int_0^{\pi/2} (7\sin^2 t \cos t + 3\sin t \cos t)dt = 23/6$

29. Represent the circular arc by $x = 3\cos t, y = 3\sin t, 0 \le t \le \pi/2$.

$$\int_C x\sqrt{y}ds = 9\sqrt{3}\int_0^{\pi/2} \sqrt{\sin\ t}\cos t\ dt = 6\sqrt{3}$$

30. $\delta(x,y) = k\sqrt{x^2+y^2}$ where k is the constant of proportionality,

$$\int_C k\sqrt{x^2+y^2}ds = k\int_0^1 e^t(\sqrt{2}e^t)\ dt = \sqrt{2}k\int_0^1 e^{2t}\ dt = (e^2-1)k/\sqrt{2}$$

31. $$\int_C \frac{kx}{1+y^2}ds = 15k\int_0^{\pi/2} \frac{\cos t}{1+9\sin^2 t}\ dt = 5k\tan^{-1}3$$

32. $\delta(x,y,z) = kz$ where k is the constant of proportionality,

$$\int_C kzds = \int_1^4 k(4\sqrt{t})(2+1/t)\ dt = 136k/3$$

33. $C : x = t^2,\ y = t,\ 0 \le t \le 1;\ W = \displaystyle\int_0^1 3t^4 dt = 3/5$

34. $W = \displaystyle\int_1^3 (t^2 + 1 - 1/t^3 + 1/t)dt = 92/9 + \ln 3$

35. $W = \displaystyle\int_0^1 (t^3 + 5t^6)dt = 27/28$

36. $C_1 : (0,0,0)$ to $(1,3,1)$; $x = t,\ y = 3t,\ z = t,\ 0 \le t \le 1$
$C_2 : (1,3,1)$ to $(2,-1,4)$; $x = 1+t,\ y = 3-4t,\ z = 1+3t,\ 0 \le t \le 1$

$$W = \int_0^1 (4t + 8t^2)dt + \int_0^1 (-11 - 17t - 11t^2)dt = -37/2$$

37. Since $\mathbf{F}$ and $\mathbf{r}$ are parallel, $\mathbf{F}\cdot\mathbf{r} = \|\mathbf{F}\|\|\mathbf{r}\|$, and since $\mathbf{F}$ is constant,

$$\int \mathbf{F}\cdot d\mathbf{r} = \int_C d(\mathbf{F}\cdot\mathbf{r}) = \int_C d(\|\mathbf{F}\|\|\mathbf{r}\|) = \sqrt{2}\int_{-4}^4 \sqrt{2}dt = 16$$

38. $\displaystyle\int_C \mathbf{F}\cdot\mathbf{r} = 0$, since $\mathbf{F}$ is perpendicular to the curve

39. $C : x = 4\cos t, y = 4\sin t, 0 \le t \le \pi/2$

$$\int_0^{\pi/2} \left(-\frac{1}{4}\sin t + \cos t\right) dt = 3/4$$

40. $C_1 : (0,3)$ to $(6,3); x = 6t, y = 3, 0 \le t \le 1$
$C_2 : (6,3)$ to $(6,0); x = 6, y = 3-3t, 0 \le t \le 1$

$$\int_0^1 \frac{6}{36t^2+9}dt + \int_0^1 \frac{-12}{36+9(1-t)^2}dt = \frac{1}{3}\tan^{-1}2 - \frac{2}{3}\tan^{-1}(1/2)$$

41. Represent the parabola by $x = t, y = t^2, 0 \le t \le 2$.

$$\int_C 3xds = \int_0^2 3t\sqrt{1+4t^2}\ dt = (17\sqrt{17}-1)/4$$

42. Represent the semicircle by $x = 2\cos t, y = 2\sin t, 0 \le t \le \pi$.

$$\int_C x^2yds = \int_0^\pi 16\cos^2 t\sin t\ dt = 32/3$$

43. **(a)** $2\pi rh = 2\pi(1)2 = 4\pi$ **(b)** $S = \int_C z(t)\,dt$

(c) $C : x = \cos t, y = \sin t, 0 \le t \le 2\pi; S = \int_0^{2\pi} (2 + (1/2)\sin 3t)\,dt = 4\pi$

44. $C : x = a\cos t, y = -a\sin t, 0 \le t \le 2\pi,$

$$\int_C \frac{x\,dy - y\,dx}{x^2 + y^2} = \int_0^{2\pi} \frac{-a^2\cos^2 t - a^2\sin^2 t}{a^2}\,dt = \int_0^{2\pi} dt = 2\pi$$

45. $W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (\lambda t^2(1-t), t - \lambda t(1-t))\cdot(1, \lambda - 2\lambda t)\,dt = -\lambda/12$, $W = 1$ when $\lambda = -12$

46. The force exerted by the farmer is $\mathbf{F} = \left(150 + 20 - \frac{1}{10}z\right)\mathbf{k} = \left(170 - \frac{3}{4\pi}t\right)\mathbf{k}$, so $\mathbf{F}\cdot d\mathbf{r} = \left(170 - \frac{1}{10}z\right)dz$, and $W = \int_0^{60}\left(170 - \frac{1}{10}z\right)dz = 10{,}020$. Note that the functions $x(z), y(z)$ are irrelevant.

EXERCISE SET 16.3

1. $\partial x/\partial y = 0 = \partial y/\partial x$, conservative so $\partial\phi/\partial x = x$ and $\partial\phi/\partial y = y$, $\phi = x^2/2 + k(y)$, $k'(y) = y$, $k(y) = y^2/2 + K$, $\phi = x^2/2 + y^2/2 + K$

2. $\partial(3y^2)/\partial y = 6y = \partial(6xy)/\partial x$, conservative so $\partial\phi/\partial x = 3y^2$ and $\partial\phi/\partial y = 6xy$, $\phi = 3xy^2 + k(y)$, $6xy + k'(y) = 6xy$, $k'(y) = 0$, $k(y) = K$, $\phi = 3xy^2 + K$

3. $\partial(x^2y)/\partial y = x^2$ and $\partial(5xy^2)/\partial x = 5y^2$, not conservative

4. $\partial(e^x\cos y)/\partial y = -e^x\sin y = \partial(-e^x\sin y)/\partial x$, conservative so $\partial\phi/\partial x = e^x\cos y$ and $\partial\phi/\partial y = -e^x\sin y$, $\phi = e^x\cos y + k(y)$, $-e^x\sin y + k'(y) = -e^x\sin y$, $k'(y) = 0$, $k(y) = K$, $\phi = e^x\cos y + K$

5. $\partial(\cos y + y\cos x)/\partial y = -\sin y + \cos x = \partial(\sin x - x\sin y)/\partial x$, conservative so $\partial\phi/\partial x = \cos y + y\cos x$ and $\partial\phi/\partial y = \sin x - x\sin y$, $\phi = x\cos y + y\sin x + k(y)$, $-x\sin y + \sin x + k'(y) = \sin x - x\sin y$, $k'(y) = 0$, $k(y) = K$, $\phi = x\cos y + y\sin x + K$

6. $\partial(x\ln y)/\partial y = x/y$ and $\partial(y\ln x)/\partial x = y/x$, not conservative

7. **(a)** $\partial(y^2)/\partial y = 2y = \partial(2xy)/\partial x$, independent of path

(b) $C : x = -1 + 2t,\ y = 2 + t,\ 0 \le t \le 1;\ \int_0^1 (4 + 14t + 6t^2)dt = 13$

(c) $\partial\phi/\partial x = y^2$ and $\partial\phi/\partial y = 2xy$, $\phi = xy^2 + k(y)$, $2xy + k'(y) = 2xy$, $k'(y) = 0$, $k(y) = K$, $\phi = xy^2 + K$. Let $K = 0$ to get $\phi(1,3) - \phi(-1,2) = 9 - (-4) = 13$

8. **(a)** $\partial(y\sin x)/\partial y = \sin x = \partial(-\cos x)/\partial x$, independent of path

(b) $C_1 : x = \pi t,\ y = 1 - 2t,\ 0 \le t \le 1;\ \int_0^1 (\pi\sin\pi t - 2\pi t\sin\pi t + 2\cos\pi t)dt = 0$

(c) $\partial\phi/\partial x = y\sin x$ and $\partial\phi/\partial y = -\cos x$, $\phi = -y\cos x + k(y)$, $-\cos x + k'(y) = -\cos x$, $k'(y) = 0$, $k(y) = K$, $\phi = -y\cos x + K$. Let $K = 0$ to get $\phi(\pi,-1) - \phi(0,1) = (-1) - (-1) = 0$

9. $\partial(3y)/\partial y = 3 = \partial(3x)/\partial x$, $\phi = 3xy$, $\phi(4,0) - \phi(1,2) = -6$

10. $\partial(e^x \sin y)/\partial y = e^x \cos y = \partial(e^x \cos y)/\partial x$, $\phi = e^x \sin y$, $\phi(1,\pi/2) - \phi(0,0) = e$

11. $\partial(2xe^y)/\partial y = 2xe^y = \partial(x^2e^y)/\partial x$, $\phi = x^2e^y$, $\phi(3,2) - \phi(0,0) = 9e^2$

12. $\partial(3x - y + 1)/\partial y = -1 = \partial[-(x + 4y + 2)]/\partial x$,
$\phi = 3x^2/2 - xy + x - 2y^2 - 2y$, $\phi(0,1) - \phi(-1,2) = 11/2$

13. $\partial(2xy^3)/\partial y = 6xy^2 = \partial(3x^2y^2)/\partial x$, $\phi = x^2y^3$, $\phi(-1,0) - \phi(2,-2) = 32$

14. $\partial(e^x \ln y - e^y/x)/\partial y = e^x/y - e^y/x = \partial(e^x/y - e^y \ln x)/\partial x$,
$\phi = e^x \ln y - e^y \ln x$, $\phi(3,3) - \phi(1,1) = 0$

15. $\phi = x^2y^2/2$, $W = \phi(0,0) - \phi(1,1) = -1/2$

16. $\phi = x^2y^3, W = \phi(4,1) - \phi(-3,0) = 16$

17. $\phi = e^{xy}$, $W = \phi(2,0) - \phi(-1,1) = 1 - e^{-1}$

18. $\phi = e^{-y}\sin x, W = \phi(-\pi/2,0) - \phi(\pi/2,1) = -1 - 1/e$

19. $\partial(e^y + ye^x)/\partial y = e^y + e^x = \partial(xe^y + e^x)/\partial x$ so $\mathbf{F}$ is conservative, $\phi(x,y) = xe^y + ye^x$ so
$\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \phi(0,\ln 2) - \phi(1,0) = \ln 2 - 1$

20. $\partial(2xy)/\partial y = 2x = \partial(x^2 + \cos y)/\partial x$ so $\mathbf{F}$ is conservative, $\phi(x,y) = x^2y + \sin y$ so
$\displaystyle\int_C \mathbf{F}\cdot dr = \phi(\pi,\pi/2) - \phi(0,0) = \pi^3/2 + 1$

21.
$$\begin{aligned}\mathbf{F}\cdot d\mathbf{r} &= [(e^y + ye^x)\mathbf{i} + (xe^y + e^x)\mathbf{j}]\cdot[(\pi/2)\cos(\pi t/2)\mathbf{i} + (1/t)\mathbf{j}]dt\\ &= \left(\frac{\pi}{2}\cos(\pi t/2)(e^y + ye^x) + (xe^y + e^x)/t\right)dt,\end{aligned}$$
so $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_1^2 \left(\frac{\pi}{2}\cos(\pi t/2)\left(t + (\ln t)e^{\sin(\pi t/2)}\right) + \left(\sin(\pi t/2) + \frac{1}{t}e^{\sin(\pi t/2)}\right)\right)dt = \ln 2 - 1$

22. $\mathbf{F}\cdot d\mathbf{r} = \left(2t^2\cos(t/3) + [t^2 + \cos(t\cos(t/3))](\cos(t/3) - (t/3)\sin(t/3))\right)dt$, so
$\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^\pi \left(2t^2\cos(t/3) + [t^2 + \cos(t\cos(t/3))](\cos(t/3) - (t/3)\sin(t/3))\right)dt = 1 + \pi^3/2$

23. No; a closed loop can be found whose tangent everywhere makes an angle $< \pi$ with the vector field, so the line integral $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} > 0$, and by Theorem 16.3.2 the vector field is not conservative.

24. The vector field is constant, say $\mathbf{F} = a\mathbf{i} + b\mathbf{j}$, so let $\phi(x,y) = ax + by$ and $\mathbf{F}$ is conservative.

25. If $\mathbf{F}$ is conservative, then $\mathbf{F} = \nabla\phi = \dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k}$ and hence $f = \dfrac{\partial\phi}{\partial x}, g = \dfrac{\partial\phi}{\partial y}$, and $h = \dfrac{\partial\phi}{\partial z}$.
Thus $\dfrac{\partial f}{\partial y} = \dfrac{\partial^2\phi}{\partial y\partial x}$ and $\dfrac{\partial g}{\partial x} = \dfrac{\partial^2\phi}{\partial x\partial y}, \dfrac{\partial f}{\partial z} = \dfrac{\partial^2\phi}{\partial z\partial x}$ and $\dfrac{\partial h}{\partial x} = \dfrac{\partial^2\phi}{\partial x\partial z}, \dfrac{\partial g}{\partial z} = \dfrac{\partial^2\phi}{\partial z\partial y}$ and $\dfrac{\partial h}{\partial y} = \dfrac{\partial^2\phi}{\partial y\partial z}$.
The result follows from the equality of mixed second partial derivatives.

26. Let $f(x,y,z) = yz, g(x,y,z) = xz, h(x,y,z) = yx^2$, then $\partial f/\partial z = y, \partial h/\partial x = 2xy \neq \partial f/\partial z$, thus by Exercise 25, $\mathbf{F} = f\mathbf{i}+g\mathbf{j}+h\mathbf{k}$ is not conservative, and by Theorem 16.3.2, $\int_C yz\,dx+xz\,dy+yx^2\,dz$ is not independent of the path.

27. $\dfrac{\partial}{\partial y}(h(x)[x\sin y + y\cos y]) = h(x)[x\cos y - y\sin y + \cos y]$

$\dfrac{\partial}{\partial x}(h(x)[x\cos y - y\sin y]) = h(x)\cos y + h'(x)[x\cos y - y\sin y]$,

equate these two partial derivatives to get $(x\cos y - y\sin y)(h'(x) - h(x)) = 0$ which holds for all x and y if $h'(x) = h(x)$, $h(x) = Ce^x$ where C is an arbitrary constant.

28. **(a)** $\dfrac{\partial}{\partial y}\dfrac{cx}{(x^2+y^2)^{3/2}} = -\dfrac{3cxy}{(x^2+y^2)^{-5/2}} = \dfrac{\partial}{\partial x}\dfrac{cy}{(x^2+y^2)^{3/2}}$ when $(x,y) \neq (0,0)$,

so by Theorem 16.3.3, $\mathbf{F}$ is conservative. Set $\partial\phi/\partial x = cx/(x^2+y^2)^{-3/2}$,

then $\phi(x,y) = -c(x^2+y^2)^{-1/2} + k(y), \partial\phi/\partial y = cy/(x^2+y^2)^{-3/2} + k'(y)$, so $k'(y) = 0$.

Thus $\phi(x,y) = -\dfrac{c}{(x^2+y^2)^{1/2}}$ is a potential function.

(b) curl $\mathbf{F} = \mathbf{0}$ is similar to Part (a), so $\mathbf{F}$ is conservative. Let

$\phi(x,y,z) = \displaystyle\int \frac{cx}{(x^2+y^2+z^2)^{3/2}}\,dx = -c(x^2+y^2+z^2)^{-1/2} + k(y,z)$. As in Part (a),

$\partial k/\partial y = \partial k/\partial z = 0$, so $\phi(x,y,z) = -c/(x^2+y^2+z^2)^{1/2}$ is a potential function for $\mathbf{F}$.

29. **(a)** See Exercise 28, $c = 1$; $W = \displaystyle\int_P^Q \mathbf{F}\cdot d\mathbf{r} = \phi(3,2,1) - \phi(1,1,2) = -\frac{1}{\sqrt{14}} + \frac{1}{\sqrt{6}}$

(b) C begins at $P(1,1,2)$ and ends at $Q(3,2,1)$ so the answer is again $W = -\dfrac{1}{\sqrt{14}} + \dfrac{1}{\sqrt{6}}$.

(c) The circle is not specified, but cannot pass through $(0,0,0)$, so Φ is continuous and differentiable on the circle. Start at any point P on the circle and return to P, so the work is $\Phi(P) - \Phi(P) = 0$.
C begins at, say, $(3,0)$ and ends at the same point so $W = 0$.

30. **(a)** $\mathbf{F}\cdot d\mathbf{r} = \left(y\dfrac{dx}{dt} - x\dfrac{dy}{dt}\right)dt$ for points on the circle $x^2+y^2 = 1$, so

$C_1 : x = \cos t, y = \sin t, 0 \le t \le \pi, \displaystyle\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_0^\pi (-\sin^2 t - \cos^2 t)\,dt = -\pi$

$C_2 : x = \cos t, y = -\sin t, 0 \le t \le \pi, \displaystyle\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_0^\pi (\sin^2 t + \cos^2 t)\,dt = \pi$

(b) $\dfrac{\partial f}{\partial y} = \dfrac{x^2-y^2}{(x^2+y^2)^2}, \dfrac{\partial g}{\partial x} = -\dfrac{y^2-x^2}{(x^2+y^2)^2} = \dfrac{\partial f}{\partial y}$

(c) The circle about the origin of radius 1, which is formed by traversing C_1 and then traversing C_2 in the reverse direction, does not lie in an open simply connected region inside which $\mathbf{F}$ is continuous, since $\mathbf{F}$ is not defined at the origin, nor can it be defined there in such a way as to make the resulting function continuous there.

31. If C is composed of smooth curves $C_1, C_2, \dots, C_n$ and curve C_i extends from (x_{i-1}, y_{i-1}) to (x_i, y_i) then $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \sum_{i=1}^{n}\int_{C_i}\mathbf{F}\cdot d\mathbf{r} = \sum_{i=1}^{n}[\phi(x_i, y_i) - \phi(x_{i-1}, y_{i-1})] = \phi(x_n, y_n) - \phi(x_0, y_0)$ where (x_0, y_0) and (x_n, y_n) are the endpoints of C.

32. $\displaystyle\int_{C_1}\mathbf{F}\cdot d\mathbf{r} + \int_{-C_2}\mathbf{F}\cdot d\mathbf{r} = 0$, but $\displaystyle\int_{-C_2}\mathbf{F}\cdot d\mathbf{r} = -\int_{C_2}\mathbf{F}\cdot d\mathbf{r}$ so $\displaystyle\int_{C_1}\mathbf{F}\cdot d\mathbf{r} = \int_{C_2}\mathbf{F}\cdot d\mathbf{r}$, thus $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path.

33. Let C_1 be an arbitrary piecewise smooth curve from (a, b) to a point (x, y_1) in the disk, and C_2 the vertical line segment from (x, y_1) to (x, y). Then

$$\phi(x, y) = \int_{C_1}\mathbf{F}\cdot d\mathbf{r} + \int_{C_2}\mathbf{F}\cdot d\mathbf{r} = \int_{(a,b)}^{(x,y_1)}\mathbf{F}\cdot d\mathbf{r} + \int_{C_2}\mathbf{F}\cdot d\mathbf{r}.$$

The first term does not depend on y;

hence $\displaystyle\frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}\int_{C_2}\mathbf{F}\cdot d\mathbf{r} = \frac{\partial}{\partial y}\int_{C_2} f(x, y)dx + g(x, y)dy.$

However, the line integral with respect to x is zero along C_2, so $\displaystyle\frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}\int_{C_2} g(x, y)\; dy.$

Express C_2 as $x = x, y = t$ where t varies from y_1 to y, then $\displaystyle\frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}\int_{y_1}^{y} g(x, t)\; dt = g(x, y).$

EXERCISE SET 16.4

1. $\displaystyle\iint_R (2x - 2y)dA = \int_0^1\int_0^1 (2x - 2y)dy\,dx = 0$; for the line integral, on $x = 0, y^2\,dx = 0, x^2\,dy = 0$; on $y = 0, y^2\,dx = x^2\,dy = 0$; on $x = 1, y^2\,dx + x^2\,dy = dy$; and on $y = 1, y^2\,dx + x^2\,dy = dx$, hence $\displaystyle\oint_C y^2\,dx + x^2\,dy = \int_0^1 dy + \int_1^0 dx = 1 - 1 = 0$

2. $\displaystyle\iint_R (1 - 1)dA = 0$; for the line integral let $x = \cos t, y = \sin t$,

$$\oint_C y\,dx + x\,dy = \int_0^{2\pi}(-\sin^2 t + \cos^2 t)dt = 0$$

3. $\displaystyle\int_{-2}^{4}\int_1^2 (2y - 3x)dy\;dx = 0$

4. $\displaystyle\int_0^{2\pi}\int_0^3 (1 + 2r\sin\theta)r\,dr\,d\theta = 9\pi$

5. $\displaystyle\int_0^{\pi/2}\int_0^{\pi/2}(-y\cos x + x\sin y)dy\,dx = 0$

6. $\displaystyle\iint_R (\sec^2 x - \tan^2 x)dA = \iint_R dA = \pi$

7. $\displaystyle\iint_R [1 - (-1)]dA = 2\iint_R dA = 8\pi$

8. $\displaystyle\int_0^1\int_{x^2}^{x}(2x - 2y)dy\,dx = 1/30$

9. $\displaystyle\iint_R \left(-\frac{y}{1+y} - \frac{1}{1+y}\right) dA = -\iint_R dA = -4$

10. $\displaystyle\int_0^{\pi/2}\int_0^4 (-r^2) r\,dr\,d\theta = -32\pi$

11. $\displaystyle\iint_R \left(-\frac{y^2}{1+y^2} - \frac{1}{1+y^2}\right) dA = -\iint_R dA = -1$

12. $\displaystyle\iint_R (\cos x\cos y - \cos x\cos y) dA = 0$

13. $\displaystyle\int_0^1\int_{x^2}^{\sqrt{x}} (y^2 - x^2) dy\,dx = 0$

14. **(a)** $\displaystyle\int_0^2\int_{x^2}^{2x} (-6x+2y) dy\,dx = -56/15$ **(b)** $\displaystyle\int_0^2\int_{x^2}^{2x} 6y\,dy\,dx = 64/5$

15. **(a)** $C : x = \cos t, y = \sin t, 0 \le t \le 2\pi$;

$$\oint_C = \int_0^{2\pi} \left(e^{\sin t}(-\sin t) + \sin t\cos t e^{\cos t}\right) dt \approx -3.550999378;$$

$$\iint_R \left[\frac{\partial}{\partial x}(ye^x) - \frac{\partial}{\partial y}e^y\right] dA = \iint_R [ye^x - e^y]\, dA$$

$$= \int_0^{2\pi}\int_0^1 \left[r\sin\theta e^{r\cos\theta} - e^{r\sin\theta}\right] r\,dr\,d\theta \approx -3.550999378$$

(b) $C_1 : x = t, y = t^2, 0 \le t \le 1; \displaystyle\int_{C_1} [e^y\,dx + ye^x\,dy] = \int_0^1 \left[e^{t^2} + 2t^3 e^t\right] dt \approx 2.589524432$

$C_2 : x = t^2, y = t, 0 \le t \le 1; \displaystyle\int_{C_2} [e^y\,dx + ye^x\,dy] = \int_0^1 \left[2te^t + te^{t^2}\right] dt = \frac{e+3}{2} \approx 2.859140914$

$\displaystyle\int_{C_1} - \int_{C_2} \approx -0.269616482; \iint_R = \int_0^1\int_{x^2}^{\sqrt{x}} [ye^x - e^y]\, dy\,dx \approx -0.269616482$

16. **(a)** $\displaystyle\oint_C x\,dy = \int_0^{2\pi} ab\cos^2 t\,dt = \pi ab$ **(b)** $\displaystyle\oint_C -y\,dx = \int_0^{2\pi} ab\sin^2 t\,dt = \pi ab$

17. $\displaystyle A = \frac{1}{2}\oint_C -y\,dx + x\,dy = \frac{1}{2}\int_0^{2\pi} (3a^2\sin^4\phi\cos^2\phi + 3a^2\cos^4\phi\sin^2\phi) d\phi$

$$= \frac{3}{2}a^2\int_0^{2\pi} \sin^2\phi\cos^2\phi\,d\phi = \frac{3}{8}a^2\int_0^{2\pi}\sin^2 2\phi\,d\phi = 3\pi a^2/8$$

18. $C_1 : (0,0)$ to $(a,0)$; $x = at, \quad y = 0, \quad 0 \le t \le 1$
$C_2 : (a,0)$ to $(0,b)$; $x = a - at, y = bt, \quad 0 \le t \le 1$
$C_3 : (0,b)$ to $(0,0)$; $x = 0, \quad y = b - bt, 0 \le t \le 1$

$$A = \oint_C x\,dy = \int_0^1 (0)dt + \int_0^1 ab(1-t)dt + \int_0^1 (0)dt = \frac{1}{2}ab$$

19. $C_1 : (0,0)$ to $(a,0)$; $x = at,\ y = 0,\ 0 \le t \le 1$

$C_2 : (a,0)$ to $(a\cos t_0, b\sin t_0)$; $x = a\cos t,\ y = b\sin t,\ 0 \le t \le t_0$

$C_3 : (a\cos t_0, b\sin t_0)$ to $(0,0)$; $x = -a(\cos t_0)t,\ y = -b(\sin t_0)t,\ -1 \le t \le 0$

$$A = \frac{1}{2}\oint_C -y\,dx + x\,dy = \frac{1}{2}\int_0^1 (0)\,dt + \frac{1}{2}\int_0^{t_0} ab\,dt + \frac{1}{2}\int_{-1}^0 (0)\,dt = \frac{1}{2}ab\,t_0$$

20. $C_1 : (0,0)$ to $(a,0)$; $x = at,\ y = 0,\ 0 \le t \le 1$

$C_2 : (a,0)$ to $(a\cosh t_0, b\sinh t_0)$; $x = a\cosh t,\ y = b\sinh t,\ 0 \le t \le t_0$

$C_3 : (a\cosh t_0, b\sinh t_0)$ to $(0,0)$; $x = -a(\cosh t_0)t,\ y = -b(\sinh t_0)t,\ -1 \le t \le 0$

$$A = \frac{1}{2}\oint_C -y\,dx + x\,dy = \frac{1}{2}\int_0^1 (0)\,dt + \frac{1}{2}\int_0^{t_0} ab\,dt + \frac{1}{2}\int_{-1}^0 (0)\,dt = \frac{1}{2}ab\,t_0$$

21. $W = \iint_R y\,dA = \int_0^{\pi}\int_0^5 r^2\sin\theta\,dr\,d\theta = 250/3$

22. We cannot apply Green's Theorem on the region enclosed by the closed curve C, since $\mathbf{F}$ does not have first order partial derivatives at the origin. However, the curve C_{x_0}, consisting of $y = x_0^3/4, x_0 \le x \le 2; x = 2, x_0^3/4 \le y \le 2$; and $y = x^3/4, x_0 \le x \le 2$ encloses a region R_{x_0} in which Green's Theorem does hold, and

$$W = \oint_C \mathbf{F}\cdot d\mathbf{r} = \lim_{x_0\to 0^+}\oint_{C_{x_0}} \mathbf{F}\cdot d\mathbf{r} = \lim_{x_0\to 0^+}\iint_{R_{x_0}} \nabla\cdot\mathbf{F}\,dA$$

$$= \lim_{x_0\to 0^+}\int_{x_0}^2\int_{x_0^3/4}^{x^3/4}\left(\frac{1}{2}x^{-1/2} - \frac{1}{2}y^{-1/2}\right)dy\,dx$$

$$= \lim_{x_0\to 0^+}\left(-\frac{18}{35}\sqrt{2} - \frac{\sqrt{2}}{4}x_0^3 + x_0^{3/2} + \frac{3}{14}x_0^{7/2} - \frac{3}{10}x_0^{5/2}\right) = -\frac{18}{35}\sqrt{2}$$

23. $\oint_C y\,dx - x\,dy = \iint_R (-2)dA = -2\int_0^{2\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta = -3\pi a^2$

24. $\bar{x} = \frac{1}{A}\iint_R x\,dA$, but $\oint_C \frac{1}{2}x^2dy = \iint_R x\,dA$ from Green's Theorem so

$\bar{x} = \frac{1}{A}\oint_C \frac{1}{2}x^2dy = \frac{1}{2A}\oint_C x^2dy$. Similarly, $\bar{y} = -\frac{1}{2A}\oint_C y^2dx$.

25. $A = \int_0^1\int_{x^3}^x dy\,dx = \frac{1}{4}$; $C_1 : x = t, y = t^3, 0 \le t \le 1$, $\int_{C_1} x^2\,dy = \int_0^1 t^2(3t^2)\,dt = \frac{3}{5}$

$C_2 : x = t, y = t, 0 \le t \le 1$; $\int_{C_2} x^2\,dy = \int_0^1 t^2\,dt = \frac{1}{3}$, $\oint_C x^2\,dy = \int_{C_1} - \int_{C_2} = \frac{3}{5} - \frac{1}{3} = \frac{4}{15}, \bar{x} = \frac{8}{15}$

$\int_C y^2\,dx = \int_0^1 t^6\,dt - \int_0^1 t^2\,dt = \frac{1}{7} - \frac{1}{3} = -\frac{4}{21}, \bar{y} = \frac{8}{21}$, centroid $\left(\frac{8}{15}, \frac{8}{21}\right)$

26. $A = \dfrac{a^2}{2}; C_1 : x = t, y = 0, 0 \le t \le a, C_2 : x = a - t, y = t, 0 \le t \le a; C_3 : x = 0, y = a - t, 0 \le t \le a;$

$$\int_{C_1} x^2\,dy = 0, \int_{C_2} x^2\,dy = \int_0^a (a-t)^2\,dt = \frac{a^3}{3}, \int_{C_3} x^2\,dy = 0, \oint_C x^2\,dy = \int_{C_1} + \int_{C_2} + \int_{C_3} = \frac{a^3}{3},\ \bar{x} = \frac{a}{3};$$

$$\int_C y^2\,dx = 0 - \int_0^a t^2\,dt + 0 = -\frac{a^3}{3},\ \bar{y} = \frac{a}{3},\ \text{centroid}\ \left(\frac{a}{3}, \frac{a}{3}\right)$$

27. $\bar{x} = 0$ from the symmetry of the region,

$C_1 : (a, 0)$ to $(-a, 0)$ along $y = \sqrt{a^2 - x^2}$; $x = a\cos t,\ y = a\sin t,\ 0 \le t \le \pi$

$C_2 : (-a, 0)$ to $(a, 0)$; $x = t,\ y = 0,\ -a \le t \le a$

$$A = \pi a^2/2, \quad \bar{y} = -\frac{1}{2A}\left[\int_0^\pi -a^3 \sin^3 t\,dt + \int_{-a}^a (0)dt\right]$$

$$= -\frac{1}{\pi a^2}\left(-\frac{4a^3}{3}\right) = \frac{4a}{3\pi};\ \text{centroid}\ \left(0, \frac{4a}{3\pi}\right)$$

28. $A = \dfrac{ab}{2}; C_1 : x = t, y = 0,\ 0 \le t \le a, C_2 : x = a, y = t,\ 0 \le t \le b;$

$C_3 : x = a - at, y = b - bt,\ 0 \le t \le 1;$

$$\int_{C_1} x^2\,dy = 0, \int_{C_2} x^2\,dy = \int_0^b a^2\,dt = ba^2, \int_{C_3} x^2\,dy = \int_0^1 a^2(1-t)^2(-b)\,dt = -\frac{ba^2}{3},$$

$$\oint_C x^2\,dy = \int_{C_1} + \int_{C_2} + \int_{C_3} = \frac{2ba^2}{3},\ \bar{x} = \frac{2a}{3};$$

$$\int_C y^2\,dx = 0 + 0 - \int_0^1 ab^2(1-t)^2\,dt = -\frac{ab^2}{3},\ \bar{y} = \frac{b}{3},\ \text{centroid}\ \left(\frac{2a}{3}, \frac{b}{3}\right)$$

29. From Green's Theorem, the given integral equals $\iint_R (1 - x^2 - y^2)dA$ where R is the region enclosed by C. The value of this integral is maximum if the integration extends over the largest region for which the integrand $1 - x^2 - y^2$ is nonnegative so we want $1 - x^2 - y^2 \ge 0$, $x^2 + y^2 \le 1$. The largest region is that bounded by the circle $x^2 + y^2 = 1$ which is the desired curve C.

30. **(a)** $C : x = a + (c - a)t,\ y = b + (d - b)t,\ 0 \le t \le 1$

$$\int_C -y\,dx + x\,dy = \int_0^1 (ad - bc)dt = ad - bc$$

(b) Let C_1, C_2, and C_3 be the line segments from (x_1, y_1) to (x_2, y_2), (x_2, y_2) to (x_3, y_3), and (x_3, y_3) to (x_1, y_1), then if C is the entire boundary consisting of C_1, C_2, and C_3

$$A = \frac{1}{2}\int_C -y\,dx + x\,dy = \frac{1}{2}\sum_{i=1}^{3}\int_{C_i} -y\,dx + x\,dy$$

$$= \frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)]$$

(c) $A = \dfrac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_ny_1 - x_1y_n)]$

(d) $A = \dfrac{1}{2}[(0 - 0) + (6 + 8) + (0 + 2) + (0 - 0)] = 8$

31. $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (x^2+y)\,dx + (4x-\cos y)\,dy = 3\iint_R dA = 3(25-2) = 69$

32. $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (e^{-x}+3y)\,dx + x\,dy = -2\iint_R dA = -2[\pi(4)^2 - \pi(2)^2] = -24\pi$

EXERCISE SET 16.5

1. R is the annular region between $x^2+y^2=1$ and $x^2+y^2=4$;

$$\iint_\sigma z^2 dS = \iint_R (x^2+y^2)\sqrt{\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}+1}\,dA$$
$$= \sqrt{2}\iint_R (x^2+y^2)dA = \sqrt{2}\int_0^{2\pi}\int_1^2 r^3 dr\,d\theta = \frac{15}{2}\pi\sqrt{2}.$$

2. $z=1-x-y$, R is the triangular region enclosed by $x+y=1$, $x=0$ and $y=0$;

$$\iint_\sigma xy\,dS = \iint_R xy\sqrt{3}\,dA = \sqrt{3}\int_0^1\int_0^{1-x} xy\,dy\,dx = \frac{\sqrt{3}}{24}.$$

3. Let $\mathbf{r}(u,v) = \cos u\mathbf{i} + v\mathbf{j} + \sin u\mathbf{k}, 0\le u\le\pi, 0\le v\le 1$. Then $\mathbf{r}_u = -\sin u\mathbf{i} + \cos u\mathbf{k}, \mathbf{r}_v = \mathbf{j}$,

$\mathbf{r}_u\times\mathbf{r}_v = -\cos u\mathbf{i} - \sin u\mathbf{k}, \|\mathbf{r}_u\times\mathbf{r}_v\| = 1, \displaystyle\iint_\sigma x^2y\,dS = \int_0^1\int_0^\pi v\cos^2 u\,du\,dv = \pi/4$

4. $z=\sqrt{4-x^2-y^2}$, R is the circular region enclosed by $x^2+y^2=3$;

$$\iint_\sigma (x^2+y^2)z\,dS = \iint_R (x^2+y^2)\sqrt{4-x^2-y^2}\sqrt{\frac{x^2}{4-x^2-y^2}+\frac{y^2}{4-x^2-y^2}+1}\,dA$$
$$= \iint_R 2(x^2+y^2)dA = 2\int_0^{2\pi}\int_0^{\sqrt{3}} r^3 dr\,d\theta = 9\pi.$$

5. If we use the projection of σ onto the xz-plane then $y=1-x$ and R is the rectangular region in the xz-plane enclosed by $x=0$, $x=1$, $z=0$ and $z=1$;

$$\iint_\sigma (x-y-z)dS = \iint_R (2x-1-z)\sqrt{2}dA = \sqrt{2}\int_0^1\int_0^1 (2x-1-z)dz\,dx = -\sqrt{2}/2$$

6. R is the triangular region enclosed by $2x+3y=6$, $x=0$, and $y=0$;

$$\iint_\sigma (x+y)dS = \iint_R (x+y)\sqrt{14}\,dA = \sqrt{14}\int_0^3\int_0^{(6-2x)/3}(x+y)dy\,dx = 5\sqrt{14}.$$

7. There are six surfaces, parametrized by projecting onto planes:
$\sigma_1 : z=0;\ 0\le x\le 1,\ 0\le y\le 1$ (onto xy-plane), $\sigma_2 : x=0;\ 0\le y\le 1,\ 0\le z\le 1$ (onto yz-plane),
$\sigma_3 : y=0;\ 0\le x\le 1,\ 0\le z\le 1$ (onto xz-plane), $\sigma_4 : z=1;\ 0\le x\le 1,\ 0\le y\le 1$ (onto xy-plane),
$\sigma_5 : x=1;\ 0\le y\le 1,\ 0\le z\le 1$ (onto yz-plane), $\sigma_6 : y=1;\ 0\le x\le 1,\ 0\le z\le 1$ (onto xz-plane).

By symmetry the integrals over σ_1, σ_2 and σ_3 are equal, as are those over σ_4, σ_5 and σ_6, and

$$\iint_{\sigma_1} (x+y+z)dS = \int_0^1 \int_0^1 (x+y)dx\,dy = 1; \quad \iint_{\sigma_4} (x+y+z)dS = \int_0^1 \int_0^1 (x+y+1)dx\,dy = 2,$$

thus, $\displaystyle\iint_{\sigma} (x+y+z)dS = 3 \cdot 1 + 3 \cdot 2 = 9.$

8. Let $\mathbf{r}(\phi,\theta) = \sin\phi\cos\theta\mathbf{i} + \sin\phi\sin\theta\mathbf{j} + \cos\phi\mathbf{k}, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi/2; \|\mathbf{r}_\phi \times \mathbf{r}_\theta\| = \sin\phi,$

$$\iint_{\sigma} (1+\cos\phi)\,dS = \int_0^{2\pi} \int_0^{\pi/2} (1+\cos\phi)\sin\phi\,d\phi\,d\theta$$
$$= 2\pi \int_0^{\pi/2} (1+\cos\phi)\sin\phi\,d\phi = 3\pi$$

9. R is the circular region enclosed by $x^2+y^2=1$;

$$\iint_{\sigma} \sqrt{x^2+y^2+z^2}\,dS = \iint_{R} \sqrt{2(x^2+y^2)}\sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1}\,dA$$
$$= \lim_{r_0 \to 0^+} 2 \iint_{R'} \sqrt{x^2+y^2}\,dA$$

where R' is the annular region enclosed by $x^2+y^2=1$ and $x^2+y^2=r_0^2$ with r_0 slightly larger than 0 because $\sqrt{\dfrac{x^2}{x^2+y^2} + \dfrac{y^2}{x^2+y^2} + 1}$ is not defined for $x^2+y^2=0$, so

$$\iint_{\sigma} \sqrt{x^2+y^2+z^2}\,dS = \lim_{r_0 \to 0^+} 2 \int_0^{2\pi} \int_{r_0}^1 r^2 dr\,d\theta = \lim_{r_0 \to 0^+} \frac{4\pi}{3}(1-r_0^3) = \frac{4\pi}{3}.$$

10. Let $\mathbf{r}(\phi,\theta) = a\sin\phi\cos\theta\,\mathbf{i} + a\sin\phi\sin\theta\,\mathbf{j} + a\cos\phi\,\mathbf{k}$,
$0 \le \theta \le 2\pi, 0 \le \phi \le \pi; \|\mathbf{r}_\phi \times \mathbf{r}_\theta\| = a^2\sin\phi,\ x^2+y^2 = a^2\sin^2\phi$

$$\iint_{\sigma} f(x,y,z) = \int_0^{2\pi} \int_0^{\pi} a^4 \sin^3\phi\,d\phi\,d\theta = \frac{8}{3}\pi a^4$$

11. **(a)** $\displaystyle\frac{\sqrt{29}}{16} \int_0^6 \int_0^{(12-2x)/3} xy(12-2x-3y)dy\,dx$

(b) $\displaystyle\frac{\sqrt{29}}{4} \int_0^3 \int_0^{(12-4z)/3} yz(12-3y-4z)dy\,dz$

(c) $\displaystyle\frac{\sqrt{29}}{9} \int_0^3 \int_0^{6-2z} xz(12-2x-4z)dx\,dz$

12. **(a)** $\displaystyle a\int_0^a \int_0^{\sqrt{a^2-x^2}} x\,dy\,dx$ **(b)** $\displaystyle a\int_0^a \int_0^{\sqrt{a^2-z^2}} z\,dy\,dz$

(c) $\displaystyle a\int_0^a \int_0^{\sqrt{a^2-z^2}} \frac{xz}{\sqrt{a^2-x^2-z^2}}dx\,dz$

13. $18\sqrt{29}/5$

14. $a^4/3$

15. $\displaystyle\int_0^4 \int_1^2 y^3 z\sqrt{4y^2+1}\,dy\,dz;\ \frac{1}{2}\int_0^4 \int_1^4 xz\sqrt{1+4x}\,dx\,dz$

16. $\displaystyle a\int_0^9 \int_{a/\sqrt{5}}^{a/\sqrt{2}} \frac{x^2 y}{\sqrt{a^2-y^2}}\,dy\,dx,\ a\int_{a/\sqrt{2}}^{2a/\sqrt{5}} \int_0^9 x^2 dx\,dz$

17. $391\sqrt{17}/15 - 5\sqrt{5}/3$

18. The region $R: 3x^2+2y^2=5$ is symmetric in y. The integrand is $x^2yz\,dS = x^2y(5-3x^2-2y^2)\sqrt{1+36x^2+16y^2}\,dy\,dx$, which is odd in y, hence $\displaystyle\iiint\limits_{\sigma} x^2yz\,dS = 0$.

19. $\displaystyle z=\sqrt{4-x^2},\ \frac{\partial z}{\partial x} = -\frac{x}{\sqrt{4-x^2}},\ \frac{\partial z}{\partial y}=0;$

$$\iint\limits_{\sigma} \delta_0 dS = \delta_0 \iint\limits_{R} \sqrt{\frac{x^2}{4-x^2}+1}\,dA = 2\delta_0\int_0^4\int_0^1 \frac{1}{\sqrt{4-x^2}}dx\,dy = \frac{4}{3}\pi\delta_0.$$

20. $z=\dfrac{1}{2}(x^2+y^2)$, R is the circular region enclosed by $x^2+y^2=8$;

$$\iint\limits_{\sigma} \delta_0 dS = \delta_0 \iint\limits_{R} \sqrt{x^2+y^2+1}\,dA = \delta_0\int_0^{2\pi}\int_0^{\sqrt{8}} \sqrt{r^2+1}\,r\,dr\,d\theta = \frac{52}{3}\pi\delta_0.$$

21. $z=4-y^2$, R is the rectangular region enclosed by $x=0$, $x=3$, $y=0$ and $y=3$;

$$\iint\limits_{\sigma} y\,dS = \iint\limits_{R} y\sqrt{4y^2+1}\,dA = \int_0^3\int_0^3 y\sqrt{4y^2+1}\,dy\,dx = \frac{1}{4}(37\sqrt{37}-1).$$

22. R is the annular region enclosed by $x^2+y^2=1$ and $x^2+y^2=16$;

$$\begin{aligned}\iint\limits_{\sigma} x^2z\,dS &= \iint\limits_{R} x^2\sqrt{x^2+y^2}\sqrt{\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}+1}\,dA \\ &= \sqrt{2}\iint\limits_{R} x^2\sqrt{x^2+y^2}\,dA = \sqrt{2}\int_0^{2\pi}\int_1^4 r^4\cos^2\theta\,dr\,d\theta = \frac{1023\sqrt{2}}{5}\pi.\end{aligned}$$

23. $\displaystyle M = \iint\limits_{\sigma} \delta(x,y,z)dS = \iint\limits_{\sigma}\delta_0 dS = \delta_0\iint\limits_{\sigma} dS = \delta_0 S$

24. $\delta(x,y,z)=|z|$; use $z=\sqrt{a^2-x^2-y^2}$, let R be the circular region enclosed by $x^2+y^2=a^2$, and σ the hemisphere above R. By the symmetry of both the surface and the density function with respect to the xy-plane we have

$$M = 2\iint\limits_{\sigma} z\,dS = 2\iint\limits_{R}\sqrt{a^2-x^2-y^2}\sqrt{\frac{x^2}{a^2-x^2-y^2}+\frac{y^2}{a^2-x^2-y^2}+1}\,dA = \lim_{r_0\to a^-} 2a\iint\limits_{R_{r_0}} dA$$

where R_{r_0} is the circular region with radius r_0 that is slightly less than a. But $\displaystyle\iint\limits_{R_{r_0}} dA$ is simply the area of the circle with radius r_0 so $M = \displaystyle\lim_{r_0\to a^-} 2a(\pi r_0^2) = 2\pi a^3$.

25. By symmetry $\bar{x} = \bar{y} = 0$.

$$\iint_\sigma dS = \iint_R \sqrt{x^2+y^2+1}\,dA = \int_0^{2\pi}\int_0^{\sqrt{8}} \sqrt{r^2+1}\,r\,dr\,d\theta = \frac{52\pi}{3},$$

$$\iint_\sigma z\,dS = \iint_R z\sqrt{x^2+y^2+1}\,dA = \frac{1}{2}\iint_R (x^2+y^2)\sqrt{x^2+y^2+1}\,dA$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^{\sqrt{8}} r^3\sqrt{r^2+1}\,dr\,d\theta = \frac{596\pi}{15}$$

so $\bar{z} = \dfrac{596\pi/15}{52\pi/3} = \dfrac{149}{65}$. The centroid is $(\bar{x},\bar{y},\bar{z}) = (0,0,149/65)$.

26. By symmetry $\bar{x} = \bar{y} = 0$.

$$\iint_\sigma dS = \iint_R \frac{2}{\sqrt{4-x^2-y^2}}dA = 2\int_0^{2\pi}\int_0^{\sqrt{3}} \frac{r}{\sqrt{4-r^2}}dr\,d\theta = 4\pi,$$

$$\iint_\sigma z\,dS = \iint_R 2\,dA = (2)(\text{area of circle of radius } \sqrt{3}) = 6\pi$$

so $\bar{z} = \dfrac{6\pi}{4\pi} = \dfrac{3}{2}$. The centroid is $(\bar{x},\bar{y},\bar{z}) = (0,0,3/2)$.

27. $\partial\mathbf{r}/\partial u = \cos v\mathbf{i} + \sin v\mathbf{j} + 3\mathbf{k}, \partial\mathbf{r}/\partial v = -u\sin v\mathbf{i} + u\cos v\mathbf{j},\ \|\partial\mathbf{r}/\partial u \times \partial\mathbf{r}/\partial v\| = \sqrt{10}u;$

$$3\sqrt{10}\iint_R u^4\sin v\cos v\,dA = 3\sqrt{10}\int_0^{\pi/2}\int_1^2 u^4\sin v\cos v\,du\,dv = 93/\sqrt{10}$$

28. $\partial\mathbf{r}/\partial u = \mathbf{j}, \partial\mathbf{r}/\partial v = -2\sin v\mathbf{i} + 2\cos v\mathbf{k},\ \|\partial\mathbf{r}/\partial u \times \partial\mathbf{r}/\partial v\| = 2;$

$$8\iint_R \frac{1}{u}dA = 8\int_0^{2\pi}\int_1^3 \frac{1}{u}du\,dv = 16\pi\ln 3$$

29. $\partial\mathbf{r}/\partial u = \cos v\mathbf{i} + \sin v\mathbf{j} + 2u\mathbf{k}, \partial\mathbf{r}/\partial v = -u\sin v\mathbf{i} + u\cos v\mathbf{j},\ \|\partial\mathbf{r}/\partial u \times \partial\mathbf{r}/\partial v\| = u\sqrt{4u^2+1};$

$$\iint_R u\,dA = \int_0^{\pi}\int_0^{\sin v} u\,du\,dv = \pi/4$$

30. $\partial\mathbf{r}/\partial u = 2\cos u\cos v\mathbf{i} + 2\cos u\sin v\mathbf{j} - 2\sin u\mathbf{k},\ \partial\mathbf{r}/\partial v = -2\sin u\sin v\mathbf{i} + 2\sin u\cos v\mathbf{j};$
$\|\partial\mathbf{r}/\partial u \times \partial\mathbf{r}/\partial v\| = 4\sin u;$

$$4\iint_R e^{-2\cos u}\sin u\,dA = 4\int_0^{2\pi}\int_0^{\pi/2} e^{-2\cos u}\sin u\,du\,dv = 4\pi(1-e^{-2})$$

31. $\partial z/\partial x = -2xe^{-x^2-y^2}, \partial z/\partial y = -2ye^{-x^2-y^2},$
$(\partial z/\partial x)^2 + (\partial z/\partial y)^2 + 1 = 4(x^2+y^2)e^{-2(x^2+y^2)} + 1$; use polar coordinates to get

$$M = \int_0^{2\pi}\int_0^3 r^2\sqrt{4r^2e^{-2r^2}+1}\,dr\,d\theta \approx 57.895751$$

32. **(b)** $A = \displaystyle\iint_\sigma dS = \int_0^{2\pi}\int_{-1}^1 \frac{1}{2}\sqrt{40u\cos(v/2) + u^2 + 4u^2\cos^2(v/2) + 100}\,du\,dv \approx 62.93768644;$

$\bar{x} \approx 0.01663836266;\ \bar{y} = \bar{z} = 0$ by symmetry

EXERCISE SET 16.6

1. **(a)** zero **(b)** zero **(c)** positive
(d) negative **(e)** zero **(f)** zero

2. **(a)** positive **(b)** zero **(c)** zero
(d) zero **(e)** negative **(f)** zero

3. **(a)** positive **(b)** zero **(c)** positive
(d) zero **(e)** positive **(f)** zero

4. 0; the flux is zero on the faces $y = 0, 1$ and $z = 0, 1$; it is 1 on $x = 1$ and -1 on $x = 0$

5. **(a)** $\mathbf{n} = -\cos v\mathbf{i} - \sin v\mathbf{j}$ **(b)** inward, by inspection

6. **(a)** $-r\cos\theta\mathbf{i} - r\sin\theta\mathbf{j} + r\mathbf{k}$ **(b)** inward, by inspection

7. $\mathbf{n} = -z_x\mathbf{i} - z_y\mathbf{j} + \mathbf{k}$, $\displaystyle\iint_R \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R (2x^2 + 2y^2 + 2(1 - x^2 - y^2))\,dS = \int_0^{2\pi}\int_0^1 2r\,dr\,d\theta = 2\pi$

8. With $z = 1 - x - y$, R is the triangular region enclosed by $x + y = 1$, $x = 0$ and $y = 0$; use upward normals to get

$$\iint_\sigma \mathbf{F}\cdot\mathbf{n}\,dS = 2\iint_R (x + y + z)dA = 2\iint_R dA = (2)(\text{area of } R) = 1.$$

9. R is the annular region enclosed by $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$;

$$\iint_\sigma \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \left(-\frac{x^2}{\sqrt{x^2+y^2}} - \frac{y^2}{\sqrt{x^2+y^2}} + 2z\right)dA$$

$$= \iint_R \sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_1^2 r^2 dr\,d\theta = \frac{14\pi}{3}.$$

10. R is the circular region enclosed by $x^2 + y^2 = 4$;

$$\iint_\sigma \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R (2y^2 - 1)dA = \int_0^{2\pi}\int_0^2 (2r^2\sin^2\theta - 1)r\,dr\,d\theta = 4\pi.$$

11. R is the circular region enclosed by $x^2 + y^2 - y = 0$; $\displaystyle\iint_\sigma \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R (-x)dA = 0$ since the region R is symmetric across the y-axis.

12. With $z = \frac{1}{2}(6 - 6x - 3y)$, R is the triangular region enclosed by $2x + y = 2$, $x = 0$, and $y = 0$;

$$\iint_\sigma \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \left(3x^2 + \frac{3}{2}yx + zx\right)dA = 3\iint_R x\,dA = 3\int_0^1\int_0^{2-2x} x\,dy\,dx = 1.$$

13. $\partial\mathbf{r}/\partial u = \cos v\mathbf{i} + \sin v\mathbf{j} - 2u\mathbf{k}$, $\partial\mathbf{r}/\partial v = -u\sin v\mathbf{i} + u\cos v\mathbf{j}$,
$\partial\mathbf{r}/\partial u \times \partial\mathbf{r}/\partial v = 2u^2\cos v\mathbf{i} + 2u^2\sin v\mathbf{j} + u\mathbf{k}$;

$$\iint_R (2u^3 + u)\,dA = \int_0^{2\pi}\int_1^2 (2u^3 + u)du\,dv = 18\pi$$

14. $\partial\mathbf{r}/\partial u = \mathbf{k}, \partial\mathbf{r}/\partial v = -2\sin v\mathbf{i} + \cos v\mathbf{j}$, $\partial\mathbf{r}/\partial u \times \partial\mathbf{r}/\partial v = -\cos v\mathbf{i} - 2\sin v\mathbf{j}$;

$$\iint_R (2\sin^2 v - e^{-\sin v}\cos v)\, dA = \int_0^{2\pi}\int_0^5 (2\sin^2 v - e^{-\sin v}\cos v) du\, dv = 10\pi$$

15. $\partial\mathbf{r}/\partial u = \cos v\mathbf{i} + \sin v\mathbf{j} + 2\mathbf{k}, \partial\mathbf{r}/\partial v = -u\sin v\mathbf{i} + u\cos v\mathbf{j}$,
$\partial\mathbf{r}/\partial u \times \partial\mathbf{r}/\partial v = -2u\cos v\mathbf{i} - 2u\sin v\mathbf{j} + u\mathbf{k}$;

$$\iint_R u^2\, dA = \int_0^{\pi}\int_0^{\sin v} u^2 du\, dv = 4/9$$

16. $\partial\mathbf{r}/\partial u = 2\cos u\cos v\mathbf{i} + 2\cos u\sin v\mathbf{j} - 2\sin u\mathbf{k}$, $\partial\mathbf{r}/\partial v = -2\sin u\sin v\mathbf{i} + 2\sin u\cos v\mathbf{j}$;
$\partial\mathbf{r}/\partial u \times \partial\mathbf{r}/\partial v = 4\sin^2 u\cos v\mathbf{i} + 4\sin^2 u\sin v\mathbf{j} + 4\sin u\cos u\mathbf{k}$;

$$\iint_R 8\sin u\, dA = 8\int_0^{2\pi}\int_0^{\pi/3} \sin u\, du\, dv = 8\pi$$

17. In each part, divide σ into the six surfaces
$\sigma_1 : x = -1$ with $|y| \le 1$, $|z| \le 1$, and $\mathbf{n} = -\mathbf{i}$, $\sigma_2 : x = 1$ with $|y| \le 1$, $|z| \le 1$, and $\mathbf{n} = \mathbf{i}$,
$\sigma_3 : y = -1$ with $|x| \le 1$, $|z| \le 1$, and $\mathbf{n} = -\mathbf{j}$, $\sigma_4 : y = 1$ with $|x| \le 1$, $|z| \le 1$, and $\mathbf{n} = \mathbf{j}$,
$\sigma_5 : z = -1$ with $|x| \le 1$, $|y| \le 1$, and $\mathbf{n} = -\mathbf{k}$, $\sigma_6 : z = 1$ with $|x| \le 1$, $|y| \le 1$, and $\mathbf{n} = \mathbf{k}$,

(a) $\displaystyle\iint_{\sigma_1} \mathbf{F}\cdot\mathbf{n}\, dS = \iint_{\sigma_1} dS = 4$, $\displaystyle\iint_{\sigma_2} \mathbf{F}\cdot\mathbf{n}\, dS = \iint_{\sigma_2} dS = 4$, and $\displaystyle\iint_{\sigma_i} \mathbf{F}\cdot\mathbf{n}\, dS = 0$ for $i = 3, 4, 5, 6$ so $\displaystyle\iint_{\sigma} \mathbf{F}\cdot\mathbf{n}\, dS = 4 + 4 + 0 + 0 + 0 + 0 = 8$.

(b) $\displaystyle\iint_{\sigma_1} \mathbf{F}\cdot\mathbf{n}\, dS = \iint_{\sigma_1} dS = 4$, similarly $\displaystyle\iint_{\sigma_i} \mathbf{F}\cdot\mathbf{n}\, dS = 4$ for $i = 2, 3, 4, 5, 6$ so
$\displaystyle\iint_{\sigma} \mathbf{F}\cdot\mathbf{n}\, dS = 4 + 4 + 4 + 4 + 4 + 4 = 24$.

(c) $\displaystyle\iint_{\sigma_1} \mathbf{F}\cdot\mathbf{n}\, dS = -\iint_{\sigma_1} dS = -4$, $\displaystyle\iint_{\sigma_2} \mathbf{F}\cdot\mathbf{n}\, dS = 4$, similarly $\displaystyle\iint_{\sigma_i} \mathbf{F}\cdot\mathbf{n}\, dS = -4$ for $i = 3, 5$ and $\displaystyle\iint_{\sigma_i} \mathbf{F}\cdot\mathbf{n}\, dS = 4$ for $i = 4, 6$ so $\displaystyle\iint_{\sigma} \mathbf{F}\cdot\mathbf{n}\, dS = -4 + 4 - 4 + 4 - 4 + 4 = 0$.

18. Decompose σ into a top σ_1 (the disk) and a bottom σ_2 (the portion of the paraboloid). Then

$$\mathbf{n}_1 = \mathbf{k}, \iint_{\sigma_1} \mathbf{F}\cdot\mathbf{n}_1\, dS = -\iint_{\sigma_1} y\, dS = -\int_0^{2\pi}\int_0^1 r^2\sin\theta\, dr\, d\theta = 0,$$

$$\mathbf{n}_2 = (2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k})/\sqrt{1 + 4x^2 + 4y^2}, \iint_{\sigma_2} \mathbf{F}\cdot\mathbf{n}_2\, dS = \iint_{\sigma_2} \frac{y(2x^2 + 2y^2 + 1)}{\sqrt{1 + 4x^2 + 4y^2}}\, dS = 0,$$

because the surface σ_2 is symmetric with respect to the xy-plane and the integrand is an odd function of y. Thus the flux is 0.

19. R is the circular region enclosed by $x^2+y^2=1$; $x=r\cos\theta, y=r\sin\theta, z=r$, $\mathbf{n}=\cos\theta\mathbf{i}+\sin\theta\mathbf{j}-\mathbf{k}$;

$$\iint_\sigma \mathbf{F}\cdot\mathbf{n}\,dS=\iint_R(\cos\theta+\sin\theta-1)\,dA=\int_0^{2\pi}\int_0^1(\cos\theta+\sin\theta-1)\,r\,dr\,d\theta=-\pi.$$

20. Let $\mathbf{r}=\cos v\mathbf{i}+u\mathbf{j}+\sin v\mathbf{k}, -2\le u\le 1, 0\le v\le 2\pi$; $\mathbf{r}_u\times\mathbf{r}_v=\cos v\mathbf{i}+\sin v\mathbf{k}$,

$$\iint_\sigma \mathbf{F}\cdot\mathbf{n}\,dS=\iint_R(\cos^2 v+\sin^2 v)\,dA=\text{area of } R=3\cdot 2\pi=6\pi$$

21. **(a)** $\mathbf{n}=\dfrac{1}{\sqrt{3}}[\mathbf{i}+\mathbf{j}+\mathbf{k}]$,

$$V=\int_\sigma \mathbf{F}\cdot\mathbf{n}\,dS=\int_0^1\int_0^{1-x}(2x-3y+1-x-y)\,dy\,dx=0\ \text{m}^3/\text{s}$$

(b) $m=0\cdot 806=0$ kg/s

22. **(a)** Let $x=3\sin\phi\cos\theta, y=3\sin\phi\sin\theta, z=3\cos\phi$, $\mathbf{n}=\sin\phi\cos\theta\mathbf{i}+\sin\phi\sin\theta\,\mathbf{j}+\cos\phi\,\mathbf{k}$, so

$$V=\int_\sigma \mathbf{F}\cdot\mathbf{n}\,dS=\iint_A 9\sin\phi\,(-3\sin^2\phi\sin\theta\cos\theta+3\sin\phi\cos\phi\sin\theta+9\sin\phi\cos\phi\cos\theta)\,dA$$

$$=\int_0^{2\pi}\int_0^3 3\sin\phi\cos\theta(-\sin\phi\sin\theta+4\cos\phi)\,r\,dr\,d\theta=0\ \text{m}^3$$

(b) $\dfrac{dm}{dt}=0\cdot 1060=0$ kg/s

23. **(a)** $G(x,y,z)=x-g(y,z)$, $\nabla G=\mathbf{i}-\dfrac{\partial g}{\partial y}\mathbf{j}-\dfrac{\partial g}{\partial z}\mathbf{k}$, apply Theorem 16.6.3:

$$\iint_\sigma \mathbf{F}\cdot\mathbf{n}dS=\iint_R \mathbf{F}\cdot\left(\mathbf{i}-\frac{\partial x}{\partial y}\mathbf{j}-\frac{\partial x}{\partial z}\mathbf{k}\right)dA, \text{ if } \sigma \text{ is oriented by front normals, and}$$

$$\iint_\sigma \mathbf{F}\cdot\mathbf{n}dS=\iint_R \mathbf{F}\cdot\left(-\mathbf{i}+\frac{\partial x}{\partial y}\mathbf{j}+\frac{\partial x}{\partial z}\mathbf{k}\right)dA, \text{ if } \sigma \text{ is oriented by back normals,}$$

where R is the projection of σ onto the yz-plane.

(b) R is the semicircular region in the yz-plane enclosed by $z=\sqrt{1-y^2}$ and $z=0$;

$$\iint_\sigma \mathbf{F}\cdot\mathbf{n}\,dS=\iint_R(-y-2yz+16z)dA=\int_{-1}^1\int_0^{\sqrt{1-y^2}}(-y-2yz+16z)dz\,dy=\frac{32}{3}.$$

24. **(a)** $G(x,y,z)=y-g(x,z)$, $\nabla G=-\dfrac{\partial g}{\partial x}\mathbf{i}+\mathbf{j}-\dfrac{\partial g}{\partial z}\mathbf{k}$, apply Theorem 16.6.3:

$$\iint_R \mathbf{F}\cdot\left(\frac{\partial y}{\partial x}\mathbf{i}-\mathbf{j}+\frac{\partial y}{\partial z}\mathbf{k}\right)dA, \ \sigma \text{ oriented by left normals,}$$

and $$\iint_R \mathbf{F}\cdot\left(-\frac{\partial y}{\partial x}\mathbf{i}+\mathbf{j}-\frac{\partial y}{\partial z}\mathbf{k}\right)dA, \ \sigma \text{ oriented by right normals,}$$

where R is the projection of σ onto the xz-plane.

(b) R is the semicircular region in the xz-plane enclosed by $z=\sqrt{1-x^2}$ and $z=0$;

$$\iint_\sigma \mathbf{F}\cdot\mathbf{n}\,dS=\iint_R(-2x^2+(x^2+z^2)-2z^2)dA=-\int_{-1}^{1}\int_0^{\sqrt{1-x^2}}(x^2+z^2)dz\,dx=-\frac{\pi}{4}.$$

25. **(a)** On the sphere, $\|\mathbf{r}\|=a$ so $\mathbf{F}=a^k\mathbf{r}$ and $\mathbf{F}\cdot\mathbf{n}=a^k\mathbf{r}\cdot(\mathbf{r}/a)=a^{k-1}\|\mathbf{r}\|^2=a^{k-1}a^2=a^{k+1}$, hence $\iint_\sigma \mathbf{F}\cdot\mathbf{n}\,dS=a^{k+1}\iint_\sigma dS=a^{k+1}(4\pi a^2)=4\pi a^{k+3}$.

(b) If $k=-3$, then $\iint_\sigma \mathbf{F}\cdot\mathbf{n}\,dS=4\pi$.

26. Let $\mathbf{r}=\sin u\cos v\mathbf{i}+\sin u\sin v\mathbf{j}+\cos u\mathbf{k}, \mathbf{r}_u\times\mathbf{r}_v=\sin^2 u\cos v\mathbf{i}+\sin^2 u\sin v\mathbf{j}+\sin u\cos u\mathbf{k}$,

$\mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)=a^2\sin^3 u\cos^2 v+\dfrac{1}{a}\sin^3 u\sin^2 v+a\sin u\cos^3 u$,

$$\begin{aligned}\iint_\sigma \mathbf{F}\cdot\mathbf{n}\,dS&=\int_0^{2\pi}\int_0^{\pi}\left(a^2\sin^3 u\cos^2 v+\frac{1}{a}\sin^3 u\sin^2 v+a\sin u\cos^3 u\right)du\,dv\\&=\frac{4}{3a}\int_0^{\pi}(a^3\cos^2 v+\sin^2 v)\,dv\\&=\frac{4\pi}{3}\left(a^2+\frac{1}{a}\right)=10 \text{ if } a\approx -1.722730, 0.459525, 1.263205\end{aligned}$$

EXERCISE SET 16.7

1. $\sigma_1: x=0, \mathbf{F}\cdot\mathbf{n}=-x=0, \iint_{\sigma_1}(0)dA=0$ $\qquad\sigma_2: x=1, \mathbf{F}\cdot\mathbf{n}=x=1, \iint_{\sigma_2}(1)dA=1$

$\sigma_3: y=0, \mathbf{F}\cdot\mathbf{n}=-y=0, \iint_{\sigma_3}(0)dA=0$ $\qquad\sigma_4: y=1, \mathbf{F}\cdot\mathbf{n}=y=1, \iint_{\sigma_4}(1)dA=1$

$\sigma_5: z=0, \mathbf{F}\cdot\mathbf{n}=-z=0, \iint_{\sigma_5}(0)dA=0$ $\qquad\sigma_6: z=1, \mathbf{F}\cdot\mathbf{n}=z=1, \iint_{\sigma_6}(1)dA=1$

$\iint_\sigma \mathbf{F}\cdot\mathbf{n}=3;\ \iiint_G \operatorname{div}\mathbf{F}dV=\iiint_G 3dV=3$

2. For any point $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ on σ let $\mathbf{n}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$; then $\mathbf{F}\cdot\mathbf{n}=x^2+y^2+z^2=1$, so $\iint_\sigma \mathbf{F}\cdot\mathbf{n}\,dS=\iint_\sigma dS=4\pi$; also $\iiint_G \operatorname{div}\mathbf{F}dV=\iiint_G 3dV=3(4\pi/3)=4\pi$

3. $\sigma_1: z=1, \mathbf{n}=\mathbf{k}, \mathbf{F}\cdot\mathbf{n}=z^2=1, \iint_{\sigma_1}(1)dS=\pi$,

$\sigma_2: \mathbf{n}=2x\mathbf{i}+2y\mathbf{j}-\mathbf{k}, \mathbf{F}\cdot\mathbf{n}=4x^2-4x^2y^2-x^4-3y^4$,

$$\iint_{\sigma_2} \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^{2\pi}\int_0^1 \left[4r^2\cos^2\theta - 4r^4\cos^2\theta\sin^2\theta - r^4\cos^4\theta - 3r^4\sin^4\theta\right] r\,dr\,d\theta = \frac{\pi}{3};$$

$$\iint_{\sigma} = \frac{4\pi}{3}$$

$$\iiint_G \text{div }\mathbf{F}dV = \iiint_G (2+z)dV = \int_0^{2\pi}\int_0^1\int_{r^2}^1 (2+z)dz\,r\,dr\,d\theta = 4\pi/3$$

4. $\sigma_1 : x = 0, \mathbf{F}\cdot\mathbf{n} = -xy = 0, \iint_{\sigma_1}(0)dA = 0$ $\qquad \sigma_2 : x = 2, \mathbf{F}\cdot\mathbf{n} = xy = 2y, \iint_{\sigma_2}(2y)dA = 8$

$\sigma_3 : y = 0, \mathbf{F}\cdot\mathbf{n} = -yz = 0, \iint_{\sigma_3}(0)dA = 0$ $\qquad \sigma_4 : y = 2, \mathbf{F}\cdot\mathbf{n} = yz = 2z, \iint_{\sigma_4}(2z)dA = 8$

$\sigma_5 : z = 0, \mathbf{F}\cdot\mathbf{n} = -xz = 0, \iint_{\sigma_5}(0)dA = 0$ $\qquad \sigma_6 : z = 2, \mathbf{F}\cdot\mathbf{n} = xz = 2x, \iint_{\sigma_6}(2x)dA = 8$

$\iint_{\sigma} \mathbf{F}\cdot\mathbf{n} = 24$; also $\iiint_G \text{div }\mathbf{F}dV = \iiint_G (y+z+x)dV = 24$

5. G is the rectangular solid; $\iiint_G \text{div }\mathbf{F}\,dV = \int_0^2\int_0^1\int_0^3 (2x-1)\,dx\,dy\,dz = 12.$

6. G is the spherical solid enclosed by σ; $\iiint_G \text{div }\mathbf{F}\,dV = \iiint_G 0\,dV = 0\iiint_G dV = 0.$

7. G is the cylindrical solid;

$$\iiint_G \text{div }\mathbf{F}\,dV = 3\iiint_G dV = (3)(\text{volume of cylinder}) = (3)[\pi a^2(1)] = 3\pi a^2.$$

8. G is the solid bounded by $z = 1 - x^2 - y^2$ and the xy-plane;

$$\iiint_G \text{div }\mathbf{F}\,dV = 3\iiint_G dV = 3\int_0^{2\pi}\int_0^1\int_0^{1-r^2} r\,dz\,dr\,d\theta = \frac{3\pi}{2}.$$

9. G is the cylindrical solid;

$$\iiint_G \text{div }\mathbf{F}\,dV = 3\iiint_G (x^2+y^2+z^2)dV = 3\int_0^{2\pi}\int_0^2\int_0^3 (r^2+z^2)r\,dz\,dr\,d\theta = 180\pi.$$

10. G is the tetrahedron; $\iiint_G \text{div }\mathbf{F}\,dV = \iiint_G x\,dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y} x\,dz\,dy\,dx = \frac{1}{24}.$

11. G is the hemispherical solid bounded by $z = \sqrt{4-x^2-y^2}$ and the xy-plane;

$$\iiint_G \text{div }\mathbf{F}\,dV = 3\iiint_G (x^2+y^2+z^2)dV = 3\int_0^{2\pi}\int_0^{\pi/2}\int_0^2 \rho^4\sin\phi\,d\rho\,d\phi\,d\theta = \frac{192\pi}{5}.$$

12. G is the hemispherical solid;

$$\iiint\limits_G \operatorname{div} \mathbf{F}\, dV = 5 \iiint\limits_G z\, dV = 5 \int_0^{2\pi} \int_0^{\pi/2} \int_0^a \rho^3 \sin\phi \cos\phi \, d\rho\, d\phi\, d\theta = \frac{5\pi a^4}{4}.$$

13. G is the conical solid;

$$\iiint\limits_G \operatorname{div} \mathbf{F}\, dV = 2 \iiint\limits_G (x+y+z) dV = 2 \int_0^{2\pi} \int_0^1 \int_r^1 (r\cos\theta + r\sin\theta + z) r\, dz\, dr\, d\theta = \frac{\pi}{2}.$$

14. G is the solid bounded by $z = 2x$ and $z = x^2 + y^2$;

$$\iiint\limits_G \operatorname{div} \mathbf{F}\, dV = \iiint\limits_G dV = 2 \int_0^{\pi/2} \int_0^{2\cos\theta} \int_{r^2}^{2r\cos\theta} r\, dz\, dr\, d\theta = \frac{\pi}{2}.$$

15. G is the solid bounded by $z = 4 - x^2$, $y + z = 5$, and the coordinate planes;

$$\iiint\limits_G \operatorname{div} \mathbf{F}\, dV = 4 \iiint\limits_G x^2 dV = 4 \int_{-2}^{2} \int_0^{4-x^2} \int_0^{5-z} x^2 dy\, dz\, dx = \frac{4608}{35}.$$

16. $$\iint\limits_\sigma \mathbf{F} \cdot \mathbf{n}\, dS = \iiint\limits_G \operatorname{div} \mathbf{F}\, dV = \iiint\limits_G 0\, dV = 0;$$

since the vector field is constant, the same amount enters as leaves.

17. $$\iint\limits_\sigma \mathbf{r} \cdot \mathbf{n}\, dS = \iiint\limits_G \operatorname{div} \mathbf{r}\, dV = 3 \iiint\limits_G dV = 3\text{vol}(G)$$

18. $$\iint\limits_\sigma \mathbf{F} \cdot \mathbf{n}\, dS = 3[\pi(3^2)(5)] = 135\pi$$

19. $$\iint\limits_\sigma \text{curl } \mathbf{F} \cdot \mathbf{n}\, dS = \iiint\limits_G \text{div}(\text{curl } \mathbf{F}) dV = \iiint\limits_G (0) dV = 0$$

20. $$\iint\limits_\sigma \nabla f \cdot \mathbf{n}\, dS = \iiint\limits_G \text{div } (\nabla f) dV = \iiint\limits_G \nabla^2 f dV$$

21. $$\iint\limits_\sigma (f\nabla g) \cdot \mathbf{n} = \iiint\limits_G \text{div } (f\nabla g) dV = \iiint\limits_G (f\nabla^2 g + \nabla f \cdot \nabla g) dV$$ by Exercise 31, Section 16.1.

22. $$\iint\limits_\sigma (f\nabla g) \cdot \mathbf{n}\, dS = \iiint\limits_G (f\nabla^2 g + \nabla f \cdot \nabla g) dV$$ by Exercise 21;

$$\iint\limits_\sigma (g\nabla f) \cdot \mathbf{n}\, dS = \iiint\limits_G (g\nabla^2 f + \nabla g \cdot \nabla f) dV$$ by interchanging f and g;

subtract to obtain the result.

23. Since $\mathbf{v}$ is constant, $\nabla \cdot \mathbf{v} = \mathbf{0}$. Let $\mathbf{F} = f\mathbf{v}$; then $\text{div}\mathbf{F} = (\nabla f)\mathbf{v}$ and by the Divergence Theorem

$$\iint_\sigma f\mathbf{v}\cdot\mathbf{n}\,dS = \iint_\sigma \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_G \text{div}\mathbf{F}\,dV = \iiint_G (\nabla f)\cdot\mathbf{v}\,dV$$

24. Let $\mathbf{r} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ so that, for $\mathbf{r} \neq \mathbf{0}$,

$$\mathbf{F}(x,y,z) = \mathbf{r}/||\mathbf{r}||^k = \frac{u}{(u^2+v^2+w^2)^{k/2}}\mathbf{i} + \frac{v}{(u^2+v^2+w^2)^{k/2}}\mathbf{j} + \frac{w}{(u^2+v^2+w^2)^{k/2}}\mathbf{k}$$

$$\frac{\partial \mathbf{F}_1}{\partial u} = \frac{u^2+v^2+w^2-ku^2}{(u^2+v^2+w^2)^{(k/2)+1}};$$ similarly for $\partial\mathbf{F}_2/\partial v, \partial\mathbf{F}_3/\partial w$, so that

$$\text{div }\mathbf{F} = \frac{3(u^2+v^2+w^2)-k(u^2+v^2+w^2)}{(u^2+v^2+w^2)^{(k/2)+1}} = 0 \text{ if and only if } k = 3.$$

25. **(a)** The flux through any cylinder whose axis is the z-axis is positive by inspection; by the Divergence Theorem, this says that the divergence cannot be negative at the origin, else the flux through a small enough cylinder would also be negative (impossible), hence the divergence at the origin must be ≥ 0.

(b) Similar to Part (a), ≤ 0.

26. **(a)** $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, div $\mathbf{F} = 3$ **(b)** $\mathbf{F} = -x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$, div $\mathbf{F} = -3$

27. div $\mathbf{F} = 0$; no sources or sinks.

28. div $\mathbf{F} = y - x$; sources where $y > x$, sinks where $y < x$.

29. div $\mathbf{F} = 3x^2 + 3y^2 + 3z^2$; sources at all points except the origin, no sinks.

30. div $\mathbf{F} = 3(x^2 + y^2 + z^2 - 1)$; sources outside the sphere $x^2 + y^2 + z^2 = 1$, sinks inside the sphere $x^2 + y^2 + z^2 = 1$.

31. Let σ_1 be the portion of the paraboloid $z = 1 - x^2 - y^2$ for $z \geq 0$, and σ_2 the portion of the plane $z = 0$ for $x^2 + y^2 \leq 1$. Then

$$\begin{aligned}\iint_{\sigma_1} \mathbf{F}\cdot\mathbf{n}\,dS &= \iint_R \mathbf{F}\cdot(2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})\,dA \\ &= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (2x[x^2y - (1-x^2-y^2)^2] + 2y(y^3 - x) + (2x + 2 - 3x^2 - 3y^2))\,dy\,dx \\ &= 3\pi/4;\end{aligned}$$

$z = 0$ and $\mathbf{n} = -\mathbf{k}$ on σ_2 so $\mathbf{F}\cdot\mathbf{n} = 1 - 2x$, $\displaystyle\iint_{\sigma_2} \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{\sigma_2}(1-2x)dS = \pi$. Thus

$\displaystyle\iint_\sigma \mathbf{F}\cdot\mathbf{n}\,dS = 3\pi/4 + \pi = 7\pi/4$. But div $\mathbf{F} = 2xy + 3y^2 + 3$ so

$$\iiint_G \text{div }\mathbf{F}\,dV = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{1-x^2-y^2}(2xy + 3y^2 + 3)\,dz\,dy\,dx = 7\pi/4.$$

EXERCISE SET 16.8

1. (a) The flow is independent of z and has no component in the direction of $\mathbf{k}$, and so by inspection the only nonzero component of the curl is in the direction of $\mathbf{k}$. However both sides of (9) are zero, as the flow is orthogonal to the curve C_a. Thus the curl is zero.

(b) Since the flow appears to be tangential to the curve C_a, it seems that the right hand side of (9) is nonzero, and thus the curl is nonzero, and points in the positive z-direction.

2. (a) The only nonzero vector component of the vector field is in the direction of $\mathbf{i}$, and it increases with y and is independent of x. Thus the curl of F is nonzero, and points in the positive z-direction. Alternatively, let $\mathbf{F} = f\mathbf{i}$, and let C be the circle of radius ϵ with positive orientation. Then $\mathbf{T} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$, and

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = -\epsilon\int_0^{2\pi} f(\epsilon,\theta)\sin\theta\,d\theta = -\epsilon\int_0^{\pi} f(\epsilon,\theta)\sin\theta\,d\theta - \epsilon\int_{-\pi}^{0} f(\epsilon,\theta)\sin\theta\,d\theta$$

$$= -\epsilon\int_0^{\pi}(f(\epsilon,\theta) - f(-\epsilon,\theta))\sin\theta\,d\theta < 0$$

because from the picture $f(\epsilon,\theta) > f(\epsilon,-\theta)$ for $0<\theta<\pi$. Thus, from (9), the curl is nonzero and points in the negative z-direction.

(b) By inspection the vector field is constant, and thus its curl is zero.

3. If σ is oriented with upward normals then C consists of three parts parametrized as

$C_1 : \mathbf{r}(t) = (1-t)\mathbf{i} + t\mathbf{j}$ for $0 \le t \le 1$, $C_2 : \mathbf{r}(t) = (1-t)\mathbf{j} + t\mathbf{k}$ for $0 \le t \le 1$,

$C_3 : \mathbf{r}(t) = t\mathbf{i} + (1-t)\mathbf{k}$ for $0 \le t \le 1$.

$$\int_{C_1}\mathbf{F}\cdot d\mathbf{r} = \int_{C_2}\mathbf{F}\cdot d\mathbf{r} = \int_{C_3}\mathbf{F}\cdot d\mathbf{r} = \int_0^1 (3t-1)dt = \frac{1}{2} \text{ so}$$

$\oint_C \mathbf{F}\cdot d\mathbf{r} = \frac{1}{2}+\frac{1}{2}+\frac{1}{2} = \frac{3}{2}$. curl $\mathbf{F} = \mathbf{i}+\mathbf{j}+\mathbf{k}$, $z = 1-x-y$, R is the triangular region in the xy-plane enclosed by $x+y=1$, $x=0$, and $y=0$;

$$\iint_\sigma (\text{curl } \mathbf{F})\cdot\mathbf{n}\,dS = 3\iint_R dA = (3)(\text{area of } R) = (3)\left[\frac{1}{2}(1)(1)\right] = \frac{3}{2}.$$

4. If σ is oriented with upward normals then C can be parametrized as $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \mathbf{k}$ for $0 \le t \le 2\pi$.

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}(\sin^2 t\cos t - \cos^2 t\sin t)dt = 0;$$

curl $\mathbf{F} = \mathbf{0}$ so $\iint_\sigma (\text{curl } \mathbf{F})\cdot\mathbf{n}\,dS = \iint_\sigma 0\,dS = 0$.

5. If σ is oriented with upward normals then C can be parametrized as $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j}$ for $0 \le t \le 2\pi$.

$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} 0\,dt = 0$; curl $\mathbf{F} = \mathbf{0}$ so $\iint_\sigma (\text{curl } \mathbf{F})\cdot\mathbf{n}\,dS = \iint_\sigma 0\,dS = 0$.

6. If σ is oriented with upward normals then C can be parametrized as $\mathbf{r}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j}$ for $0 \le t \le 2\pi$.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (9\sin^2 t + 9\cos^2 t)dt = 9\int_0^{2\pi} dt = 18\pi.$$

curl $\mathbf{F} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, R is the circular region in the xy-plane enclosed by $x^2 + y^2 = 9$;

$$\iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_R (-4x + 4y + 2)dA = \int_0^{2\pi} \int_0^3 (-4r\cos\theta + 4r\sin\theta + 2)r\, dr\, d\theta = 18\pi.$$

7. Take σ as the part of the plane $z = 0$ for $x^2 + y^2 \le 1$ with $\mathbf{n} = \mathbf{k}$; curl $\mathbf{F} = -3y^2\mathbf{i} + 2z\mathbf{j} + 2\mathbf{k}$,

$$\iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = 2\iint_\sigma dS = (2)(\text{area of circle}) = (2)[\pi(1)^2] = 2\pi.$$

8. curl $\mathbf{F} = x\mathbf{i} + (x - y)\mathbf{j} + 6xy^2\mathbf{k}$;

$$\iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_R (x - y - 6xy^2)dA = \int_0^1 \int_0^3 (x - y - 6xy^2)dy\, dx = -30.$$

9. C is the boundary of R and curl $\mathbf{F} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, so

$$\oint \mathbf{F} \cdot \mathbf{r} = \iint_R \text{curl } \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R 4\, dA = 4(\text{area of } R) = 16\pi$$

10. curl $\mathbf{F} = -4\mathbf{i} - 6\mathbf{j} + 6y\mathbf{k}$, $z = y/2$ oriented with upward normals, R is the triangular region in the xy-plane enclosed by $x + y = 2$, $x = 0$, and $y = 0$;

$$\iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_R (3 + 6y)dA = \int_0^2 \int_0^{2-x} (3 + 6y)dy\, dx = 14.$$

11. curl $\mathbf{F} = x\mathbf{k}$, take σ as part of the plane $z = y$ oriented with upward normals, R is the circular region in the xy-plane enclosed by $x^2 + y^2 - y = 0$;

$$\iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_R x\, dA = \int_0^\pi \int_0^{\sin\theta} r^2\cos\theta\, dr\, d\theta = 0.$$

12. curl $\mathbf{F} = -y\mathbf{i} - z\mathbf{j} - x\mathbf{k}$, $z = 1 - x - y$ oriented with upward normals, R is the triangular region in the xy-plane enclosed by $x + y = 1$, $x = 0$ and $y = 0$;

$$\iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_R (-y - z - x)dA = -\iint_R dA = -\frac{1}{2}(1)(1) = -\frac{1}{2}.$$

13. curl $\mathbf{F} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, take σ as the part of the plane $z = 0$ with $x^2 + y^2 \le a^2$ and $\mathbf{n} = \mathbf{k}$;

$$\iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_\sigma dS = \text{ area of circle } = \pi a^2.$$

14. curl $\mathbf{F} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, take σ as the part of the plane $z = 1/\sqrt{2}$ with $x^2 + y^2 \le 1/2$ and $\mathbf{n} = \mathbf{k}$.

$$\iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_\sigma dS = \text{ area of circle } = \frac{\pi}{2}.$$

15. **(a)** Take σ as the part of the plane $2x+y+2z=2$ in the first octant, oriented with downward normals; curl $\mathbf{F} = -x\mathbf{i} + (y-1)\mathbf{j} - \mathbf{k}$,

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_\sigma (\text{curl }\mathbf{F})\cdot\mathbf{n}\,dS$$
$$= \iint_R \left(x - \frac{1}{2}y + \frac{3}{2}\right) dA = \int_0^1 \int_0^{2-2x} \left(x - \frac{1}{2}y + \frac{3}{2}\right) dy\,dx = \frac{3}{2}.$$

(b) At the origin curl $\mathbf{F} = -\mathbf{j} - \mathbf{k}$ and with $\mathbf{n} = \mathbf{k}$, curl $\mathbf{F}(0,0,0)\cdot\mathbf{n} = (-\mathbf{j}-\mathbf{k})\cdot\mathbf{k} = -1$.

(c) The rotation of $\mathbf{F}$ has its maximum value at the origin about the unit vector in the same direction as curl $\mathbf{F}(0,0,0)$ so $\mathbf{n} = -\frac{1}{\sqrt{2}}\mathbf{j} - \frac{1}{\sqrt{2}}\mathbf{k}$.

16. **(a)** Using the hint, the orientation of the curve C with respect to the surface σ_1 is the opposite of the orientation of C with respect to the surface σ_2.
Thus in the expressions

$$\iint_{\sigma_1} (\text{curl }\mathbf{F})\cdot\mathbf{n}\,dS = \int_C \mathbf{F}\cdot\mathbf{T}\,dS \text{ and } \iint_{\sigma_2} (\text{curl }\mathbf{F})\cdot\mathbf{n}\,dS = \oint_C \mathbf{F}\cdot\mathbf{T}\,dS,$$

the two line integrals have oppositely oriented tangents $\mathbf{T}$. Hence

$$\iint_\sigma (\text{curl }\mathbf{F})\cdot\mathbf{n}\,dS = \iint_{\sigma_1} (\text{curl }\mathbf{F})\cdot\mathbf{n}\,dS + \iint_{\sigma_2} (\text{curl }\mathbf{F})\cdot\mathbf{n}\,dS = \mathbf{0}.$$

(b) The flux of the curl field through the boundary of a solid is zero.

17. Since $\oint_C \mathbf{E}\cdot\mathbf{r}d\mathbf{r} = \iint_\sigma \text{curl }\mathbf{E}\cdot\mathbf{n}\,dS$, it follows that $\iint_\sigma \text{curl }\mathbf{E}\cdot\mathbf{n}dS = -\iint_\sigma \frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{n}dS$. This relationship holds for any surface σ, hence curl $\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$.

18. Parametrize C by $x=\cos t, y=\sin t, 0\le t\le 2\pi$. But $\mathbf{F} = x^2y\mathbf{i} + (y^3-x)\mathbf{j} + (2x-1)\mathbf{k}$ along C so $\oint_C \mathbf{F}\cdot d\mathbf{r} = -5\pi/4$. Since curl $\mathbf{F} = (-2z-2)\mathbf{j} + (-1-x^2)\mathbf{k}$,

$$\iint_\sigma (\text{curl }\mathbf{F})\cdot\mathbf{n}\,dS = \iint_R (\text{curl }\mathbf{F})\cdot(2x\mathbf{i}+2y\mathbf{j}+\mathbf{k})\,dA$$
$$= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} [2y(2x^2+2y^2-4)-1-x^2]\,dy\,dx = -5\pi/4$$

CHAPTER 16 SUPPLEMENTARY EXERCISES

2. **(b)** $\frac{c}{\|\mathbf{r}-\mathbf{r}_0\|^3}(\mathbf{r}-\mathbf{r}_0)$ **(c)** $c\frac{(x-x_0)\mathbf{i}+(y-y_0)\mathbf{j}+(z-z_0)\mathbf{k}}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}$

3. **(a)** $\int_a^b \left[f(x(t),y(t))\frac{dx}{dt} + g(x(t),y(t))\frac{dy}{dt}\right] dt$

(b) $\displaystyle\int_a^b f(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2}\, dt$

4. **(a)** $\displaystyle M = \int\limits_C \delta(x, y, z)\, ds$ **(b)** $\displaystyle L = \int\limits_C ds$ **(c)** $\displaystyle S = \iint\limits_\sigma dS$

(d) $\displaystyle A = \oint\limits_C x\, dy = -\oint\limits_C y\, dx = \frac{1}{2}\oint\limits_C -y\, dx + x\, dy$

11. $\displaystyle\iint\limits_\sigma f(x, y, z)dS = \iint\limits_R f(x(u, v), y(u, v), z(u, v))\|\mathbf{r}_u \times \mathbf{r}_v\|\, du\, dv$

13. Let O be the origin, P the point with polar coordinates $\theta = \alpha, r = f(\alpha)$, and Q the point with polar coordinates $\theta = \beta, r = f(\beta)$. Let

$C_1 : O$ to P; $x = t\cos\alpha,\ y = t\sin\alpha,\ 0 \le t \le f(\alpha), -y\dfrac{dx}{dt} + x\dfrac{dy}{dt} = 0$

$C_2 : P$ to Q; $x = f(t)\cos t,\ y = f(t)\sin t,\ \alpha \le \theta \le \beta, -y\dfrac{dx}{dt} + x\dfrac{dy}{dt} = f(t)^2$

$C_3 : Q$ to O; $x = -t\cos\beta,\ y = -t\sin\beta,\ -f(\beta) \le t \le 0, -y\dfrac{dx}{dt} + x\dfrac{dy}{dt} = 0$

$\displaystyle A = \frac{1}{2}\oint_C -y\, dx + x\, dy = \frac{1}{2}\int_\alpha^\beta f(t)^2\, dt$; set $t = \theta$ and $r = f(\theta) = f(t)$, $\displaystyle A = \frac{1}{2}\int_\alpha^\beta r^2\, d\theta.$

14. **(a)** $\displaystyle\mathbf{F}(x, y, z) = \frac{qQ(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{4\pi\epsilon_0(x^2 + y^2 + z^2)^{3/2}}$

(b) $\mathbf{F} = \nabla\phi$, where $\displaystyle\phi = -\frac{qQ}{4\pi\epsilon_0(x^2 + y^2 + z^2)^{1/2}}$, so $\displaystyle W = \phi(3, 1, 5) - \phi(3, 0, 0) = \frac{qQ}{4\pi\epsilon_0}\left(\frac{1}{3} - \frac{1}{\sqrt{35}}\right)$.

$C : x = 3, y = t, z = 5t, 0 \le t \le 1;\ \mathbf{F}\cdot d\mathbf{r} = \dfrac{qQ[0 + t + 25t]\, dt}{4\pi\epsilon_0(9 + t^2 + 25t^2)^{3/2}}$

$$W = \int_0^1 \frac{26qQt\, dt}{4\pi\epsilon_0(26t^2 + 9)^{3/2}} = \frac{qQ}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{35}} - \frac{1}{3}\right)$$

15. **(a)** Assume the mass M is located at the origin and the mass m at (x, y, z), then

$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \mathbf{F}(x, y, z) = -\dfrac{GmM}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{r}$,

$$W = -\int_{t_1}^{t_2} \frac{GmM}{(x^2 + y^2 + z^2)^{3/2}}\left(x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt}\right) dt$$

$$= GmM(x^2 + y^2 + z^2)^{-1/2}\Big]_{t_1}^{t_2} = GmM\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

(b) $\displaystyle W = 3.99 \times 10^5 \times 10^3\left[\frac{1}{7170} - \frac{1}{6970}\right] \approx -1596.801594\ \text{km}^2\text{kg/s}^2 \approx -1.597 \times 10^9\ \text{J}$

16. $\displaystyle\text{div}\,\mathbf{F} = \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{x^2 - y^2}{(x^2 + y^2)^2} + \frac{1}{(x^2 + y^2)} = \frac{1}{x^2 + y^2}$, the level surface of div $\mathbf{F} = 1$ is the cylinder about the z-axis of radius 1.

17. $\bar{x} = 0$ by symmetry; by Exercise 16, $\bar{y} = -\dfrac{1}{2A}\displaystyle\int_C y^2\,dx; C_1 : y = 0, -a \le x \le a, y^2\,dx = 0;$

$C_2 : x = a\cos\theta, y = a\sin\theta, 0 \le \theta \le \pi$, so

$$\bar{y} = -\frac{1}{2(\pi a^2/2)}\int_0^{\pi} a^2\sin^2\theta(-a\sin\theta)\,d\theta = \frac{4a}{3\pi}$$

18. $\bar{y} = \bar{x}$ by symmetry; by Exercise 16, $\bar{x} = \dfrac{1}{2A}\displaystyle\int_C x^2\,dy; C_1 : y = 0, 0 \le x \le a, x^2\,dy = 0;$

$C_2 : x = a\cos\theta, y = a\sin\theta, 0 \le \theta \le \pi/2; C_3 : x = 0, x^2\,dy = 0;$

$$\bar{x} = \frac{1}{2(\pi a^2/4)}\int_0^{\pi/2} a^2(\cos^2\theta)a\cos\theta\,d\theta = \frac{4a}{3\pi}$$

19. $\bar{y} = 0$ by symmetry; $\bar{x} = \dfrac{1}{2A}\displaystyle\int_C x^2\,dy; A = \alpha a^2; C_1 : x = t\cos\alpha, y = -t\sin\alpha, 0 \le t \le a;$

$C_2 : x = a\cos\theta, y = a\sin\theta, -\alpha \le \theta \le \alpha;$

$C_3 : x = t\cos\alpha, y = t\sin\alpha, 0 \le t \le a$ (reverse orientation);

$$\begin{aligned}2A\bar{x} &= -\int_0^a t^2\cos^2\alpha\sin\alpha\,dt + \int_{-\alpha}^{\alpha} a^3\cos^3\theta\,d\theta - \int_0^a t^2\cos^2\alpha\sin\alpha\,dt,\\ &= -\frac{2a^3}{3}\cos^2\alpha\sin\alpha + 2a^3\int_0^{\alpha}\cos^3\theta\,d\theta = -\frac{2a^3}{3}\cos^2\alpha\sin\alpha + 2a^3\left[\sin\alpha - \frac{1}{3}\sin^3\alpha\right]\\ &= \frac{4}{3}a^3\sin\alpha;\ \text{since} A = \alpha a^2, \bar{x} = \frac{2a}{3}\frac{\sin\alpha}{\alpha}\end{aligned}$$

20. $A = \displaystyle\int_0^a \left(b - \frac{b}{a^2}x^2\right)dx = \frac{2ab}{3}, C_1 : x = t, y = bt^2/a^2, 0 \le t \le a;$

$C_2 : x = a - t, y = b, 0 \le t \le a, x^2\,dy = 0; C_3 : x = 0, y = b - t, 0 \le t \le b, x^2\,dy = y^2\,dx = 0;$

$$2A\bar{x} = \int_0^a t^2(2bt/a^2)\,dt = \frac{a^2b}{2},\ \bar{x} = \frac{3a}{8};$$

$$2A\bar{y} = -\int_0^a (bt^2/a^2)^2\,dt + \int_0^a b^2\,dt = -\frac{ab^2}{5} + ab^2 = \frac{4ab^2}{5}, \bar{y} = \frac{3b}{5}$$

21. **(a)** $\displaystyle\int_C f(x)\,dx + g(y)\,dy = \iint_R \left(\frac{\partial}{\partial x}g(y) - \frac{\partial}{\partial y}f(x)\right)dA = 0$

(b) $W = \displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C f(x)\,dx + g(y)\,dy = 0$, so the work done by the vector field around any simple closed curve is zero. The field is conservative.

22. **(a)** Let $\mathbf{r} = d\cos\theta\mathbf{i} + d\sin\theta\mathbf{j} + z\mathbf{k}$ in cylindrical coordinates, so

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{d\theta}\frac{d\theta}{dt} = \omega(-d\sin\theta\mathbf{i} + d\cos\theta\mathbf{j}), \mathbf{v} = \frac{d\mathbf{r}}{dt} = \omega\mathbf{k}\times\mathbf{r} = \boldsymbol{\omega}\times\mathbf{r}.$$

(b) From Part (a), $\mathbf{v} = \omega d(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) = -\omega y\mathbf{i} + \omega x\mathbf{j}$

(c) From Part (b), curl $\mathbf{v} = 2\omega\mathbf{k} = 2\boldsymbol{\omega}$

(d) No; from Exercise 34 in Section 16.1, if ϕ were a potential function for $\mathbf{v}$, then curl $(\nabla\phi) = $ curl $\mathbf{v} = \mathbf{0}$, contradicting Part (c) above.

23. Yes; by imagining a normal vector sliding around the surface it is evident that the surface has two sides.

24. $D_{\mathbf{n}}\phi = \mathbf{n}\cdot\nabla\phi$, so $\displaystyle\iint_\sigma D_{\mathbf{n}}\phi\,dS = \iint_\sigma \mathbf{n}\cdot\nabla\phi\,dS = \iiint_G \nabla\cdot(\nabla\phi)\,dV$

$$= \iiint_G \left[\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right] dV$$

25. By Exercise 24, $\displaystyle\iint_\sigma D_{\mathbf{n}}f\,dS = -\iiint_G [f_{xx} + f_{yy} + f_{zz}]\,dV = -6\iiint_G dV = -6\text{vol}(G) = -8\pi$

26. **(a)** $f_y - g_x = e^{xy} + xye^{xy} - e^{xy} - xye^{xy} = 0$ so the vector field is conservative.

(b) $\phi_x = ye^{xy} - 1, \phi = e^{xy} - x + k(x), \phi_y = xe^{xy}$, let $k(x) = 0; \phi(x,y) = e^{xy} - x$

(c) $\displaystyle W = \int_C \mathbf{F}\cdot d\mathbf{r} = \phi(x(8\pi), y(8\pi)) - \phi(x(0), y(0)) = \phi(8\pi, 0) - \phi(0,0) = -8\pi$

27. **(a)** If $h(x)\mathbf{F}$ is conservative, then $\dfrac{\partial}{\partial y}(yh(x)) = \dfrac{\partial}{\partial x}(-2xh(x))$, or $h(x) = -2h(x) - 2xh'(x)$ which has the general solution $x^3h(x)^2 = C_1, h(x) = Cx^{-3/2}$, so $C\dfrac{y}{x^{3/2}}\mathbf{i} - C\dfrac{2}{x^{1/2}}\mathbf{j}$ is conservative, with potential function $\phi = -2Cy/\sqrt{x}$.

(b) If $g(y)\mathbf{F}(x,y)$ is conservative then $\dfrac{\partial}{\partial y}(yg(y)) = \dfrac{\partial}{\partial x}(-2xg(y))$, or $g(y) + yg'(y) = -2g(y)$, with general solution $g(y) = C/y^3$, so $\mathbf{F} = C\dfrac{1}{y^2}\mathbf{i} - C\dfrac{2x}{y^3}\mathbf{j}$ is conservative, with potential function Cx/y^2.

28. A computation of curl $\mathbf{F}$ shows that curl $\mathbf{F} = \mathbf{0}$ if and only if the three given equations hold. Moreover the equations hold if $\mathbf{F}$ is conservative, so it remains to show that $\mathbf{F}$ is conservative if curl $\mathbf{F} = \mathbf{0}$. Let C by any simple closed curve in the region. Since the region is simply connected, there is a piecewise smooth, oriented surface σ in the region with boundary C. By Stokes' Theorem, $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_\sigma (\text{curl } \mathbf{F})\cdot\mathbf{n}\,dS = \iint_\sigma 0\,dS = 0.$

By the 3-space analog of Theorem 16.3.2, $\mathbf{F}$ is conservative.

29. **(a)** conservative, $\phi(x,y,z) = xz^2 - e^{-y}$ **(b)** not conservative, $f_y \neq g_x$

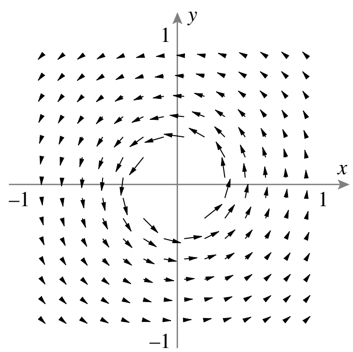
30. **(a)** conservative, $\phi(x,y,z) = -\cos x + yz$ **(b)** not conservative, $f_z \neq h_x$

CHAPTER 16 HORIZON MODULE

1. (a) If $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ denotes the position vector, then $\mathbf{F}_1 \cdot \mathbf{r} = 0$ by inspection, so the velocity field is tangent to the circle. The relationship $\mathbf{F}_1 \times \mathbf{r} = -\dfrac{k}{2\pi}\mathbf{k}$ indicates that $\mathbf{r}, \mathbf{F}_1, \mathbf{k}$ is a right-handed system, so the flow is counterclockwise. The polar form $\mathbf{F}_1 = -\dfrac{k}{2\pi r}(\sin\theta\mathbf{i} - \cos\theta\mathbf{j})$ shows that the speed is the constant $\dfrac{k}{2\pi r}$ on a circle of radius r; and it also shows that the speed is proportional to $\dfrac{1}{r}$ with constant of proportionality $k/(2\pi)$.

(b) Since $\|\mathbf{F}_1\| = \dfrac{k}{2\pi r}$, when $r = 1$ we get $k = 2\pi\|\mathbf{F}_1\|$

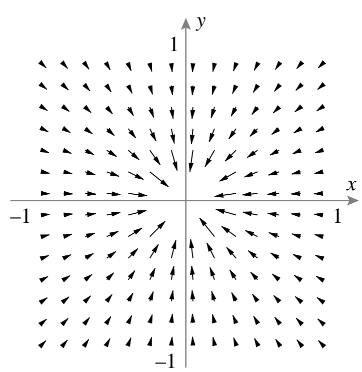
2.



3. (a) $\mathbf{F}_2 = -\dfrac{q}{2\pi\|\mathbf{r}\|^2}\mathbf{r}$ so $\mathbf{F}_2$ is directed toward the origin, and $\|\mathbf{F}_2\| = \dfrac{q}{2\pi r}$ is constant for constant r, and the speed is inversely proportional to the distance from the origin (constant of proportionality $\dfrac{q}{2\pi}$). Since the velocity vector is directed toward the origin, the fluid flows towards the origin, which must therefore be a sink.

(b) From Part (a) when $r = 1$, $q = 2\pi\|\mathbf{F}_2\|$.

4.



5. (b) The magnitudes of the field vectors increase, and their directions become more tangent to circles about the origin.

(c) The magnitudes of the field vectors increase, and their directions tend more towards the origin.

6. **(a)** The inward component is $\mathbf{F}_2$, so at $r = 20, 15 = \|\mathbf{F}_2\| = \dfrac{q}{2\pi(20)}$, so $q = 600\pi$; the tangential component is $\mathbf{F}_1$, so at $r = 20, 45 = \|\mathbf{F}_1\| = \dfrac{k}{2\pi(20)}$, so $k = 1800\pi$.

(b) $\mathbf{F} = -\dfrac{1}{x^2+y^2}[(300x + 900y)\mathbf{i} + (300y - 900x)\mathbf{j}]$

(c) $\|\mathbf{F}\| = \dfrac{300\sqrt{10}}{r} \le 5$ km/hr if $r \ge 60\sqrt{10} \approx 189.7$ km.

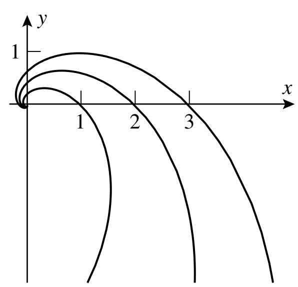
7. $\mathbf{F} = -\dfrac{1}{2\pi r}[(q\cos\theta + k\sin\theta)\mathbf{i} + (q\sin\theta - k\cos\theta)\mathbf{j}] = -\dfrac{q}{2\pi r}\mathbf{u}_r + \dfrac{k}{2\pi r}\mathbf{u}_\theta = -\dfrac{1}{2\pi r}(q\mathbf{u}_r - k\mathbf{u}_\theta)$

8. $\mathbf{F} \cdot \nabla\psi = -\dfrac{1}{2\pi r}(q\mathbf{u}_r - k\mathbf{u}_\theta) \cdot \left(\dfrac{k}{r}\mathbf{u}_r + \dfrac{1}{r}q\mathbf{u}_\theta\right) = -\dfrac{1}{2\pi r}\left(q\dfrac{k}{r} - k\dfrac{1}{r}q\right) = 0$, since $\mathbf{u}_r$ and $\mathbf{u}_\theta$ are orthogonal unit vectors.

9. From the hypotheses of Exercise 8, $\psi = k\ln r + G(\theta)$, $\dfrac{\partial}{\partial\theta}\psi = G'(\theta) = q$;
let $G = q\theta, \psi = k\ln r + q\theta$

10. The streamline $\psi = c$ becomes $k\ln r + q\theta = c, \ln r = -q\theta/k + c/k$,
$r = e^{-q\theta/k}e^{c/k} = \kappa e^{-q\theta/k}$, where $\kappa > 0$.

11.



12. $q = 600\pi, k = 1800\pi, r = \kappa e^{-\theta/3}$; at $r = 20, \theta = \pi/4, \kappa = re^{\theta/3} = 20e^{\pi/12} \approx 25.985$; the desired streamline has the polar equation $r = 25.985e^{-\theta/3}$.