CHAPTER 15
Multiple Integrals

EXERCISE SET 15.1

1. $\int_0^1 \int_0^2 (x+3)dy\,dx = \int_0^1 (2x+6)dx = 7$

2. $\int_1^3 \int_{-1}^1 (2x-4y)dy\,dx = \int_1^3 4x\,dx = 16$

3. $\int_2^4 \int_0^1 x^2 y\,dx\,dy = \int_2^4 \frac{1}{3}y\,dy = 2$

4. $\int_{-2}^0 \int_{-1}^2 (x^2+y^2)dx\,dy = \int_{-2}^0 (3+3y^2)dy = 14$

5. $\int_0^{\ln 3} \int_0^{\ln 2} e^{x+y} dy\,dx = \int_0^{\ln 3} e^x dx = 2$

6. $\int_0^2 \int_0^1 y \sin x\,dy\,dx = \int_0^2 \frac{1}{2}\sin x\,dx = (1-\cos 2)/2$

7. $\int_{-1}^0 \int_2^5 dx\,dy = \int_{-1}^0 3\,dy = 3$

8. $\int_4^6 \int_{-3}^7 dy\,dx = \int_4^6 10dx = 20$

9. $\int_0^1 \int_0^1 \frac{x}{(xy+1)^2} dy\,dx = \int_0^1 \left(1 - \frac{1}{x+1}\right) dx = 1 - \ln 2$

10. $\int_{\pi/2}^{\pi} \int_1^2 x \cos xy\,dy\,dx = \int_{\pi/2}^{\pi} (\sin 2x - \sin x)dx = -2$

11. $\int_0^{\ln 2} \int_0^1 xy\,e^{y^2 x} dy\,dx = \int_0^{\ln 2} \frac{1}{2}(e^x - 1)dx = (1 - \ln 2)/2$

12. $\int_3^4 \int_1^2 \frac{1}{(x+y)^2} dy\,dx = \int_3^4 \left(\frac{1}{x+1} - \frac{1}{x+2}\right) dx = \ln(25/24)$

13. $\int_{-1}^1 \int_{-2}^2 4xy^3 dy\,dx = \int_{-1}^1 0\,dx = 0$

14. $\int_0^1 \int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}} dy\,dx = \int_0^1 [x(x^2+2)^{1/2} - x(x^2+1)^{1/2}]dx = (3\sqrt{3} - 4\sqrt{2} + 1)/3$

15. $\int_0^1 \int_2^3 x\sqrt{1-x^2}\,dy\,dx = \int_0^1 x(1-x^2)^{1/2} dx = 1/3$

16. $\int_0^{\pi/2} \int_0^{\pi/3} (x \sin y - y \sin x)dy\,dx = \int_0^{\pi/2} \left(\frac{x}{2} - \frac{\pi^2}{18}\sin x\right) dx = \pi^2/144$

17. **(a)** $x_k^* = k/2 - 1/4, k = 1,2,3,4; y_l^* = l/2 - 1/4, l = 1,2,3,4,$

$$\int\int_R f(x,y)\,dxdy \approx \sum_{k=1}^4 \sum_{l=1}^4 f(x_k^*, y_l^*)\Delta A_{kl} = \sum_{k=1}^4 \sum_{l=1}^4 [(k/2-1/4)^2 + (l/2-1/4)](1/2)^2 = 37/4$$

(b) $\int_0^2 \int_0^2 (x^2+y)\,dxdy = 28/3$; the error is $|37/4 - 28/3| = 1/12$

18. **(a)** $x_k^* = k/2 - 1/4, k = 1,2,3,4; y_l^* = l/2 - 1/4, l = 1,2,3,4,$

$$\int\int_R f(x,y)\,dxdy \approx \sum_{k=1}^{4}\sum_{l=1}^{4} f(x_k^*, y_l^*)\Delta A_{kl} = \sum_{k=1}^{4}\sum_{l=1}^{4}[(k/2-1/4) - 2(l/2-1/4)](1/2)^2 = -4$$

(b) $\int_0^2\int_0^2 (x-2y)\,dxdy = -4$; the error is zero

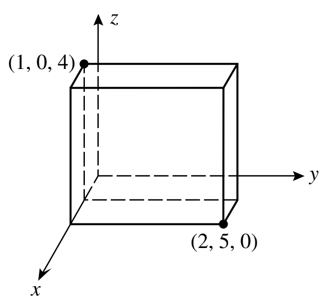
19. $V = \int_3^5\int_1^2 (2x+y)dy\,dx = \int_3^5 (2x+3/2)dx = 19$

20. $V = \int_1^3\int_0^2 (3x^3+3x^2y)dy\,dx = \int_1^3 (6x^3+6x^2)dx = 172$

21. $V = \int_0^2\int_0^3 x^2 dy\,dx = \int_0^2 3x^2 dx = 8$

22. $V = \int_0^3\int_0^4 5(1-x/3)dy\,dx = \int_0^3 5(4-4x/3)dx = 30$

23. **(a)**



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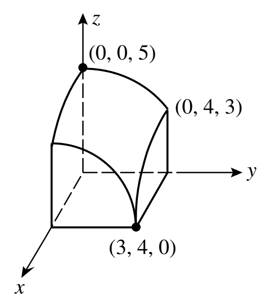
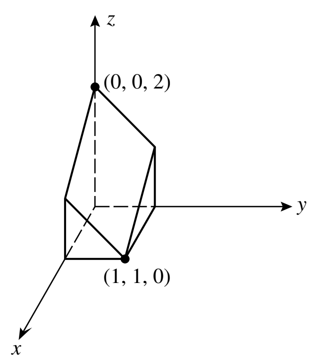
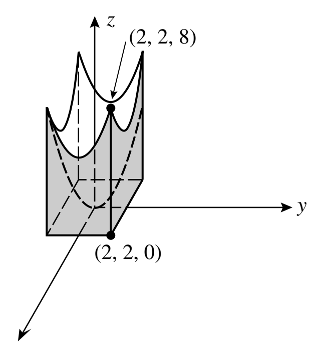


24. **(a)**



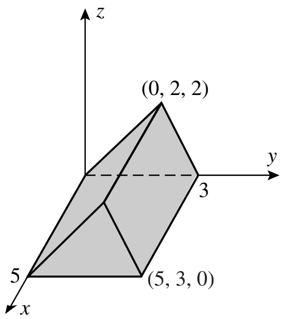
(b)



25. $\int_0^{1/2}\int_0^{\pi} x\cos(xy)\cos^2\pi x\,dy\,dx = \int_0^{1/2}\cos^2\pi x\sin(xy)\Big]_0^{\pi} dx$

$$= \int_0^{1/2}\cos^2\pi x\sin\pi x\,dx = -\frac{1}{3\pi}\cos^3\pi x\Big]_0^{1/2} = \frac{1}{3\pi}$$

26. (a)



(b) $V = \int_0^5 \int_0^2 y\,dy\,dx + \int_0^5 \int_2^3 (-2y+6)\,dy\,dx$

$= 10 + 5 = 15$

27. $f_{\text{ave}} = \dfrac{2}{\pi}\int_0^{\pi/2}\int_0^1 y\sin xy\,dx\,dy = \dfrac{2}{\pi}\int_0^{\pi/2}\left(-\cos xy\Big]_{x=0}^{x=1}\right)dy = \dfrac{2}{\pi}\int_0^{\pi/2}(1-\cos y)\,dy = 1 - \dfrac{2}{\pi}$

28. average $= \dfrac{1}{3}\int_0^3\int_0^1 x(x^2+y)^{1/2}dx\,dy = \int_0^3 \dfrac{1}{9}[(1+y)^{3/2} - y^{3/2}]dy = 2(31 - 9\sqrt{3})/45$

29. $T_{\text{ave}} = \dfrac{1}{2}\int_0^1\int_0^2 \left(10 - 8x^2 - 2y^2\right)dy\,dx = \dfrac{1}{2}\int_0^1\left(\dfrac{44}{3} - 16x^2\right)dx = \left(\dfrac{14}{3}\right)^{\circ}$

30. $f_{\text{ave}} = \dfrac{1}{A(R)}\int_a^b\int_c^d k\,dy\,dx = \dfrac{1}{A(R)}(b-a)(d-c)k = k$

31. 1.381737122

32. 2.230985141

33. $\displaystyle\iint_R f(x,y)dA = \int_a^b\left[\int_c^d g(x)h(y)dy\right]dx = \int_a^b g(x)\left[\int_c^d h(y)dy\right]dx$

$\displaystyle= \left[\int_a^b g(x)dx\right]\left[\int_c^d h(y)dy\right]$

34. The integral of $\tan x$ (an odd function) over the interval $[-1,1]$ is zero.

35. The first integral equals 1/2, the second equals $-1/2$. No, because the integrand is not continuous.

EXERCISE SET 15.2

1. $\displaystyle\int_0^1\int_{x^2}^x xy^2dy\,dx = \int_0^1 \frac{1}{3}(x^4 - x^7)dx = 1/40$

2. $\displaystyle\int_1^{3/2}\int_y^{3-y} y\,dx\,dy = \int_1^{3/2}(3y - 2y^2)dy = 7/24$

3. $\displaystyle\int_0^3\int_0^{\sqrt{9-y^2}} y\,dx\,dy = \int_0^3 y\sqrt{9-y^2}\,dy = 9$

4. $\displaystyle\int_{1/4}^1\int_{x^2}^x \sqrt{x/y}\,dy\,dx = \int_{1/4}^1\int_{x^2}^x x^{1/2}y^{-1/2}dy\,dx = \int_{1/4}^1 2(x - x^{3/2})dx = 13/80$

5. $\displaystyle\int_{\sqrt{\pi}}^{\sqrt{2\pi}}\int_0^{x^3}\sin(y/x)dy\,dx=\int_{\sqrt{\pi}}^{\sqrt{2\pi}}[-x\cos(x^2)+x]dx=\pi/2$

6. $\displaystyle\int_{-1}^{1}\int_{-x^2}^{x^2}(x^2-y)dy\,dx=\int_{-1}^{1}2x^4dx=4/5$

7. $\displaystyle\int_{\pi/2}^{\pi}\int_0^{x^2}\frac{1}{x}\cos(y/x)dy\,dx=\int_{\pi/2}^{\pi}\sin x\,dx=1$

8. $\displaystyle\int_0^1\int_0^x e^{x^2}dy\,dx=\int_0^1 xe^{x^2}dx=(e-1)/2$

9. $\displaystyle\int_0^1\int_0^x y\sqrt{x^2-y^2}\,dy\,dx=\int_0^1\frac{1}{3}x^3dx=1/12$

10. $\displaystyle\int_1^2\int_0^{y^2}e^{x/y^2}dx\,dy=\int_1^2(e-1)y^2dy=7(e-1)/3$

11. **(a)** $\displaystyle\int_0^2\int_0^{x^2}xy\,dy\,dx=\int_0^2\frac{1}{2}x^5dx=\frac{16}{3}$

(b) $\displaystyle\int_1^3\int_{-(y-5)/2}^{(y+7)/2}xy\,dx\,dy=\int_1^3(3y^2+3y)dy=38$

12. **(a)** $\displaystyle\int_0^1\int_{x^2}^{\sqrt{x}}(x+y)dy\,dx=\int_0^1(x^{3/2}+x/2-x^3-x^4/2)dx=3/10$

(b) $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}x\,dy\,dx+\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}y\,dy\,dx=\int_{-1}^{1}2x\sqrt{1-x^2}\,dx+0=0$

13. **(a)** $\displaystyle\int_4^8\int_{16/x}^{x}x^2dy\,dx=\int_4^8(x^3-16x)dx=576$

(b) $$\int_2^4\int_{16/y}^{8}x^2\,dxdy+\int_4^8\int_y^8x^2\,dx\,dy=\int_4^8\left[\frac{512}{3}-\frac{4096}{3y^3}\right]dy+\int_4^8\frac{512-y^3}{3}dy$$
$$=\frac{640}{3}+\frac{1088}{3}=576$$

14. **(a)** $\displaystyle\int_1^2\int_0^y xy^2dx\,dy=\int_1^2\frac{1}{2}y^4dy=31/10$

(b) $\displaystyle\int_0^1\int_1^2 xy^2\,dydx+\int_1^2\int_x^2 xy^2\,dydx=\int_0^1 7x/3\,dx+\int_1^2\frac{8x-x^4}{3}dx=7/6+29/15=31/10$

15. **(a)** $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(3x-2y)dy\,dx=\int_{-1}^{1}6x\sqrt{1-x^2}\,dx=0$

(b) $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}(3x-2y)\,dxdy=\int_{-1}^{1}-4y\sqrt{1-y^2}\,dy=0$

16. **(a)** $\displaystyle\int_0^5\int_{5-x}^{\sqrt{25-x^2}}y\,dy\,dx=\int_0^5(5x-x^2)dx=125/6$

(b) $\displaystyle\int_0^5\int_{5-y}^{\sqrt{25-y^2}}y\,dxdy=\int_0^5 y\left(\sqrt{25-y^2}-5+y\right)dy=125/6$

17. $\displaystyle\int_0^4\int_0^{\sqrt{y}}x(1+y^2)^{-1/2}dx\,dy=\int_0^4\frac{1}{2}y(1+y^2)^{-1/2}dy=(\sqrt{17}-1)/2$

18. $\int_0^{\pi}\int_0^{x} x\cos y\,dy\,dx = \int_0^{\pi} x\sin x\,dx = \pi$

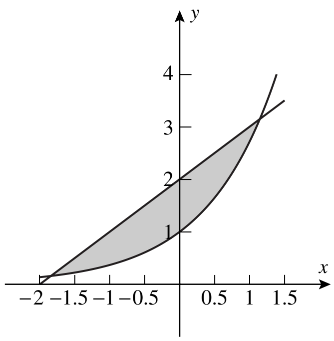
19. $\int_0^{2}\int_{y^2}^{6-y} xy\,dx\,dy = \int_0^{2}\frac{1}{2}(36y-12y^2+y^3-y^5)dy = 50/3$

20. $\int_0^{\pi/4}\int_{\sin y}^{1/\sqrt{2}} x\,dx\,dy = \int_0^{\pi/4}\frac{1}{4}\cos 2y\,dy = 1/8$

21. $\int_0^{1}\int_{x^3}^{x}(x-1)dy\ dx = \int_0^{1}(-x^4+x^3+x^2-x)dx = -7/60$

22. $\int_0^{1/\sqrt{2}}\int_x^{2x} x^2dy\,dx + \int_{1/\sqrt{2}}^{1}\int_x^{1/x} x^2dy\,dx = \int_0^{1/\sqrt{2}} x^3dx + \int_{1/\sqrt{2}}^{1}(x-x^3)dx = 1/8$

23. **(a)**

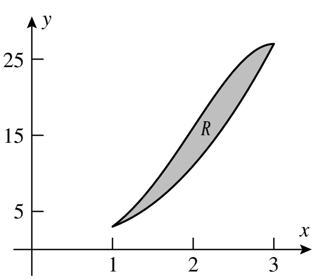


(b) $x = (-1.8414, 0.1586), (1.1462, 3.1462)$

(c) $\iint_R x\,dA \approx \int_{-1.8414}^{1.1462}\int_{e^x}^{x+2} x\,dydx = \int_{-1.8414}^{1.1462} x(x+2-e^x)\,dx \approx -0.4044$

(d) $\iint_R x\,dA \approx \int_{0.1586}^{3.1462}\int_{y-2}^{\ln y} x\,dxdy = \int_{0.1586}^{3.1462}\left[\frac{\ln^2 y}{2} - \frac{(y-2)^2}{2}\right] dy \approx -0.4044$

24. **(a)**



(b) $(1,3), (3,27)$

(c) $\int_1^{3}\int_{3-4x+4x^2}^{4x^3-x^4} x\,dy\,dx = \int_1^{3} x[(4x^3-x^4)-(3-4x+4x^2)]\,dx = \frac{224}{15}$

25. $A = \int_0^{\pi/4}\int_{\sin x}^{\cos x} dy\,dx = \int_0^{\pi/4}(\cos x - \sin x)dx = \sqrt{2}-1$

26. $A = \int_{-4}^{1}\int_{3y-4}^{-y^2} dx\,dy = \int_{-4}^{1}(-y^2-3y+4)dy = 125/6$

27. $A = \int_{-3}^{3}\int_{1-y^2/9}^{9-y^2} dx\,dy = \int_{-3}^{3} 8(1-y^2/9)dy = 32$

28. $A = \int_{0}^{1}\int_{\sinh x}^{\cosh x} dy\,dx = \int_{0}^{1}(\cosh x - \sinh x)dx = 1 - e^{-1}$

29. $\int_{0}^{4}\int_{0}^{6-3x/2}(3 - 3x/4 - y/2)\,dy\,dx = \int_{0}^{4}[(3-3x/4)(6-3x/2) - (6-3x/2)^2/4]\,dx = 12$

30. $\int_{0}^{2}\int_{0}^{\sqrt{4-x^2}}\sqrt{4-x^2}\,dy\,dx = \int_{0}^{2}(4-x^2)\,dx = 16/3$

31. $V = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}(3-x)dy\,dx = \int_{-3}^{3}(6\sqrt{9-x^2} - 2x\sqrt{9-x^2})dx = 27\pi$

32. $V = \int_{0}^{1}\int_{x^2}^{x}(x^2+3y^2)dy\,dx = \int_{0}^{1}(2x^3 - x^4 - x^6)dx = 11/70$

33. $V = \int_{0}^{3}\int_{0}^{2}(9x^2+y^2)dy\,dx = \int_{0}^{3}(18x^2 + 8/3)dx = 170$

34. $V = \int_{-1}^{1}\int_{y^2}^{1}(1-x)dx\,dy = \int_{-1}^{1}(1/2 - y^2 + y^4/2)dy = 8/15$

35. $V = \int_{-3/2}^{3/2}\int_{-\sqrt{9-4x^2}}^{\sqrt{9-4x^2}}(y+3)dy\,dx = \int_{-3/2}^{3/2} 6\sqrt{9-4x^2}\,dx = 27\pi/2$

36. $V = \int_{0}^{3}\int_{y^2/3}^{3}(9-x^2)dx\,dy = \int_{0}^{3}(18 - 3y^2 + y^6/81)dy = 216/7$

37. $V = 8\int_{0}^{5}\int_{0}^{\sqrt{25-x^2}}\sqrt{25-x^2}dy\,dx = 8\int_{0}^{5}(25-x^2)dx = 2000/3$

38. $V = 2\int_{0}^{2}\int_{0}^{\sqrt{1-(y-1)^2}}(x^2+y^2)dx\,dy = 2\int_{0}^{2}\left(\frac{1}{3}[1-(y-1)^2]^{3/2} + y^2[1-(y-1)^2]^{1/2}\right)dy,$

let $y - 1 = \sin\theta$ to get $V = 2\int_{-\pi/2}^{\pi/2}\left[\frac{1}{3}\cos^3\theta + (1+\sin\theta)^2\cos\theta\right]\cos\theta\,d\theta$ which eventually yields $V = 3\pi/2$

39. $V = 4\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}}(1-x^2-y^2)dy\,dx = \frac{8}{3}\int_{0}^{1}(1-x^2)^{3/2}dx = \pi/2$

40. $V = \int_{0}^{2}\int_{0}^{\sqrt{4-x^2}}(x^2+y^2)dy\,dx = \int_{0}^{2}\left[x^2\sqrt{4-x^2} + \frac{1}{3}(4-x^2)^{3/2}\right]dx = 2\pi$

41. $\int_{0}^{\sqrt{2}}\int_{y^2}^{2} f(x,y)dx\,dy$ **42.** $\int_{0}^{8}\int_{0}^{x/2} f(x,y)dy\,dx$ **43.** $\int_{1}^{e^2}\int_{\ln x}^{2} f(x,y)dy\,dx$

44. $\displaystyle\int_0^1 \int_{e^y}^{e} f(x,y)dx\,dy$ **45.** $\displaystyle\int_0^{\pi/2} \int_0^{\sin x} f(x,y)dy\,dx$ **46.** $\displaystyle\int_0^1 \int_{x^2}^{\sqrt{x}} f(x,y)dy\,dx$

47. $\displaystyle\int_0^4 \int_0^{y/4} e^{-y^2}dx\,dy = \int_0^4 \frac{1}{4}ye^{-y^2}dy = (1-e^{-16})/8$

48. $\displaystyle\int_0^1 \int_0^{2x} \cos(x^2)dy\,dx = \int_0^1 2x\cos(x^2)dx = \sin 1$

49. $\displaystyle\int_0^2 \int_0^{x^2} e^{x^3}dy\,dx = \int_0^2 x^2e^{x^3}dx = (e^8-1)/3$

50. $\displaystyle\int_0^{\ln 3} \int_{e^y}^{3} x\,dx\,dy = \frac{1}{2}\int_0^{\ln 3}(9-e^{2y})dy = \frac{1}{2}(9\ln 3-4)$

51. $\displaystyle\int_0^2 \int_0^{y^2} \sin(y^3)dx\,dy = \int_0^2 y^2\sin(y^3)dy = (1-\cos 8)/3$

52. $\displaystyle\int_0^1 \int_{e^x}^{e} x\,dy\,dx = \int_0^1 (ex - xe^x)dx = e/2-1$

53. **(a)** $\displaystyle\int_0^4 \int_{\sqrt{x}}^{2} \sin \pi y^3\,dy\,dx$; the inner integral is non-elementary.

$$\int_0^2 \int_0^{y^2} \sin\left(\pi y^3\right)\,dx\,dy = \int_0^2 y^2\sin\left(\pi y^3\right)\,dy = -\frac{1}{3\pi}\cos\left(\pi y^3\right)\Big]_0^2 = 0$$

(b) $\displaystyle\int_0^1 \int_{\sin^{-1}y}^{\pi/2} \sec^2(\cos x)dx\,dy$; the inner integral is non-elementary.

$$\int_0^{\pi/2} \int_0^{\sin x} \sec^2(\cos x)dy\,dx = \int_0^{\pi/2} \sec^2(\cos x)\sin x\,dx = \tan 1$$

54. $\displaystyle V = 4\int_0^2 \int_0^{\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx = 4\int_0^2 \left(x^2\sqrt{4-x^2}+\frac{1}{3}(4-x^2)^{3/2}\right)\,dx \qquad (x = 2\sin\theta)$

$$= \int_0^{\pi/2}\left(\frac{64}{3}+\frac{64}{3}\sin^2\theta-\frac{128}{3}\sin^4\theta\right)\,d\theta = \frac{64}{3}\frac{\pi}{2}+\frac{64}{3}\frac{\pi}{4}-\frac{128}{3}\frac{\pi}{2}\frac{1\cdot 3}{2\cdot 4} = 8\pi$$

55. The region is symmetric with respect to the y-axis, and the integrand is an odd function of x, hence the answer is zero.

56. This is the volume in the first octant under the surface $z = \sqrt{1-x^2-y^2}$, so 1/8 of the volume of the sphere of radius 1, thus $\frac{\pi}{6}$.

57. Area of triangle is 1/2, so $\displaystyle\bar{f} = 2\int_0^1 \int_x^1 \frac{1}{1+x^2}dy\,dx = 2\int_0^1\left[\frac{1}{1+x^2}-\frac{x}{1+x^2}\right]dx = \frac{\pi}{2}-\ln 2$

58. Area $\displaystyle = \int_0^2 (3x-x^2-x)\,dx = 4/3$, so

$$\bar{f} = \frac{3}{4}\int_0^2 \int_x^{3x-x^2} (x^2-xy)dy\,dx = \frac{3}{4}\int_0^2(-2x^3+2x^4-x^5/2)dx = -\frac{3}{4}\frac{8}{15} = -\frac{2}{5}$$

59. $T_{\text{ave}} = \dfrac{1}{A(R)} \displaystyle\iint_R (5xy + x^2)\, dA$. The diamond has corners $(\pm 2, 0), (0, \pm 4)$ and thus has area $A(R) = 4\dfrac{1}{2}2(4) = 16\text{m}^2$. Since $5xy$ is an odd function of x (as well as y), $\displaystyle\iint_R 5xy\, dA = 0$. Since x^2 is an even function of both x and y,

$$T_{\text{ave}} = \frac{4}{16} \iint_{\substack{R \\ x,y>0}} x^2\, dA = \frac{1}{4}\int_0^2 \int_0^{4-2x} x^2\, dydx = \frac{1}{4}\int_0^2 (4-2x)x^2\, dx = \frac{1}{4}\left(\frac{4}{3}x^3 - \frac{1}{2}x^4\right)\Big]_0^2 = \frac{2}{3}^{\circ}\text{C}$$

60. The area of the lens is $\pi R^2 = 4\pi$ and the average thickness T_{ave} is

$$T_{\text{ave}} = \frac{4}{4\pi}\int_0^2 \int_0^{\sqrt{4-x^2}} \left(1 - (x^2+y^2)/4\right)\, dydx = \frac{1}{\pi}\int_0^2 \frac{1}{6}(4-x^2)^{3/2} dx \qquad (x = 2\cos\theta)$$

$$= \frac{8}{3\pi}\int_0^{\pi} \sin^4\theta\, d\theta = \frac{8}{3\pi}\frac{1\cdot 3}{2\cdot 4}\frac{\pi}{2} = \frac{1}{2} \text{ in}$$

61. $y = \sin x$ and $y = x/2$ intersect at $x = 0$ and $x = a = 1.895494$, so

$$V = \int_0^a \int_{x/2}^{\sin x} \sqrt{1+x+y}\; dy\; dx = 0.676089$$

EXERCISE SET 15.3

1. $\displaystyle\int_0^{\pi/2}\int_0^{\sin\theta} r\cos\theta dr\, d\theta = \int_0^{\pi/2} \frac{1}{2}\sin^2\theta\cos\theta\, d\theta = 1/6$

2. $\displaystyle\int_0^{\pi}\int_0^{1+\cos\theta} r\, dr\, d\theta = \int_0^{\pi} \frac{1}{2}(1+\cos\theta)^2 d\theta = 3\pi/4$

3. $\displaystyle\int_0^{\pi/2}\int_0^{a\sin\theta} r^2 dr\, d\theta = \int_0^{\pi/2} \frac{a^3}{3}\sin^3\theta\, d\theta = \frac{2}{9}a^3$

4. $\displaystyle\int_0^{\pi/6}\int_0^{\cos 3\theta} r\, dr\, d\theta = \int_0^{\pi/6} \frac{1}{2}\cos^2 3\theta\, d\theta = \pi/24$

5. $\displaystyle\int_0^{\pi}\int_0^{1-\sin\theta} r^2\cos\theta\, dr\, d\theta = \int_0^{\pi} \frac{1}{3}(1-\sin\theta)^3\cos\theta\, d\theta = 0$

6. $\displaystyle\int_0^{\pi/2}\int_0^{\cos\theta} r^3 dr\, d\theta = \int_0^{\pi/2} \frac{1}{4}\cos^4\theta\, d\theta = 3\pi/64$

7. $\displaystyle A = \int_0^{2\pi}\int_0^{1-\cos\theta} r\, dr\, d\theta = \int_0^{2\pi} \frac{1}{2}(1-\cos\theta)^2 d\theta = 3\pi/2$

8. $\displaystyle A = 4\int_0^{\pi/2}\int_0^{\sin 2\theta} r\, dr\, d\theta = 2\int_0^{\pi/2} \sin^2 2\theta\, d\theta = \pi/2$

9. $\displaystyle A = \int_{\pi/4}^{\pi/2}\int_{\sin 2\theta}^{1} r\, dr\, d\theta = \int_{\pi/4}^{\pi/2} \frac{1}{2}(1-\sin^2 2\theta) d\theta = \pi/16$

10. $A = 2\int_0^{\pi/3}\int_{\sec\theta}^{2} r\,dr\,d\theta = \int_0^{\pi/3}(4-\sec^2\theta)d\theta = 4\pi/3 - \sqrt{3}$

11. $A = 2\int_{\pi/6}^{\pi/2}\int_2^{4\sin\theta} r\,dr\,d\theta = \int_{\pi/6}^{\pi/2}(16\sin^2\theta - 4)d\theta = 4\pi/3 + 2\sqrt{3}$

12. $A = 2\int_{\pi/2}^{\pi}\int_{1+\cos\theta}^{1} r\,dr\,d\theta = \int_{\pi/2}^{\pi}(-2\cos\theta - \cos^2\theta)d\theta = 2 - \pi/4$

13. $V = 8\int_0^{\pi/2}\int_1^3 r\sqrt{9-r^2}\,dr\,d\theta = \frac{128}{3}\sqrt{2}\int_0^{\pi/2} d\theta = \frac{64}{3}\sqrt{2}\pi$

14. $V = 2\int_0^{\pi/2}\int_0^{2\sin\theta} r^2 dr\,d\theta = \frac{16}{3}\int_0^{\pi/2}\sin^3\theta\,d\theta = 32/9$

15. $V = 2\int_0^{\pi/2}\int_0^{\cos\theta}(1-r^2)r\,dr\,d\theta = \frac{1}{2}\int_0^{\pi/2}(2\cos^2\theta - \cos^4\theta)d\theta = 5\pi/32$

16. $V = 4\int_0^{\pi/2}\int_1^3 dr\,d\theta = 8\int_0^{\pi/2} d\theta = 4\pi$

17. $V = \int_0^{\pi/2}\int_0^{3\sin\theta} r^2\sin\theta\,dr\,d\theta = 9\int_0^{\pi/2}\sin^4\theta\,d\theta = 27\pi/16$

18. $V = 4\int_0^{\pi/2}\int_{2\cos\theta}^{2} r\sqrt{4-r^2}\,dr\,d\theta + \int_{\pi/2}^{\pi}\int_0^2 r\sqrt{4-r^2}\,dr\,d\theta$

$= \frac{32}{3}\int_0^{\pi/2}\sin^3\theta\,d\theta + \frac{32}{3}\int_{\pi/2}^{\pi} d\theta = \frac{64}{9} + \frac{16}{3}\pi$

19. $\int_0^{2\pi}\int_0^1 e^{-r^2} r\,dr\,d\theta = \frac{1}{2}(1-e^{-1})\int_0^{2\pi} d\theta = (1-e^{-1})\pi$

20. $\int_0^{\pi/2}\int_0^3 r\sqrt{9-r^2}\,dr\,d\theta = 9\int_0^{\pi/2} d\theta = 9\pi/2$

21. $\int_0^{\pi/4}\int_0^2 \frac{1}{1+r^2} r\,dr\,d\theta = \frac{1}{2}\ln 5\int_0^{\pi/4} d\theta = \frac{\pi}{8}\ln 5$

22. $\int_{\pi/4}^{\pi/2}\int_0^{2\cos\theta} 2r^2\sin\theta\,dr\,d\theta = \frac{16}{3}\int_{\pi/4}^{\pi/2}\cos^3\theta\sin\theta\,d\theta = 1/3$

23. $\int_0^{\pi/2}\int_0^1 r^3 dr\,d\theta = \frac{1}{4}\int_0^{\pi/2} d\theta = \pi/8$

24. $\int_0^{2\pi}\int_0^2 e^{-r^2} r\,dr\,d\theta = \frac{1}{2}(1-e^{-4})\int_0^{2\pi} d\theta = (1-e^{-4})\pi$

25. $\int_0^{\pi/2}\int_0^{2\cos\theta} r^2 dr\,d\theta = \frac{8}{3}\int_0^{\pi/2}\cos^3\theta\,d\theta = 16/9$

26. $\int_0^{\pi/2}\int_0^1 \cos(r^2)r\,dr\,d\theta = \frac{1}{2}\sin 1\int_0^{\pi/2} d\theta = \frac{\pi}{4}\sin 1$

27. $\int_0^{\pi/2}\int_0^a \frac{r}{(1+r^2)^{3/2}}dr\,d\theta = \frac{\pi}{2}\left(1-1/\sqrt{1+a^2}\right)$

28. $\int_0^{\pi/4}\int_0^{\sec\theta\tan\theta} r^2 dr\,d\theta = \frac{1}{3}\int_0^{\pi/4}\sec^3\theta\tan^3\theta\,d\theta = 2(\sqrt{2}+1)/45$

29. $\int_0^{\pi/4}\int_0^2 \frac{r}{\sqrt{1+r^2}}dr\,d\theta = \frac{\pi}{4}(\sqrt{5}-1)$

30. $\int_{\tan^{-1}(3/4)}^{\pi/2}\int_{3\csc\theta}^5 r\,dr\,d\theta = \frac{1}{2}\int_{\tan^{-1}(3/4)}^{\pi/2}(25-9\csc^2\theta)d\theta$

$= \frac{25}{2}\left[\frac{\pi}{2}-\tan^{-1}(3/4)\right]-6 = \frac{25}{2}\tan^{-1}(4/3)-6$

31. $V = \int_0^{2\pi}\int_0^a hr\,dr\,d\theta = \int_0^{2\pi} h\frac{a^2}{2}\,d\theta = \pi a^2 h$

32. **(a)** $V = 8\int_0^{\pi/2}\int_0^a \frac{c}{a}(a^2-r^2)^{1/2}\,r\,dr\,d\theta = -\frac{4c}{3a}\pi(a^2-r^2)^{3/2}\Big]_0^a = \frac{4}{3}\pi a^2 c$

(b) $V \approx \frac{4}{3}\pi(6378.1370)^2 6356.5231 \approx 1{,}083{,}168{,}200{,}000\text{ km}^3$

33. $V = 2\int_0^{\pi/2}\int_0^{a\sin\theta}\frac{c}{a}(a^2-r^2)^{1/2}r\,dr\,d\theta = \frac{2}{3}a^2c\int_0^{\pi/2}(1-\cos^3\theta)d\theta = (3\pi-4)a^2c/9$

34. $A = 4\int_0^{\pi/4}\int_0^{a\sqrt{2\cos 2\theta}} r\,dr\,d\theta = 4a^2\int_0^{\pi/4}\cos 2\theta\,d\theta = 2a^2$

35. $A = \int_{\pi/6}^{\pi/4}\int_{\sqrt{8\cos 2\theta}}^{4\sin\theta} r\,dr\,d\theta + \int_{\pi/4}^{\pi/2}\int_0^{4\sin\theta} r\,dr\,d\theta$

$= \int_{\pi/6}^{\pi/4}(8\sin^2\theta - 4\cos 2\theta)d\theta + \int_{\pi/4}^{\pi/2} 8\sin^2\theta\,d\theta = 4\pi/3 + 2\sqrt{3} - 2$

36. $A = \int_0^{\phi}\int_0^{2a\sin\theta} r\,dr\,d\theta = 2a^2\int_0^{\phi}\sin^2\theta\,d\theta = a^2\phi - \frac{1}{2}a^2\sin 2\phi$

37. **(a)** $I^2 = \left[\int_0^{+\infty} e^{-x^2}dx\right]\left[\int_0^{+\infty} e^{-y^2}dy\right] = \int_0^{+\infty}\left[\int_0^{+\infty} e^{-x^2}dx\right]e^{-y^2}dy$

$= \int_0^{+\infty}\int_0^{+\infty} e^{-x^2}e^{-y^2}dx\,dy = \int_0^{+\infty}\int_0^{+\infty} e^{-(x^2+y^2)}dx\,dy$

(b) $I^2 = \int_0^{\pi/2}\int_0^{+\infty} e^{-r^2}r\,dr\,d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta = \pi/4$ **(c)** $I = \sqrt{\pi}/2$

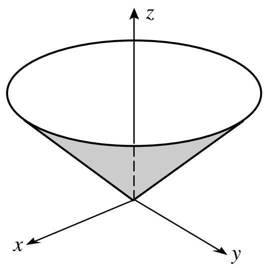
38. **(a)** 1.173108605 **(b)** $\int_0^{\pi}\int_0^1 re^{-r^4}\,dr\,d\theta = \pi\int_0^1 re^{-r^4}\,dr \approx 1.173108605$

39. $V = \int_0^{2\pi}\int_0^R D(r)r\,dr\,d\theta = \int_0^{2\pi}\int_0^R ke^{-r}r\,dr\,d\theta = -2\pi k(1+r)e^{-r}\Big]_0^R = 2\pi k[1-(R+1)e^{-R}]$

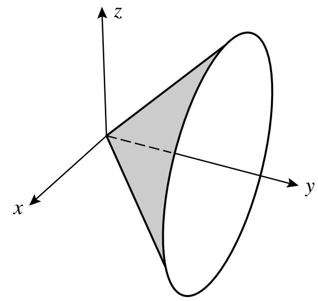
40. $\int_{\tan^{-1}(1/3)}^{\tan^{-1}(2)}\int_0^2 r^3\cos^2\theta\,dr\,d\theta = 4\int_{\tan^{-1}(1/3)}^{\tan^{-1}(2)}\cos^2\theta\,d\theta = \frac{1}{5} + 2[\tan^{-1}(2) - \tan^{-1}(1/3)] = \frac{1}{5} + \frac{\pi}{2}$

EXERCISE SET 15.4

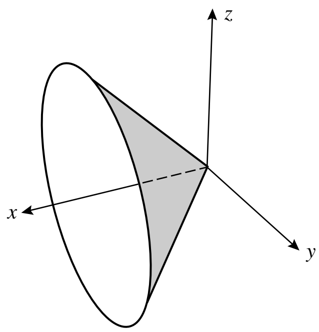
1. (a)



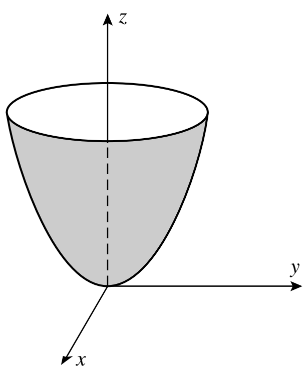
(b)



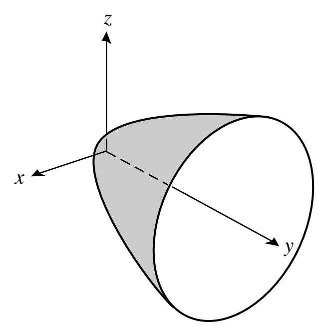
(c)



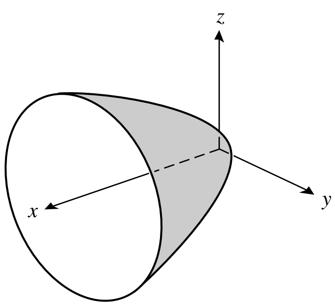
2. (a)



(b)



(c)



3. **(a)** $x = u, y = v, z = \frac{5}{2} + \frac{3}{2}u - 2v$ **(b)** $x = u, y = v, z = u^2$

4. **(a)** $x = u, y = v, z = \frac{v}{1+u^2}$ **(b)** $x = u, y = v, z = \frac{1}{3}v^2 - \frac{5}{3}$

5. **(a)** $x = 5\cos u, y = 5\sin u, z = v; 0 \le u \le 2\pi, 0 \le v \le 1$
(b) $x = 2\cos u, y = v, z = 2\sin u; 0 \le u \le 2\pi, 1 \le v \le 3$

6. **(a)** $x = u, y = 1 - u, z = v; -1 \le v \le 1$ **(b)** $x = u, y = 5 + 2v, z = v; 0 \le u \le 3$

7. $x = u, y = \sin u \cos v, z = \sin u \sin v$

8. $x = u, y = e^u \cos v, z = e^u \sin v$

9. $x = r\cos\theta, y = r\sin\theta, z = \frac{1}{1+r^2}$

10. $x = r\cos\theta, y = r\sin\theta, z = e^{-r^2}$

11. $x = r\cos\theta, y = r\sin\theta, z = 2r^2\cos\theta\sin\theta$

12. $x = r\cos\theta, y = r\sin\theta, z = r^2(\cos^2\theta - \sin^2\theta)$

13. $x = r\cos\theta, y = r\sin\theta, z = \sqrt{9 - r^2}; r \le \sqrt{5}$

14. $x = r\cos\theta, y = r\sin\theta, z = r; r \le 3$

15. $x = \frac{1}{2}\rho\cos\theta, y = \frac{1}{2}\rho\sin\theta, z = \frac{\sqrt{3}}{2}\rho$

16. $x = 3\cos\theta, y = 3\sin\theta, z = 3\cot\phi$

17. $z = x - 2y$; a plane

18. $y = x^2 + z^2, 0 \le y \le 4$; part of a circular paraboloid

19. $(x/3)^2 + (y/2)^2 = 1; 2 \le z \le 4$; part of an elliptic cylinder

20. $z = x^2 + y^2; 0 \le z \le 4$; part of a circular paraboloid

21. $(x/3)^2 + (y/4)^2 = z^2; 0 \le z \le 1$; part of an elliptic cone

22. $x^2 + (y/2)^2 + (z/3)^2 = 1$; an ellipsoid

23. **(a)** $x = r\cos\theta, y = r\sin\theta, z = r, 0 \le r \le 2; x = u, y = v, z = \sqrt{u^2 + v^2}; 0 \le u^2 + v^2 \le 4$

24. **(a)** I: $x = r\cos\theta, y = r\sin\theta, z = r^2, 0 \le r \le \sqrt{2}$; II: $x = u, y = v, z = u^2 + v^2; u^2 + v^2 \le 2$

25. **(a)** $0 \le u \le 3, 0 \le v \le \pi$ **(b)** $0 \le u \le 4, -\pi/2 \le v \le \pi/2$

26. **(a)** $0 \le u \le 6, -\pi \le v \le 0$ **(b)** $0 \le u \le 5, \pi/2 \le v \le 3\pi/2$

27. **(a)** $0 \le \phi \le \pi/2, 0 \le \theta \le 2\pi$ **(b)** $0 \le \phi \le \pi, 0 \le \theta \le \pi$

28. **(a)** $\pi/2 \le \phi \le \pi, 0 \le \theta \le 2\pi$ **(b)** $0 \le \theta \le \pi/2, 0 \le \phi \le \pi/2$

29. $u = 1, v = 2, \mathbf{r}_u \times \mathbf{r}_v = -2\mathbf{i} - 4\mathbf{j} + \mathbf{k}; 2x + 4y - z = 5$

30. $u = 1, v = 2, \mathbf{r}_u \times \mathbf{r}_v = -4\mathbf{i} - 2\mathbf{j} + 8\mathbf{k}; 2x + y - 4z = -6$

31. $u = 0, v = 1, \mathbf{r}_u \times \mathbf{r}_v = 6\mathbf{k}; z = 0$

32. $\mathbf{r}_u \times \mathbf{r}_v = 2\mathbf{i} - \mathbf{j} - 3\mathbf{k}; 2x - y - 3z = -4$

33. $\mathbf{r}_u \times \mathbf{r}_v = (\sqrt{2}/2)\mathbf{i} - (\sqrt{2}/2)\mathbf{j} + (1/2)\mathbf{k}; x - y + \dfrac{\sqrt{2}}{2}z = \dfrac{\pi\sqrt{2}}{8}$

34. $\mathbf{r}_u \times \mathbf{r}_v = 2\mathbf{i} - \ln 2\mathbf{k}; 2x - (\ln 2)z = 0$

35. $z = \sqrt{9-y^2},\ z_x = 0,\ z_y = -y/\sqrt{9-y^2},\ z_x^2 + z_y^2 + 1 = 9/(9-y^2),$

$$S = \int_0^2 \int_{-3}^3 \frac{3}{\sqrt{9-y^2}}\, dy\, dx = \int_0^2 3\pi\, dx = 6\pi$$

36. $z = 8 - 2x - 2y,\ z_x^2 + z_y^2 + 1 = 4 + 4 + 1 = 9,\ S = \displaystyle\int_0^4 \int_0^{4-x} 3\, dy\, dx = \int_0^4 3(4-x)dx = 24$

37. $z^2 = 4x^2 + 4y^2,\ 2zz_x = 8x$ so $z_x = 4x/z$, similarly $z_y = 4y/z$ thus

$$z_x^2 + z_y^2 + 1 = (16x^2 + 16y^2)/z^2 + 1 = 5,\ S = \int_0^1 \int_{x^2}^x \sqrt{5}\, dy\, dx = \sqrt{5}\int_0^1 (x - x^2)dx = \sqrt{5}/6$$

38. $z^2 = x^2 + y^2,\ z_x = x/z,\ z_y = y/z,\ z_x^2 + z_y^2 + 1 = (z^2 + y^2)/z^2 + 1 = 2,$

$$S = \iint_R \sqrt{2}\, dA = 2\int_0^{\pi/2} \int_0^{2\cos\theta} \sqrt{2}\, r\, dr\, d\theta = 4\sqrt{2}\int_0^{\pi/2} \cos^2\theta\, d\theta = \sqrt{2}\pi$$

39. $z_x = -2x,\ z_y = -2y,\ z_x^2 + z_y^2 + 1 = 4x^2 + 4y^2 + 1,$

$$S = \iint_R \sqrt{4x^2 + 4y^2 + 1}\, dA = \int_0^{2\pi} \int_0^1 r\sqrt{4r^2 + 1}\, dr\, d\theta$$

$$= \frac{1}{12}(5\sqrt{5} - 1)\int_0^{2\pi} d\theta = (5\sqrt{5} - 1)\pi/6$$

40. $z_x = 2,\ z_y = 2y,\ z_x^2 + z_y^2 + 1 = 5 + 4y^2,$

$$S = \int_0^1 \int_0^y \sqrt{5 + 4y^2}\, dx\, dy = \int_0^1 y\sqrt{5 + 4y^2}\, dy = (27 - 5\sqrt{5})/12$$

41. $\partial\mathbf{r}/\partial u = \cos v\mathbf{i} + \sin v\mathbf{j} + 2u\mathbf{k},\ \partial\mathbf{r}/\partial v = -u\sin v\mathbf{i} + u\cos v\mathbf{j},$

$$\|\partial\mathbf{r}/\partial u \times \partial\mathbf{r}/\partial v\| = u\sqrt{4u^2 + 1};\ S = \int_0^{2\pi} \int_1^2 u\sqrt{4u^2 + 1}\, du\ dv = (17\sqrt{17} - 5\sqrt{5})\pi/6$$

42. $\partial\mathbf{r}/\partial u = \cos v\mathbf{i} + \sin v\mathbf{j} + \mathbf{k},\ \partial\mathbf{r}/\partial v = -u\sin v\mathbf{i} + u\cos v\mathbf{j},$

$$\|\partial\mathbf{r}/\partial u \times \partial\mathbf{r}/\partial v\| = \sqrt{2}u;\ S = \int_0^{\pi/2} \int_0^{2v} \sqrt{2}\ u\ du\ dv = \frac{\sqrt{2}}{12}\pi^3$$

43. $z_x = y,\ z_y = x,\ z_x^2 + z_y^2 + 1 = x^2 + y^2 + 1,$

$$S = \iint_R \sqrt{x^2 + y^2 + 1}\, dA = \int_0^{\pi/6} \int_0^3 r\sqrt{r^2 + 1}\, dr\, d\theta = \frac{1}{3}(10\sqrt{10} - 1)\int_0^{\pi/6} d\theta = (10\sqrt{10} - 1)\pi/18$$

44. $z_x = x,\ z_y = y,\ z_x^2 + z_y^2 + 1 = x^2 + y^2 + 1$,

$$S = \iint_R \sqrt{x^2+y^2+1}\,dA = \int_0^{2\pi}\int_0^{\sqrt{8}} r\sqrt{r^2+1}\,dr\,d\theta = \frac{26}{3}\int_0^{2\pi} d\theta = 52\pi/3$$

45. On the sphere, $z_x = -x/z$ and $z_y = -y/z$ so $z_x^2 + z_y^2 + 1 = (x^2+y^2+z^2)/z^2 = 16/(16-x^2-y^2)$; the planes $z = 1$ and $z = 2$ intersect the sphere along the circles $x^2 + y^2 = 15$ and $x^2 + y^2 = 12$;

$$S = \iint_R \frac{4}{\sqrt{16-x^2-y^2}}dA = \int_0^{2\pi}\int_{\sqrt{12}}^{\sqrt{15}} \frac{4r}{\sqrt{16-r^2}}\,dr\,d\theta = 4\int_0^{2\pi} d\theta = 8\pi$$

46. On the sphere, $z_x = -x/z$ and $z_y = -y/z$ so $z_x^2 + z_y^2 + 1 = (x^2+y^2+z^2)/z^2 = 8/(8-x^2-y^2)$; the cone cuts the sphere in the circle $x^2 + y^2 = 4$;

$$S = \int_0^{2\pi}\int_0^2 \frac{2\sqrt{2}r}{\sqrt{8-r^2}}dr\,d\theta = (8-4\sqrt{2})\int_0^{2\pi} d\theta = 8(2-\sqrt{2})\pi$$

47. $\mathbf{r}(u,v) = a\cos u\sin v\mathbf{i} + a\sin u\sin v\mathbf{j} + a\cos v\mathbf{k},\ \|\mathbf{r}_u \times \mathbf{r}_v\| = a^2\sin v$,

$$S = \int_0^{\pi}\int_0^{2\pi} a^2\sin v\,du\,dv = 2\pi a^2\int_0^{\pi}\sin v\,dv = 4\pi a^2$$

48. $\mathbf{r} = r\cos u\mathbf{i} + r\sin u\mathbf{j} + v\mathbf{k},\ \|\mathbf{r}_u \times \mathbf{r}_v\| = r;\ S = \int_0^h\int_0^{2\pi} r\,du\,dv = 2\pi rh$

49. $z_x = \dfrac{h}{a}\dfrac{x}{\sqrt{x^2+y^2}},\ z_y = \dfrac{h}{a}\dfrac{y}{\sqrt{x^2+y^2}},\ z_x^2 + z_y^2 + 1 = \dfrac{h^2x^2+h^2y^2}{a^2(x^2+y^2)} + 1 = (a^2+h^2)/a^2$,

$$S = \int_0^{2\pi}\int_0^a \frac{\sqrt{a^2+h^2}}{a}r\,dr\,d\theta = \frac{1}{2}a\sqrt{a^2+h^2}\int_0^{2\pi} d\theta = \pi a\sqrt{a^2+h^2}$$

50. Revolving a point $(a_0, 0, b_0)$ of the xz-plane around the z-axis generates a circle, an equation of which is $\mathbf{r} = a_0\cos u\mathbf{i} + a_0\sin u\mathbf{j} + b_0\mathbf{k}, 0 \le u \le 2\pi$. A point on the circle $(x-a)^2 + z^2 = b^2$ which generates the torus can be written $\mathbf{r} = (a + b\cos v)\mathbf{i} + b\sin v\mathbf{k}, 0 \le v \le 2\pi$. Set $a_0 = a + b\cos v$ and $b_0 = a + b\sin v$ and use the first result: any point on the torus can thus be written in the form $\mathbf{r} = (a + b\cos v)\cos u\mathbf{i} + (a + b\cos v)\sin u\mathbf{j} + b\sin v\mathbf{k}$, which yields the result.

51. $\partial\mathbf{r}/\partial u = -(a + b\cos v)\sin u\mathbf{i} + (a + b\cos v)\cos u\mathbf{j}$,

$\partial\mathbf{r}/\partial v = -b\sin v\cos u\mathbf{i} - b\sin v\sin u\mathbf{j} + b\cos v\mathbf{k},\ \|\partial\mathbf{r}/\partial u \times \partial\mathbf{r}/\partial v\| = b(a + b\cos v)$;

$$S = \int_0^{2\pi}\int_0^{2\pi} b(a + b\cos v)du\ dv = 4\pi^2 ab$$

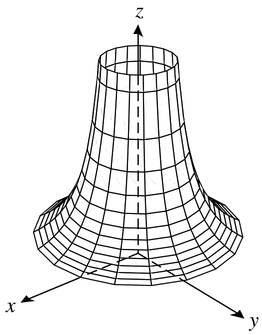
52. $\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{u^2+1}; S = \int_0^{4\pi}\int_0^5 \sqrt{u^2+1}\,du\,dv = 4\pi\int_0^5 \sqrt{u^2+1}\,du = 174.7199011$

53. $z = -1$ when $v \approx 0.27955, z = 1$ when $v \approx 2.86204, \|\mathbf{r}_u \times \mathbf{r}_v\| = |\cos v|$;

$$S = \int_0^{2\pi}\int_{0.27955}^{2.86204} |\cos v|\,dv\,du \approx 9.099$$

54. **(a)** $x = v\cos u, y = v\sin u, z = f(v)$, for example **(b)** $x = v\cos u, y = v\sin u, z = 1/v^2$

(c)



55. $(x/a)^2 + (y/b)^2 + (z/c)^2 = \cos^2 v(\cos^2 u + \sin^2 u) + \sin^2 v = 1$, ellipsoid

56. $(x/a)^2 + (y/b)^2 - (z/c)^2 = \cos^2 u \cosh^2 v + \sin^2 u \cosh^2 v - \sinh^2 v = 1$, hyperboloid of one sheet

57. $(x/a)^2 + (y/b)^2 - (z/c)^2 = \sinh^2 v + \cosh^2 v(\sinh^2 u - \cosh^2 u) = -1$, hyperboloid of two sheets

EXERCISE SET 15.5

1. $\displaystyle\int_{-1}^{1}\int_0^2\int_0^1 (x^2+y^2+z^2)dx\,dy\,dz = \int_{-1}^{1}\int_0^2 (1/3+y^2+z^2)dy\,dz = \int_{-1}^{1}(10/3+2z^2)dz = 8$

2. $\displaystyle\int_{1/3}^{1/2}\int_0^{\pi}\int_0^1 zx\sin xy\,dz\,dy\,dx = \int_{1/3}^{1/2}\int_0^{\pi}\frac{1}{2}x\sin xy\,dy\,dx = \int_{1/3}^{1/2}\frac{1}{2}(1-\cos\pi x)dx = \frac{1}{12}+\frac{\sqrt{3}-2}{4\pi}$

3. $\displaystyle\int_0^2\int_{-1}^{y^2}\int_{-1}^{z} yz\,dx\,dz\,dy = \int_0^2\int_{-1}^{y^2}(yz^2+yz)dz\,dy = \int_0^2\left(\frac{1}{3}y^7+\frac{1}{2}y^5-\frac{1}{6}y\right)dy = \frac{47}{3}$

4. $\displaystyle\int_0^{\pi/4}\int_0^1\int_0^{x^2} x\cos y\,dz\,dx\,dy = \int_0^{\pi/4}\int_0^1 x^3\cos y\,dx\,dy = \int_0^{\pi/4}\frac{1}{4}\cos y\,dy = \sqrt{2}/8$

5. $\displaystyle\int_0^3\int_0^{\sqrt{9-z^2}}\int_0^x xy\,dy\,dx\,dz = \int_0^3\int_0^{\sqrt{9-z^2}}\frac{1}{2}x^3dx\,dz = \int_0^3\frac{1}{8}(81-18z^2+z^4)dz = 81/5$

6. $\displaystyle\int_1^3\int_x^{x^2}\int_0^{\ln z} xe^y\,dy\,dz\,dx = \int_1^3\int_x^{x^2}(xz-x)dz\,dx = \int_1^3\left(\frac{1}{2}x^5-\frac{3}{2}x^3+x^2\right)dx = 118/3$

7. $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}}\int_{-5+x^2+y^2}^{3-x^2-y^2} x\,dz\,dy\,dx = \int_0^2\int_0^{\sqrt{4-x^2}}[2x(4-x^2)-2xy^2]dy\,dx$

$\displaystyle = \int_0^2\frac{4}{3}x(4-x^2)^{3/2}dx = 128/15$

8. $\displaystyle\int_1^2\int_z^2\int_0^{\sqrt{3}y}\frac{y}{x^2+y^2}dx\,dy\,dz = \int_1^2\int_z^2\frac{\pi}{3}dy\,dz = \int_1^2\frac{\pi}{3}(2-z)dz = \pi/6$

9. $\displaystyle\int_0^{\pi}\int_0^1\int_0^{\pi/6} xy\sin yz\,dz\,dy\,dx = \int_0^{\pi}\int_0^1 x[1-\cos(\pi y/6)]dy\,dx = \int_0^{\pi}(1-3/\pi)x\,dx = \pi(\pi-3)/2$

10. $\displaystyle\int_{-1}^1\int_0^{1-x^2}\int_0^y y\,dz\,dy\,dx = \int_{-1}^1\int_0^{1-x^2} y^2dy\,dx = \int_{-1}^1 \frac{1}{3}(1-x^2)^3dx = 32/105$

11. $\displaystyle\int_0^{\sqrt{2}}\int_0^x\int_0^{2-x^2} xyz\,dz\,dy\,dx = \int_0^{\sqrt{2}}\int_0^x \frac{1}{2}xy(2-x^2)^2dy\,dx = \int_0^{\sqrt{2}}\frac{1}{4}x^3(2-x^2)^2dx = 1/6$

12. $\displaystyle\int_{\pi/6}^{\pi/2}\int_y^{\pi/2}\int_0^{xy}\cos(z/y)dz\,dx\,dy = \int_{\pi/6}^{\pi/2}\int_y^{\pi/2} y\sin x\,dx\,dy = \int_{\pi/6}^{\pi/2} y\cos y\,dy = (5\pi-6\sqrt{3})/12$

13. $\displaystyle\int_0^3\int_1^2\int_{-2}^1 \frac{\sqrt{x+z^2}}{y}\,dzdydx \approx 9.425$

14. $\displaystyle 8\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} e^{-x^2-y^2-z^2}\,dz\,dy\,dx \approx 2.381$

15. $\displaystyle V = \int_0^4\int_0^{(4-x)/2}\int_0^{(12-3x-6y)/4} dz\,dy\,dx = \int_0^4\int_0^{(4-x)/2}\frac{1}{4}(12-3x-6y)dy\,dx$

$\displaystyle = \int_0^4 \frac{3}{16}(4-x)^2dx = 4$

16. $\displaystyle V = \int_0^1\int_0^{1-x}\int_0^{\sqrt{y}} dz\,dy\,dx = \int_0^1\int_0^{1-x}\sqrt{y}\,dy\,dx = \int_0^1\frac{2}{3}(1-x)^{3/2}dx = 4/15$

17. $\displaystyle V = 2\int_0^2\int_{x^2}^4\int_0^{4-y} dz\,dy\,dx = 2\int_0^2\int_{x^2}^4(4-y)dy\,dx = 2\int_0^2\left(8-4x^2+\frac{1}{2}x^4\right)dx = 256/15$

18. $\displaystyle V = \int_0^1\int_0^y\int_0^{\sqrt{1-y^2}} dz\,dx\,dy = \int_0^1\int_0^y\sqrt{1-y^2}\,dx\,dy = \int_0^1 y\sqrt{1-y^2}\,dy = 1/3$

19. The projection of the curve of intersection onto the xy-plane is $x^2+y^2=1$,

$\displaystyle V = 4\int_0^1\int_0^{\sqrt{1-x^2}}\int_{4x^2+y^2}^{4-3y^2} dz\,dy\,dx$

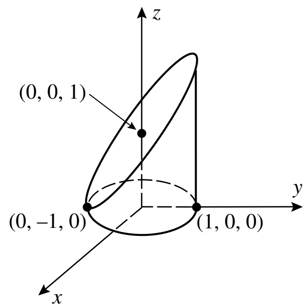
20. The projection of the curve of intersection onto the xy-plane is $2x^2+y^2=4$,

$\displaystyle V = 4\int_0^{\sqrt{2}}\int_0^{\sqrt{4-2x^2}}\int_{3x^2+y^2}^{8-x^2-y^2} dz\,dy\,dx$

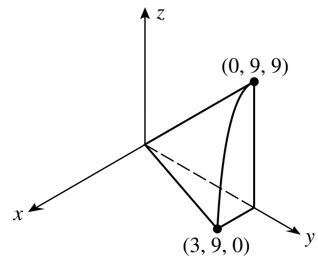
21. $\displaystyle V = 2\int_{-3}^3\int_0^{\sqrt{9-x^2}/3}\int_0^{x+3} dz\,dy\,dx$

22. $\displaystyle V = 8\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2}} dz\,dy\,dx$

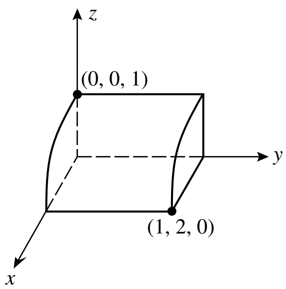
23. **(a)**



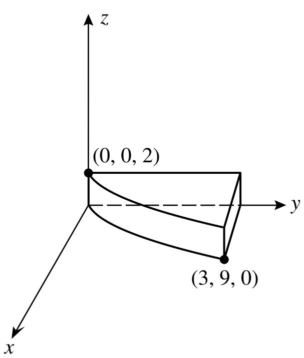
(b)



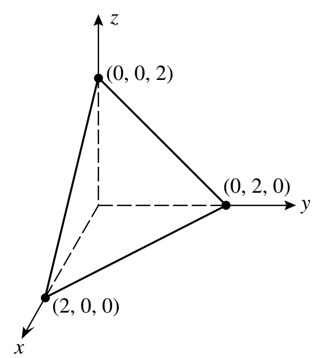
(c)



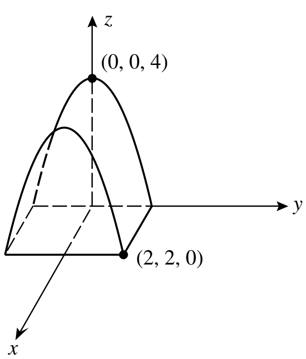
24. **(a)**



(b)



(c)



25. $V = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\, dy\, dx = 1/6,\ f_{\text{ave}} = 6 \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x+y+z)\, dz\, dy\, dx = \dfrac{3}{4}$

26. The integrand is an odd function of each of x, y, and z, so the answer is zero.

27. The volume $V = \dfrac{3\pi}{\sqrt{2}}$, and thus

$$r_{\text{ave}} = \frac{\sqrt{2}}{3\pi} \iiint_G \sqrt{x^2+y^2+z^2}\, dV = \frac{\sqrt{2}}{3\pi} \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} \int_{5x^2+5y^2}^{6-7x^2-y^2} \sqrt{x^2+y^2+z^2}\, dz dy dx \approx 3.291$$

28. $V = 1, d_{\text{ave}} = \dfrac{1}{V}\displaystyle\int_0^1\int_0^1\int_0^1 \sqrt{(x-z)^2+(y-z)^2+z^2}\,dxdydz \approx 0.771$

29. **(a)** $\displaystyle\int_0^a\int_0^{b(1-x/a)}\int_0^{c(1-x/a-y/b)} dz\,dy\,dx,\ \int_0^b\int_0^{a(1-y/b)}\int_0^{c(1-x/a-y/b)} dz\,dx\,dy,$

$\displaystyle\int_0^c\int_0^{a(1-z/c)}\int_0^{b(1-x/a-z/c)} dy\,dx\,dz,\ \int_0^a\int_0^{c(1-x/a)}\int_0^{b(1-x/a-z/c)} dy\,dz\,dx,$

$\displaystyle\int_0^c\int_0^{b(1-z/c)}\int_0^{a(1-y/b-z/c)} dx\,dy\,dz,\ \int_0^b\int_0^{c(1-y/b)}\int_0^{a(1-y/b-z/c)} dx\,dz\,dy$

(b) Use the first integral in Part (a) to get

$$\int_0^a\int_0^{b(1-x/a)} c\left(1-\frac{x}{a}-\frac{y}{b}\right)dy\,dx = \int_0^a \frac{1}{2}bc\left(1-\frac{x}{a}\right)^2 dx = \frac{1}{6}abc$$

30. $V = 8\displaystyle\int_0^a\int_0^{b\sqrt{1-x^2/a^2}}\int_0^{c\sqrt{1-x^2/a^2-y^2/b^2}} dz\,dy\,dx$

31. **(a)** $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^5 f(x,y,z)\,dzdydx$

(b) $\displaystyle\int_0^9\int_0^{3-\sqrt{x}}\int_y^{3-\sqrt{x}} f(x,y,z)\,dzdydx$ **(c)** $\displaystyle\int_0^2\int_0^{4-x^2}\int_y^{8-y} f(x,y,z)\,dzdydx$

32. **(a)** $\displaystyle\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2-y^2}} f(x,y,z)dz\,dy\,dx$

(b) $\displaystyle\int_0^4\int_0^{x/2}\int_0^2 f(x,y,z)dz\,dy\,dx$ **(c)** $\displaystyle\int_0^2\int_0^{4-x^2}\int_{x^2}^{4-y} f(x,y,z)dz\,dy\,dx$

33. **(a)** At any point outside the closed sphere $\{x^2+y^2+z^2 \le 1\}$ the integrand is negative, so to maximize the integral it suffices to include all points inside the sphere; hence the maximum value is taken on the region $G = \{x^2+y^2+z^2 \le 1\}$.

(b) 4.934802202

(c) $\displaystyle\int_0^{2\pi}\int_0^{\pi}\int_0^1 (1-\rho^2)\rho\,d\rho\,d\phi\,d\theta = \frac{\pi^2}{2}$

34.
$$\begin{aligned}\int_a^b\int_c^d\int_k^\ell f(x)g(y)h(z)dz\,dy\,dx &= \int_a^b\int_c^d f(x)g(y)\left[\int_k^\ell h(z)dz\right]dy\,dx\\ &= \left[\int_a^b f(x)\left[\int_c^d g(y)dy\right]dx\right]\left[\int_k^\ell h(z)dz\right]\\ &= \left[\int_a^b f(x)dx\right]\left[\int_c^d g(y)dy\right]\left[\int_k^\ell h(z)dz\right]\end{aligned}$$

35. **(a)** $\displaystyle\left[\int_{-1}^1 x\,dx\right]\left[\int_0^1 y^2dy\right]\left[\int_0^{\pi/2}\sin z\,dz\right] = (0)(1/3)(1) = 0$

(b) $\displaystyle\left[\int_0^1 e^{2x}dx\right]\left[\int_0^{\ln 3} e^y dy\right]\left[\int_0^{\ln 2} e^{-z}dz\right] = [(e^2-1)/2](2)(1/2) = (e^2-1)/2$

EXERCISE SET 15.6

1. Let a be the unknown coordinate of the fulcrum; then the total moment about the fulcrum is $5(0-a)+10(5-a)+20(10-a)=0$ for equilibrium, so $250-35a=0$, $a=50/7$. The fulcrum should be placed $50/7$ units to the right of m_1.

2. At equilibrium, $10(0-4)+3(2-4)+4(3-4)+m(6-4)=0, m=25$

3. $A=1,\ \overline{x}=\int_0^1\int_0^1 x\,dy\,dx=\frac{1}{2},\ \overline{y}=\int_0^1\int_0^1 y\,dy\,dx=\frac{1}{2}$

4. $A=2, \overline{x}=\frac{1}{2}\iint\limits_G x\,dy\,dx$, and the region of integration is symmetric with respect to the x-axes and the integrand is an odd function of x, so $\overline{x}=0$. Likewise, $\overline{y}=0$.

5. $A=1/2,\ \iint\limits_R x\,dA=\int_0^1\int_0^x x\,dy\,dx=1/3,\ \iint\limits_R y\,dA=\int_0^1\int_0^x y\,dy\,dx=1/6;$

centroid $(2/3,1/3)$

6. $A=\int_0^1\int_0^{x^2} dy\,dx=1/3,\ \iint\limits_R x\,dA=\int_0^1\int_0^{x^2} x\,dy\,dx=1/4,$

$\iint\limits_R y\,dA=\int_0^1\int_0^{x^2} y\,dy\,dx=1/10$; centroid $(3/4,3/10)$

7. $A=\int_0^1\int_x^{2-x^2} dy\,dx=7/6,\ \iint\limits_R x\,dA=\int_0^1\int_x^{2-x^2} x\,dy\,dx=5/12,$

$\iint\limits_R y\,dA=\int_0^1\int_x^{2-x^2} y\,dy\,dx=19/15$; centroid $(5/14,38/35)$

8. $A=\frac{\pi}{4},\iint\limits_R x\,dA=\int_0^1\int_0^{\sqrt{1-x^2}} x\,dy\,dx=\frac{1}{3},\ \overline{x}=\frac{4}{3\pi},\ \overline{y}=\frac{4}{3\pi}$ by symmetry

9. $\overline{x}=0$ from the symmetry of the region,

$A=\frac{1}{2}\pi(b^2-a^2),\ \iint\limits_R y\,dA=\int_0^{\pi}\int_a^b r^2\sin\theta\,dr\,d\theta=\frac{2}{3}(b^3-a^3);$ centroid $\overline{x}=0,\ \overline{y}=\frac{4(b^3-a^3)}{3\pi(b^2-a^2)}$.

10. $\overline{y}=0$ from the symmetry of the region, $A=\pi a^2/2,$

$\iint\limits_R x\,dA=\int_{-\pi/2}^{\pi/2}\int_0^a r^2\cos\theta\,dr\,d\theta=2a^3/3$; centroid $\left(\frac{4a}{3\pi},0\right)$

11. $M=\iint\limits_R \delta(x,y)dA=\int_0^1\int_0^1 |x+y-1|\,dxdy$

$=\int_0^1\left[\int_0^{1-x}(1-x-y)\,dy+\int_{1-x}^1(x+y-1)\,dy\right]dx=\frac{1}{3}$

$$\overline{x} = 3\int_0^1\int_0^1 x\delta(x,y)\,dy\,dx = 3\int_0^1\left[\int_0^{1-x} x(1-x-y)\,dy + \int_{1-x}^1 x(x+y-1)\,dy\right]dx = \frac{1}{2}$$

By symmetry, $\overline{y} = \dfrac{1}{2}$ as well; center of gravity $(1/2, 1/2)$

12. $\overline{x} = \dfrac{1}{M}\displaystyle\iint_G x\delta(x,y)\,dA$, and the integrand is an odd function of x while the region is symmetric with respect to the y-axis, thus $\overline{x} = 0$; likewise $\overline{y} = 0$.

13. $M = \displaystyle\int_0^1\int_0^{\sqrt{x}}(x+y)dy\,dx = 13/20$, $M_x = \displaystyle\int_0^1\int_0^{\sqrt{x}}(x+y)y\,dy\,dx = 3/10$,

$M_y = \displaystyle\int_0^1\int_0^{\sqrt{x}}(x+y)x\,dy\,dx = 19/42$, $\overline{x} = M_y/M = 190/273$, $\overline{y} = M_x/M = 6/13$;

the mass is $13/20$ and the center of gravity is at $(190/273, 6/13)$.

14. $M = \displaystyle\int_0^\pi\int_0^{\sin x} y\,dy\,dx = \pi/4$, $\overline{x} = \pi/2$ from the symmetry of the density and the region,

$M_x = \displaystyle\int_0^\pi\int_0^{\sin x} y^2dy\,dx = 4/9$, $\overline{y} = M_x/M = \dfrac{16}{9\pi}$; mass $\pi/4$, center of gravity $\left(\dfrac{\pi}{2}, \dfrac{16}{9\pi}\right)$.

15. $M = \displaystyle\int_0^{\pi/2}\int_0^a r^3\sin\theta\cos\theta\,dr\,d\theta = a^4/8$, $\overline{x} = \overline{y}$ from the symmetry of the density and the region, $M_y = \displaystyle\int_0^{\pi/2}\int_0^a r^4\sin\theta\cos^2\theta\,dr\,d\theta = a^5/15$, $\overline{x} = 8a/15$; mass $a^4/8$, center of gravity $(8a/15, 8a/15)$.

16. $M = \displaystyle\int_0^\pi\int_0^1 r^3dr\,d\theta = \pi/4$, $\overline{x} = 0$ from the symmetry of density and region,

$M_x = \displaystyle\int_0^\pi\int_0^1 r^4\sin\theta\,dr\,d\theta = 2/5$, $\overline{y} = \dfrac{8}{5\pi}$; mass $\pi/4$, center of gravity $\left(0, \dfrac{8}{5\pi}\right)$.

17. $V = 1, \overline{x} = \displaystyle\int_0^1\int_0^1\int_0^1 x\,dz\,dy\,dx = \dfrac{1}{2}$, similarly $\overline{y} = \overline{z} = \dfrac{1}{2}$; centroid $\left(\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{1}{2}\right)$

18. symmetry, $\displaystyle\iiint_G z\,dz\,dy\,dx = \int_0^2\int_0^{2\pi}\int_0^1 rz\,dr\,d\theta\,dz = 2\pi$, centroid $= (0,0,1)$

19. $\overline{x} = \overline{y} = \overline{z}$ from the symmetry of the region, $V = 1/6$,

$\overline{x} = \dfrac{1}{V}\displaystyle\int_0^1\int_0^{1-x}\int_0^{1-x-y} x\,dz\,dy\,dx = (6)(1/24) = 1/4$; centroid $(1/4, 1/4, 1/4)$

20. The solid is described by $-1 \le y \le 1, 0 \le z \le 1-y^2, 0 \le x \le 1-z$;

$V = \displaystyle\int_{-1}^1\int_0^{1-y^2}\int_0^{1-z} dx\,dz\,dy = \dfrac{4}{5}, \overline{x} = \dfrac{1}{V}\displaystyle\int_{-1}^1\int_0^{1-y^2}\int_0^{1-z} x\,dx\,dz\,dy = \dfrac{5}{14}, \overline{y} = 0$ by symmetry,

$\overline{z} = \dfrac{1}{V}\displaystyle\int_{-1}^1\int_0^{1-y^2}\int_0^{1-z} z\,dx\,dz\,dy = \dfrac{2}{7}$; the centroid is $\left(\dfrac{5}{14}, 0, \dfrac{2}{7}\right)$.

21. $\overline{x} = 1/2$ and $\overline{y} = 0$ from the symmetry of the region,

$$V = \int_0^1 \int_{-1}^1 \int_{y^2}^1 dz\,dy\,dx = 4/3,\ \overline{z} = \frac{1}{V}\iiint\limits_G z\,dV = (3/4)(4/5) = 3/5;\text{ centroid } (1/2, 0, 3/5)$$

22. $\overline{x} = \overline{y}$ from the symmetry of the region,

$$V = \int_0^2 \int_0^2 \int_0^{xy} dz\,dy\,dx = 4,\ \overline{x} = \frac{1}{V}\iiint\limits_G x\,dV = (1/4)(16/3) = 4/3,$$

$$\overline{z} = \frac{1}{V}\iiint\limits_G z\,dV = (1/4)(32/9) = 8/9;\text{ centroid } (4/3, 4/3, 8/9)$$

23. $\overline{x} = \overline{y} = \overline{z}$ from the symmetry of the region, $V = \pi a^3/6$,

$$\overline{x} = \frac{1}{V}\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} x\,dz\,dy\,dx = \frac{1}{V}\int_0^a \int_0^{\sqrt{a^2-x^2}} x\sqrt{a^2-x^2-y^2}\,dy\,dx$$

$$= \frac{1}{V}\int_0^{\pi/2} \int_0^a r^2\sqrt{a^2-r^2}\cos\theta\,dr\,d\theta = \frac{6}{\pi a^3}(\pi a^4/16) = 3a/8;\ \text{centroid } (3a/8, 3a/8, 3a/8)$$

24. $\overline{x} = \overline{y} = 0$ from the symmetry of the region, $V = 2\pi a^3/3$

$$\overline{z} = \frac{1}{V}\int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} z\,dz\,dy\,dx = \frac{1}{V}\int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{1}{2}(a^2-x^2-y^2)dy\,dx$$

$$= \frac{1}{V}\int_0^{2\pi} \int_0^a \frac{1}{2}(a^2-r^2)r\,dr\,d\theta = \frac{3}{2\pi a^3}(\pi a^4/4) = 3a/8;\ \text{centroid } (0, 0, 3a/8)$$

25. $M = \int_0^a \int_0^a \int_0^a (a-x)dz\,dy\,dx = a^4/2$, $\overline{y} = \overline{z} = a/2$ from the symmetry of density and region, $\overline{x} = \frac{1}{M}\int_0^a \int_0^a \int_0^a x(a-x)dz\,dy\,dx = (2/a^4)(a^5/6) = a/3$;

mass $a^4/2$, center of gravity $(a/3, a/2, a/2)$

26. $M = \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^h (h-z)dz\,dy\,dx = \frac{1}{2}\pi a^2 h^2$, $\overline{x} = \overline{y} = 0$ from the symmetry of density and region, $\overline{z} = \frac{1}{M}\iiint\limits_G z(h-z)dV = \frac{2}{\pi a^2 h^2}(\pi a^2 h^3/6) = h/3$;

mass $\pi a^2 h^2/2$, center of gravity $(0, 0, h/3)$

27. $M = \int_{-1}^1 \int_0^1 \int_0^{1-y^2} yz\,dz\,dy\,dx = 1/6$, $\overline{x} = 0$ by the symmetry of density and region,

$$\overline{y} = \frac{1}{M}\iiint\limits_G y^2 z\,dV = (6)(8/105) = 16/35,\ \overline{z} = \frac{1}{M}\iiint\limits_G yz^2 dV = (6)(1/12) = 1/2;$$

mass $1/6$, center of gravity $(0, 16/35, 1/2)$

28. $M = \int_0^3 \int_0^{9-x^2} \int_0^1 xz\,dz\,dy\,dx = 81/8$, $\bar{x} = \frac{1}{M}\iiint\limits_G x^2 z\,dV = (8/81)(81/5) = 8/5$,

$\bar{y} = \frac{1}{M}\iiint\limits_G xyz\,dV = (8/81)(243/8) = 3$, $\bar{z} = \frac{1}{M}\iiint\limits_G xz^2 dV = (8/81)(27/4) = 2/3$;

mass 81/8, center of gravity $(8/5, 3, 2/3)$

29. **(a)** $M = \int_0^1 \int_0^1 k(x^2+y^2)dy\,dx = 2k/3$, $\bar{x} = \bar{y}$ from the symmetry of density and region,

$\bar{x} = \frac{1}{M}\iint\limits_R kx(x^2+y^2)dA = \frac{3}{2k}(5k/12) = 5/8$; center of gravity $(5/8, 5/8)$

(b) $\bar{y} = 1/2$ from the symmetry of density and region,

$M = \int_0^1 \int_0^1 kx\,dy\,dx = k/2$, $\bar{x} = \frac{1}{M}\iint\limits_R kx^2 dA = (2/k)(k/3) = 2/3$,

center of gravity $(2/3, 1/2)$

30. **(a)** $\bar{x} = \bar{y} = \bar{z}$ from the symmetry of density and region,

$M = \int_0^1 \int_0^1 \int_0^1 k(x^2+y^2+z^2)dz\,dy\,dx = k$,

$\bar{x} = \frac{1}{M}\iiint\limits_G kx(x^2+y^2+z^2)dV = (1/k)(7k/12) = 7/12$; center of gravity $(7/12, 7/12, 7/12)$

(b) $\bar{x} = \bar{y} = \bar{z}$ from the symmetry of density and region,

$M = \int_0^1 \int_0^1 \int_0^1 k(x+y+z)dz\,dy\,dx = 3k/2$,

$\bar{x} = \frac{1}{M}\iiint\limits_G kx(x+y+z)dV = \frac{2}{3k}(5k/6) = 5/9$; center of gravity $(5/9, 5/9, 5/9)$

31. $V = \iiint\limits_G dV = \int_0^{\pi} \int_0^{\sin x} \int_0^{1/(1+x^2+y^2)} dz\ dy\ dx = 0.666633$,

$\bar{x} = \frac{1}{V}\iiint\limits_G x dV = 1.177406$, $\bar{y} = \frac{1}{V}\iiint\limits_G y dV = 0.353554$, $\bar{z} = \frac{1}{V}\iiint\limits_G z dV = 0.231557$

32. **(b)** Use polar coordinates for x and y to get

$$V = \iiint\limits_G dV = \int_0^{2\pi} \int_0^a \int_0^{1/(1+r^2)} r\ dz\ dr\ d\theta = \pi \ln(1+a^2),$$

$$\bar{z} = \frac{1}{V}\iiint\limits_G z dV = \frac{a^2}{2(1+a^2)\ln(1+a^2)}$$

Thus $\lim_{a\to 0^+} \bar{z} = \frac{1}{2}$; $\lim_{a\to +\infty} \bar{z} = 0$.

$\lim_{a\to 0^+} \bar{z} = \frac{1}{2}$; $\lim_{a\to +\infty} \bar{z} = 0$

(c) Solve $\bar{z} = 1/4$ for a to obtain $a \approx 1.980291$.

33. Let $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\,dr\,d\theta$ in formulas (11) and (12).

34. $\bar{x} = 0$ from the symmetry of the region, $A = \int_0^{2\pi}\int_0^{a(1+\sin\theta)} r\,dr\,d\theta = 3\pi a^2/2$,

$\bar{y} = \dfrac{1}{A}\int_0^{2\pi}\int_0^{a(1+\sin\theta)} r^2\sin\theta\,dr\,d\theta = \dfrac{2}{3\pi a^2}(5\pi a^3/4) = 5a/6$; centroid $(0, 5a/6)$

35. $\bar{x} = \bar{y}$ from the symmetry of the region, $A = \int_0^{\pi/2}\int_0^{\sin 2\theta} r\,dr\,d\theta = \pi/8$,

$\bar{x} = \dfrac{1}{A}\int_0^{\pi/2}\int_0^{\sin 2\theta} r^2\cos\theta\,dr\,d\theta = (8/\pi)(16/105) = \dfrac{128}{105\pi}$; centroid $\left(\dfrac{128}{105\pi}, \dfrac{128}{105\pi}\right)$

36. $\bar{x} = 3/2$ and $\bar{y} = 1$ from the symmetry of the region,

$\iint_R x\,dA = \bar{x}A = (3/2)(6) = 9$, $\iint_R y\,dA = \bar{y}A = (1)(6) = 6$

37. $\bar{x} = 0$ from the symmetry of the region, $\pi a^2/2$ is the area of the semicircle, $2\pi\bar{y}$ is the distance traveled by the centroid to generate the sphere so $4\pi a^3/3 = (\pi a^2/2)(2\pi\bar{y})$, $\bar{y} = 4a/(3\pi)$

38. **(a)** $V = \left[\dfrac{1}{2}\pi a^2\right]\left[2\pi\left(a + \dfrac{4a}{3\pi}\right)\right] = \dfrac{1}{3}\pi(3\pi + 4)a^3$

(b) the distance between the centroid and the line is $\dfrac{\sqrt{2}}{2}\left(a + \dfrac{4a}{3\pi}\right)$ so

$$V = \left[\frac{1}{2}\pi a^2\right]\left[2\pi\frac{\sqrt{2}}{2}\left(a + \frac{4a}{3\pi}\right)\right] = \frac{1}{6}\sqrt{2}\pi(3\pi + 4)a^3$$

39. $\bar{x} = k$ so $V = (\pi ab)(2\pi k) = 2\pi^2 abk$

40. $\bar{y} = 4$ from the symmetry of the region,

$A = \int_{-2}^{2}\int_{x^2}^{8-x^2} dy\,dx = 64/3$ so $V = (64/3)[2\pi(4)] = 512\pi/3$

41. The region generates a cone of volume $\dfrac{1}{3}\pi ab^2$ when it is revolved about the x-axis, the area of the region is $\dfrac{1}{2}ab$ so $\dfrac{1}{3}\pi ab^2 = \left(\dfrac{1}{2}ab\right)(2\pi\bar{y})$, $\bar{y} = b/3$. A cone of volume $\dfrac{1}{3}\pi a^2 b$ is generated when the region is revolved about the y-axis so $\dfrac{1}{3}\pi a^2 b = \left(\dfrac{1}{2}ab\right)(2\pi\bar{x})$, $\bar{x} = a/3$. The centroid is $(a/3, b/3)$.

42. $I_x = \int_0^a\int_0^b y^2\delta\,dy\,dx = \dfrac{1}{3}\delta ab^3$, $I_y = \int_0^a\int_0^b x^2\delta\,dy\,dx = \dfrac{1}{3}\delta a^3 b$,

$I_z = \int_0^a\int_0^b (x^2 + y^2)\delta\,dy\,dx = \dfrac{1}{3}\delta ab(a^2 + b^2)$

43. $I_x = \int_0^{2\pi}\int_0^a r^3\sin^2\theta\,\delta\,dr\,d\theta = \delta\pi a^4/4$; $I_y = \int_0^{2\pi}\int_0^a r^3\cos^2\theta\,\delta\,dr\,d\theta = \delta\pi a^4/4 = I_x$;

$I_z = I_x + I_y = \delta\pi a^4/2$

EXERCISE SET 15.7

1. $\int_0^{2\pi}\int_0^1\int_0^{\sqrt{1-r^2}} zr\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \frac{1}{2}(1-r^2)r\,dr\,d\theta = \int_0^{2\pi}\frac{1}{8}d\theta = \pi/4$

2. $\int_0^{\pi/2}\int_0^{\cos\theta}\int_0^{r^2} r\sin\theta\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^{\cos\theta} r^3\sin\theta\,dr\,d\theta = \int_0^{\pi/2}\frac{1}{4}\cos^4\theta\sin\theta\,d\theta = 1/20$

3. $\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho^3\sin\phi\cos\phi\,d\rho\,d\phi\,d\theta = \int_0^{\pi/2}\int_0^{\pi/2}\frac{1}{4}\sin\phi\cos\phi\,d\phi\,d\theta = \int_0^{\pi/2}\frac{1}{8}d\theta = \pi/16$

4. $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{a\sec\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/4}\frac{1}{3}a^3\sec^3\phi\sin\phi\,d\phi\,d\theta = \int_0^{2\pi}\frac{1}{6}a^3 d\theta = \pi a^3/3$

5. $V = \int_0^{2\pi}\int_0^3\int_{r^2}^9 r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^3 r(9-r^2)dr\,d\theta = \int_0^{2\pi}\frac{81}{4}d\theta = 81\pi/2$

6. $V = 2\int_0^{2\pi}\int_0^2\int_0^{\sqrt{9-r^2}} r\,dz\,dr\,d\theta = 2\int_0^{2\pi}\int_0^2 r\sqrt{9-r^2}dr\,d\theta$

$$= \frac{2}{3}(27-5\sqrt{5})\int_0^{2\pi}d\theta = 4(27-5\sqrt{5})\pi/3$$

7. $r^2+z^2=20$ intersects $z=r^2$ in a circle of radius 2; the volume consists of two portions, one inside the cylinder $r=\sqrt{20}$ and one outside that cylinder:

$$V = \int_0^{2\pi}\int_0^2\int_{-\sqrt{20-r^2}}^{r^2} r\,dz\,dr\,d\theta + \int_0^{2\pi}\int_2^{\sqrt{20}}\int_{-\sqrt{20-r^2}}^{\sqrt{20-r^2}} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 r\left(r^2+\sqrt{20-r^2}\right)dr\,d\theta + \int_0^{2\pi}\int_2^{\sqrt{20}} 2r\sqrt{20-r^2}\,dr\,d\theta$$

$$= \frac{4}{3}(10\sqrt{5}-13)\int_0^{2\pi}d\theta + \frac{128}{3}\int_0^{2\pi}d\theta = \frac{152}{3}\pi + \frac{80}{3}\pi\sqrt{5}$$

8. $z=hr/a$ intersects $z=h$ in a circle of radius a,

$$V = \int_0^{2\pi}\int_0^a\int_{hr/a}^h r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^a \frac{h}{a}(ar-r^2)dr\,d\theta = \int_0^{2\pi}\frac{1}{6}a^2h\,d\theta = \pi a^2h/3$$

9. $V = \int_0^{2\pi}\int_0^{\pi/3}\int_0^4\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/3}\frac{64}{3}\sin\phi\,d\phi\,d\theta = \frac{32}{3}\int_0^{2\pi}d\theta = 64\pi/3$

10. $V = \int_0^{2\pi}\int_0^{\pi/4}\int_1^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/4}\frac{7}{3}\sin\phi\,d\phi\,d\theta = \frac{7}{6}(2-\sqrt{2})\int_0^{2\pi}d\theta = 7(2-\sqrt{2})\pi/3$

11. In spherical coordinates the sphere and the plane $z=a$ are $\rho=2a$ and $\rho=a\sec\phi$, respectively. They intersect at $\phi=\pi/3$,

$$V = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{a\sec\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta + \int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_0^{2a}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3}\frac{1}{3}a^3\sec^3\phi\sin\phi\,d\phi\,d\theta + \int_0^{2\pi}\int_{\pi/3}^{\pi/2}\frac{8}{3}a^3\sin\phi\,d\phi\,d\theta$$

$$= \frac{1}{2}a^3\int_0^{2\pi}d\theta + \frac{4}{3}a^3\int_0^{2\pi}d\theta = 11\pi a^3/3$$

12. $V = \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^3 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_{\pi/4}^{\pi/2} 9\sin\phi \, d\phi \, d\theta = \frac{9\sqrt{2}}{2}\int_0^{2\pi} d\theta = 9\sqrt{2}\pi$

13. $\int_0^{\pi/2}\int_0^a\int_0^{a^2-r^2} r^3\cos^2\theta \, dz \, dr \, d\theta = \int_0^{\pi/2}\int_0^a (a^2r^3 - r^5)\cos^2\theta \, dr \, d\theta$

$$= \frac{1}{12}a^6\int_0^{\pi/2}\cos^2\theta \, d\theta = \pi a^6/48$$

14. $\int_0^{\pi}\int_0^{\pi/2}\int_0^1 e^{-\rho^3}\rho^2\sin\phi \, d\rho \, d\phi \, d\theta = \frac{1}{3}(1-e^{-1})\int_0^{\pi}\int_0^{\pi/2}\sin\phi \, d\phi \, d\theta = (1-e^{-1})\pi/3$

15. $\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sqrt{8}} \rho^4\cos^2\phi\sin\phi \, d\rho \, d\phi \, d\theta = 32(2\sqrt{2}-1)\pi/15$

16. $\int_0^{2\pi}\int_0^{\pi}\int_0^3 \rho^3\sin\phi \, d\rho \, d\phi \, d\theta = 81\pi$

17. **(a)** $\int_{\pi/3}^{\pi/2}\int_1^4\int_{-2}^2 \frac{r\tan^3\theta}{\sqrt{1+z^2}} \, dz \, dr \, d\theta = \left(\int_{\pi/6}^{\pi/3}\tan^3\theta \, d\theta\right)\left(\int_1^4 r \, dr\right)\left(\int_{-2}^2 \frac{1}{\sqrt{1+z^2}} \, dz\right)$

$$= \left(\frac{4}{3} - \frac{1}{2}\ln 3\right)\frac{15}{2}\left(-2\ln(\sqrt{5}-2)\right) = \frac{5}{2}(-8+3\ln 3)\ln(\sqrt{5}-2)$$

(b) $\int_{\pi/3}^{\pi/2}\int_1^4\int_{-2}^2 \frac{y\tan^3 z}{\sqrt{1+x^2}} \, dx \, dy \, dz$; the region is a rectangular solid with sides $\pi/6$, 3, 4.

18. $\int_0^{\pi/2}\int_0^{\pi/4} \frac{1}{18}\cos^{37}\theta\cos\phi \, d\phi \, d\theta = \frac{\sqrt{2}}{36}\int_0^{\pi/2}\cos^{37}\theta \, d\theta = \frac{4{,}294{,}967{,}296}{755{,}505{,}013{,}725}\sqrt{2} \approx 0.008040$

19. **(a)** $V = 2\int_0^{2\pi}\int_0^a\int_0^{\sqrt{a^2-r^2}} r \, dz \, dr \, d\theta = 4\pi a^3/3$

(b) $V = \int_0^{2\pi}\int_0^{\pi}\int_0^a \rho^2\sin\phi \, d\rho \, d\phi \, d\theta = 4\pi a^3/3$

20. **(a)** $\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} xyz \, dz \, dy \, dx$

$$= \int_0^2\int_0^{\sqrt{4-x^2}} \frac{1}{2}xy(4-x^2-y^2)dy \, dx = \frac{1}{8}\int_0^2 x(4-x^2)^2dx = 4/3$$

(b) $\int_0^{\pi/2}\int_0^2\int_0^{\sqrt{4-r^2}} r^3 z\sin\theta\cos\theta \, dz \, dr \, d\theta$

$$= \int_0^{\pi/2}\int_0^2 \frac{1}{2}(4r^3 - r^5)\sin\theta\cos\theta \, dr \, d\theta = \frac{8}{3}\int_0^{\pi/2}\sin\theta\cos\theta \, d\theta = 4/3$$

(c) $\int_0^{\pi/2}\int_0^{\pi/2}\int_0^2 \rho^5\sin^3\phi\cos\phi\sin\theta\cos\theta \, d\rho \, d\phi \, d\theta$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \frac{32}{3}\sin^3\phi\cos\phi\sin\theta\cos\theta \, d\phi \, d\theta = \frac{8}{3}\int_0^{\pi/2}\sin\theta\cos\theta \, d\theta = 4/3$$

21. $M = \int_0^{2\pi}\int_0^3\int_r^3 (3-z)r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^3 \frac{1}{2}r(3-r)^2 dr\,d\theta = \frac{27}{8}\int_0^{2\pi} d\theta = 27\pi/4$

22. $M = \int_0^{2\pi}\int_0^a\int_0^h k\,zr\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^a \frac{1}{2}kh^2 r\,dr\,d\theta = \frac{1}{4}ka^2h^2\int_0^{2\pi} d\theta = \pi ka^2h^2/2$

23. $M = \int_0^{2\pi}\int_0^{\pi}\int_0^a k\rho^3\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi}\frac{1}{4}ka^4\sin\phi\,d\phi\,d\theta = \frac{1}{2}ka^4\int_0^{2\pi}d\theta = \pi ka^4$

24. $M = \int_0^{2\pi}\int_0^{\pi}\int_1^2 \rho\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi}\frac{3}{2}\sin\phi\,d\phi\,d\theta = 3\int_0^{2\pi}d\theta = 6\pi$

25. $\bar{x} = \bar{y} = 0$ from the symmetry of the region,

$V = \int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (r\sqrt{2-r^2} - r^3)dr\,d\theta = (8\sqrt{2}-7)\pi/6,$

$\bar{z} = \frac{1}{V}\int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} zr\,dz\,dr\,d\theta = \frac{6}{(8\sqrt{2}-7)\pi}(7\pi/12) = 7/(16\sqrt{2}-14);$

centroid $\left(0, 0, \frac{7}{16\sqrt{2}-14}\right)$

26. $\bar{x} = \bar{y} = 0$ from the symmetry of the region, $V = 8\pi/3$,

$\bar{z} = \frac{1}{V}\int_0^{2\pi}\int_0^2\int_r^2 zr\,dz\,dr\,d\theta = \frac{3}{8\pi}(4\pi) = 3/2$; centroid $(0, 0, 3/2)$

27. $\bar{x} = \bar{y} = \bar{z}$ from the symmetry of the region, $V = \pi a^3/6$,

$\bar{z} = \frac{1}{V}\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \rho^3\cos\phi\sin\phi\,d\rho\,d\phi\,d\theta = \frac{6}{\pi a^3}(\pi a^4/16) = 3a/8;$

centroid $(3a/8, 3a/8, 3a/8)$

28. $\bar{x} = \bar{y} = 0$ from the symmetry of the region, $V = \int_0^{2\pi}\int_0^{\pi/3}\int_0^4 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 64\pi/3,$

$\bar{z} = \frac{1}{V}\int_0^{2\pi}\int_0^{\pi/3}\int_0^4 \rho^3\cos\phi\sin\phi\,d\rho\,d\phi\,d\theta = \frac{3}{64\pi}(48\pi) = 9/4$; centroid $(0, 0, 9/4)$

29. $\bar{y} = 0$ from the symmetry of the region, $V = 2\int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^{r^2} r\,dz\,dr\,d\theta = 3\pi/2,$

$\bar{x} = \frac{2}{V}\int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^{r^2} r^2\cos\theta\,dz\,dr\,d\theta = \frac{4}{3\pi}(\pi) = 4/3,$

$\bar{z} = \frac{2}{V}\int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^{r^2} rz\,dz\,dr\,d\theta = \frac{4}{3\pi}(5\pi/6) = 10/9$; centroid $(4/3, 0, 10/9)$

30. $M = \int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^{4-r^2} zr\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^{2\cos\theta}\frac{1}{2}r(4-r^2)^2 dr\,d\theta$

$= \frac{16}{3}\int_0^{\pi/2}(1-\sin^6\theta)d\theta = (16/3)(11\pi/32) = 11\pi/6$

31. $V = \int_0^{\pi/2}\int_{\pi/6}^{\pi/3}\int_0^2 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{\pi/2}\int_{\pi/6}^{\pi/3} \frac{8}{3}\sin\phi\, d\phi\, d\theta = \frac{4}{3}(\sqrt{3}-1)\int_0^{\pi/2} d\theta$
$= 2(\sqrt{3}-1)\pi/3$

32. $M = \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^3 \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/4} \frac{1}{4}\sin\phi\, d\phi\, d\theta = \frac{1}{8}(2-\sqrt{2})\int_0^{2\pi} d\theta = (2-\sqrt{2})\pi/4$

33. $\bar{x} = \bar{y} = 0$ from the symmetry of density and region,
$M = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} (r^2+z^2)r\, dz\, dr\, d\theta = \pi/4$,
$\bar{z} = \frac{1}{M}\int_0^{2\pi}\int_0^1\int_0^{1-r^2} z(r^2+z^2)r\, dz\, dr\, d\theta = (4/\pi)(11\pi/120) = 11/30$; center of gravity $(0, 0, 11/30)$

34. $\bar{x} = \bar{y} = 0$ from the symmetry of density and region, $M = \int_0^{2\pi}\int_0^1\int_0^r zr\, dz\, dr\, d\theta = \pi/4$,
$\bar{z} = \frac{1}{M}\int_0^{2\pi}\int_0^1\int_0^r z^2 r\, dz\, dr\, d\theta = (4/\pi)(2\pi/15) = 8/15$; center of gravity $(0, 0, 8/15)$

35. $\bar{x} = \bar{y} = 0$ from the symmetry of density and region,
$M = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a k\rho^3 \sin\phi\, d\rho\, d\phi\, d\theta = \pi k a^4/2$,
$\bar{z} = \frac{1}{M}\int_0^{2\pi}\int_0^{\pi/2}\int_0^a k\rho^4 \sin\phi\cos\phi\, d\rho\, d\phi\, d\theta = \frac{2}{\pi k a^4}(\pi k a^5/5) = 2a/5$; center of gravity $(0, 0, 2a/5)$

36. $\bar{x} = \bar{z} = 0$ from the symmetry of the region, $V = 54\pi/3 - 16\pi/3 = 38\pi/3$,
$\bar{y} = \frac{1}{V}\int_0^{\pi}\int_0^{\pi}\int_2^3 \rho^3 \sin^2\phi \sin\theta\, d\rho\, d\phi\, d\theta = \frac{1}{V}\int_0^{\pi}\int_0^{\pi} \frac{65}{4}\sin^2\phi\sin\theta\, d\phi\, d\theta$
$= \frac{1}{V}\int_0^{\pi} \frac{65\pi}{8}\sin\theta\, d\theta = \frac{3}{38\pi}(65\pi/4) = 195/152$; centroid $(0, 195/152, 0)$

37. $M = \int_0^{2\pi}\int_0^{\pi}\int_0^R \delta_0 e^{-(\rho/R)^3}\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi} \frac{1}{3}(1-e^{-1})R^3\delta_0 \sin\phi\, d\phi\, d\theta$
$= \frac{4}{3}\pi(1-e^{-1})\delta_0 R^3$

38. **(a)** The sphere and cone intersect in a circle of radius $\rho_0 \sin\phi_0$,

$$V = \int_{\theta_1}^{\theta_2}\int_0^{\rho_0\sin\phi_0}\int_{r\cot\phi_0}^{\sqrt{\rho_0^2-r^2}} r\, dz\, dr\, d\theta = \int_{\theta_1}^{\theta_2}\int_0^{\rho_0\sin\phi_0} \left(r\sqrt{\rho_0^2-r^2} - r^2\cot\phi_0\right) dr\, d\theta$$

$$= \int_{\theta_1}^{\theta_2} \frac{1}{3}\rho_0^3(1-\cos^3\phi_0 - \sin^3\phi_0\cot\phi_0)d\theta = \frac{1}{3}\rho_0^3(1-\cos^3\phi_0 - \sin^2\phi_0\cos\phi_0)(\theta_2-\theta_1)$$

$$= \frac{1}{3}\rho_0^3(1-\cos\phi_0)(\theta_2-\theta_1).$$

(b) From Part (a), the volume of the solid bounded by $\theta = \theta_1$, $\theta = \theta_2$, $\phi = \phi_1$, $\phi = \phi_2$, and $\rho = \rho_0$ is $\frac{1}{3}\rho_0^3(1-\cos\phi_2)(\theta_2-\theta_1) - \frac{1}{3}\rho_0^3(1-\cos\phi_1)(\theta_2-\theta_1) = \frac{1}{3}\rho_0^3(\cos\phi_1-\cos\phi_2)(\theta_2-\theta_1)$ so the volume of the spherical wedge between $\rho=\rho_1$ and $\rho=\rho_2$ is

$$\Delta V = \frac{1}{3}\rho_2^3(\cos\phi_1-\cos\phi_2)(\theta_2-\theta_1) - \frac{1}{3}\rho_1^3(\cos\phi_1-\cos\phi_2)(\theta_2-\theta_1)$$
$$= \frac{1}{3}(\rho_2^3-\rho_1^3)(\cos\phi_1-\cos\phi_2)(\theta_2-\theta_1)$$

(c) $\frac{d}{d\phi}\cos\phi = -\sin\phi$ so from the Mean-Value Theorem $\cos\phi_2 - \cos\phi_1 = -(\phi_2-\phi_1)\sin\phi^*$ where ϕ^* is between ϕ_1 and ϕ_2. Similarly $\frac{d}{d\rho}\rho^3 = 3\rho^2$ so $\rho_2^3-\rho_1^3 = 3\rho^{*2}(\rho_2-\rho_1)$ where ρ^* is between ρ_1 and ρ_2. Thus $\cos\phi_1-\cos\phi_2 = \sin\phi^*\Delta\phi$ and $\rho_2^3-\rho_1^3 = 3\rho^{*2}\Delta\rho$ so $\Delta V = \rho^{*2}\sin\phi^*\Delta\rho\Delta\phi\Delta\theta$.

39. $I_z = \int_0^{2\pi}\int_0^a\int_0^h r^2\delta\, r\,dz\,dr\,d\theta = \delta\int_0^{2\pi}\int_0^a\int_0^h r^3 dz\,dr\,d\theta = \frac{1}{2}\delta\pi a^4 h$

40. $I_y = \int_0^{2\pi}\int_0^a\int_0^h (r^2\cos^2\theta + z^2)\delta r\,dz\,dr\,d\theta = \delta\int_0^{2\pi}\int_0^a (hr^3\cos^2\theta + \frac{1}{3}h^3 r)dr\,d\theta$

$$= \delta\int_0^{2\pi}\left(\frac{1}{4}a^4 h\cos^2\theta + \frac{1}{6}a^2h^3\right)d\theta = \delta\left(\frac{\pi}{4}a^4h + \frac{\pi}{3}a^2h^3\right)$$

41. $I_z = \int_0^{2\pi}\int_{a_1}^{a_2}\int_0^h r^2\delta\, r\,dz\,dr\,d\theta = \delta\int_0^{2\pi}\int_{a_1}^{a_2}\int_0^h r^3 dz\,dr\,d\theta = \frac{1}{2}\delta\pi h(a_2^4 - a_1^4)$

42. $I_z = \int_0^{2\pi}\int_0^{\pi}\int_0^a (\rho^2\sin^2\phi)\delta\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \delta\int_0^{2\pi}\int_0^{\pi}\int_0^a \rho^4\sin^3\phi\,d\rho\,d\phi\,d\theta = \frac{8}{15}\delta\pi a^5$

EXERCISE SET 15.8

1. $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 4 \\ 3 & -5 \end{vmatrix} = -17$

2. $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 4v \\ 4u & -1 \end{vmatrix} = -1-16uv$

3. $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \cos u & -\sin v \\ \sin u & \cos v \end{vmatrix} = \cos u\cos v + \sin u\sin v = \cos(u-v)$

4. $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{2(v^2-u^2)}{(u^2+v^2)^2} & -\frac{4uv}{(u^2+v^2)^2} \\ \frac{4uv}{(u^2+v^2)^2} & \frac{2(v^2-u^2)}{(u^2+v^2)^2} \end{vmatrix} = 4/(u^2+v^2)^2$

5. $x = \frac{2}{9}u + \frac{5}{9}v$, $y = -\frac{1}{9}u + \frac{2}{9}v$; $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2/9 & 5/9 \\ -1/9 & 2/9 \end{vmatrix} = \frac{1}{9}$

6. $x = \ln u$, $y = uv$; $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/u & 0 \\ v & u \end{vmatrix} = 1$

7. $x = \sqrt{u+v}/\sqrt{2},\ y = \sqrt{v-u}/\sqrt{2};\ \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{1}{2\sqrt{2}\sqrt{u+v}} & \dfrac{1}{2\sqrt{2}\sqrt{u+v}} \\ -\dfrac{1}{2\sqrt{2}\sqrt{v-u}} & \dfrac{1}{2\sqrt{2}\sqrt{v-u}} \end{vmatrix} = \dfrac{1}{4\sqrt{v^2-u^2}}$

8. $x = u^{3/2}/v^{1/2},\ y = v^{1/2}/u^{1/2};\ \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{3u^{1/2}}{2v^{1/2}} & -\dfrac{u^{3/2}}{2v^{3/2}} \\ -\dfrac{v^{1/2}}{2u^{3/2}} & \dfrac{1}{2u^{1/2}v^{1/2}} \end{vmatrix} = \dfrac{1}{2v}$

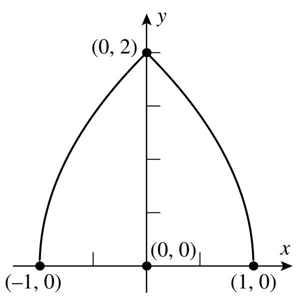
9. $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 3 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{vmatrix} = 5$

10. $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix} = u^2v$

11. $y = v, x = u/y = u/v, z = w - x = w - u/v; \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1/v & -u/v^2 & 0 \\ 0 & 1 & 0 \\ -1/v & u/v^2 & 1 \end{vmatrix} = 1/v$

12. $x = (v+w)/2, y = (u-w)/2, z = (u-v)/2, \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & -1/2 \\ 1/2 & -1/2 & 0 \end{vmatrix} = -\dfrac{1}{4}$

13.



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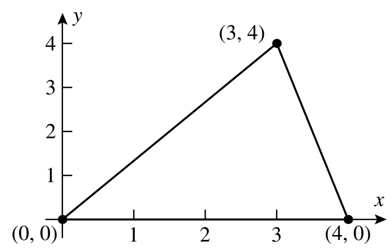

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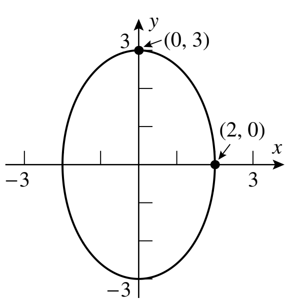

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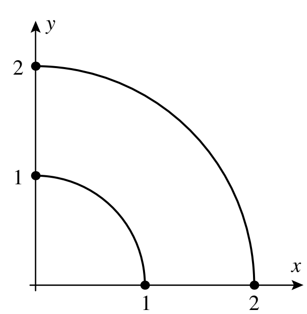


17. $x = \dfrac{1}{5}u + \dfrac{2}{5}v,\ y = -\dfrac{2}{5}u + \dfrac{1}{5}v,\ \dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{5};\ \ \dfrac{1}{5}\iint\limits_S \dfrac{u}{v}dA_{uv} = \dfrac{1}{5}\int_1^3\int_1^4 \dfrac{u}{v}\ du\ dv = \dfrac{3}{2}\ln 3$

18. $x = \frac{1}{2}u + \frac{1}{2}v,\ y = \frac{1}{2}u - \frac{1}{2}v,\ \frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2};\ \frac{1}{2}\iint\limits_S ve^{uv}dA_{uv} = \frac{1}{2}\int_1^4\int_0^1 ve^{uv}du\,dv = \frac{1}{2}(e^4 - e - 3)$

19. $x = u + v,\ y = u - v,\ \frac{\partial(x,y)}{\partial(u,v)} = -2$; the boundary curves of the region S in the uv-plane are $v = 0, v = u$, and $u = 1$ so $2\iint\limits_S \sin u \cos v dA_{uv} = 2\int_0^1\int_0^u \sin u \cos v\ dv\ du = 1 - \frac{1}{2}\sin 2$

20. $x = \sqrt{v/u},\ y = \sqrt{uv}$ so, from Example 3, $\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2u}$; the boundary curves of the region S in the uv-plane are $u = 1, u = 3, v = 1$, and $v = 4$ so $\iint\limits_S uv^2\left(\frac{1}{2u}\right)dA_{uv} = \frac{1}{2}\int_1^4\int_1^3 v^2du\ dv = 21$

21. $x = 3u, y = 4v, \frac{\partial(x,y)}{\partial(u,v)} = 12$; S is the region in the uv-plane enclosed by the circle $u^2 + v^2 = 1$. Use polar coordinates to obtain $\iint\limits_S 12\sqrt{u^2+v^2}(12)\,dA_{uv} = 144\int_0^{2\pi}\int_0^1 r^2dr\ d\theta = 96\pi$

22. $x = 2u, y = v, \frac{\partial(x,y)}{\partial(u,v)} = 2$; S is the region in the uv-plane enclosed by the circle $u^2 + v^2 = 1$. Use polar coordinates to obtain $\iint\limits_S e^{-(4u^2+4v^2)}(2)\,dA_{uv} = 2\int_0^{2\pi}\int_0^1 re^{-4r^2}dr\ d\theta = (1 - e^{-4})\pi/2$

23. Let S be the region in the uv-plane bounded by $u^2 + v^2 = 1$, so $u = 2x, v = 3y$,

$x = u/2, y = v/3, \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & 0 \\ 0 & 1/3 \end{vmatrix} = 1/6$, use polar coordinates to get

$$\frac{1}{6}\iint\limits_S \sin(u^2+v^2)du\,dv = \frac{1}{6}\int_0^{\pi/2}\int_0^1 r\sin r^2\,dr\,d\theta = \frac{\pi}{24}(-\cos r^2)\Big]_0^1 = \frac{\pi}{24}(1-\cos 1)$$

24. $u = x/a, v = y/b, x = au, y = bv;\ \frac{\partial(x,y)}{\partial(u,v)} = ab;\ A = ab\int_0^{2\pi}\int_0^1 r\,dr\,d\theta = \pi ab$

25. $x = u/3, y = v/2, z = w, \frac{\partial(x,y,z)}{\partial(u,v,w)} = 1/6$; S is the region in uvw-space enclosed by the sphere $u^2 + v^2 + w^2 = 36$ so

$$\begin{aligned}\iiint\limits_S \frac{u^2}{9}\frac{1}{6}dV_{uvw} &= \frac{1}{54}\int_0^{2\pi}\int_0^{\pi}\int_0^6 (\rho\sin\phi\cos\theta)^2\rho^2\sin\phi\ d\rho\ d\phi\ d\theta \\ &= \frac{1}{54}\int_0^{2\pi}\int_0^{\pi}\int_0^6 \rho^4\sin^3\phi\cos^2\theta d\rho\ d\phi\ d\theta = \frac{192}{5}\pi\end{aligned}$$

26. Let G_1 be the region $u^2+v^2+w^2 \le 1$, with $x = au, y = bv, z = cw$, $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = abc$; then use spherical coordinates in uvw-space:

$$I_x = \iiint_G (y^2+z^2)dx\,dy\,dz = abc\iiint_{G_1}(b^2v^2+c^2w^2)\,du\,dv\,dw$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^1 abc(b^2\sin^2\phi\sin^2\theta + c^2\cos^2\phi)\rho^4\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\frac{abc}{15}(4b^2\sin^2\theta + 2c^2)d\theta = \frac{4}{15}\pi abc(b^2+c^2)$$

27. $u = \theta = \cot^{-1}(x/y), v = r = \sqrt{x^2+y^2}$

28. $u = r = \sqrt{x^2+y^2}, v = (\theta+\pi/2)/\pi = (1/\pi)\tan^{-1}(y/x) + 1/2$

29. $u = \dfrac{3}{7}x - \dfrac{2}{7}y, v = -\dfrac{1}{7}x + \dfrac{3}{7}y$

30. $u = -x + \dfrac{4}{3}y, v = y$

31. Let $u = y - 4x, v = y + 4x$, then $x = \dfrac{1}{8}(v-u), y = \dfrac{1}{2}(v+u)$ so $\dfrac{\partial(x,y)}{\partial(u,v)} = -\dfrac{1}{8}$;

$$\frac{1}{8}\iint_S \frac{u}{v}dA_{uv} = \frac{1}{8}\int_2^5\int_0^2 \frac{u}{v}du\ dv = \frac{1}{4}\ln\frac{5}{2}$$

32. Let $u = y + x, v = y - x$, then $x = \dfrac{1}{2}(u-v), y = \dfrac{1}{2}(u+v)$ so $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{2}$;

$$-\frac{1}{2}\iint_S uv\ dA_{uv} = -\frac{1}{2}\int_0^2\int_0^1 uv\ du\ dv = -\frac{1}{2}$$

33. Let $u = x - y, v = x + y$, then $x = \dfrac{1}{2}(v+u), y = \dfrac{1}{2}(v-u)$ so $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{2}$; the boundary curves of the region S in the uv-plane are $u = 0, v = u$, and $v = \pi/4$; thus

$$\frac{1}{2}\iint_S \frac{\sin u}{\cos v}dA_{uv} = \frac{1}{2}\int_0^{\pi/4}\int_0^v \frac{\sin u}{\cos v}du\ dv = \frac{1}{2}[\ln(\sqrt{2}+1) - \pi/4]$$

34. Let $u = y - x, v = y + x$, then $x = \dfrac{1}{2}(v-u), y = \dfrac{1}{2}(u+v)$ so $\dfrac{\partial(x,y)}{\partial(u,v)} = -\dfrac{1}{2}$; the boundary curves of the region S in the uv-plane are $v = -u, v = u, v = 1$, and $v = 4$; thus

$$\frac{1}{2}\iint_S e^{u/v}dA_{uv} = \frac{1}{2}\int_1^4\int_{-v}^v e^{u/v}du\ dv = \frac{15}{4}(e - e^{-1})$$

35. Let $u = y/x, v = x/y^2$, then $x = 1/(u^2v), y = 1/(uv)$ so $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{u^4v^3}$;

$$\iint_S \frac{1}{u^4v^3}dA_{uv} = \int_1^4\int_1^2 \frac{1}{u^4v^3}du\ dv = 35/256$$

36. Let $x = 3u, y = 2v, \dfrac{\partial(x,y)}{\partial(u,v)} = 6$; S is the region in the uv-plane enclosed by the circle $u^2 + v^2 = 1$

so $\displaystyle\iint\limits_R (9 - x - y)dA = \iint\limits_S 6(9 - 3u - 2v)dA_{uv} = 6\int_0^{2\pi}\int_0^1 (9 - 3r\cos\theta - 2r\sin\theta)r\,dr\,d\theta = 54\pi$

37. $x = u, y = w/u, z = v + w/u, \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = -\dfrac{1}{u}$;

$$\iiint\limits_S \frac{v^2w}{u}dV_{uvw} = \int_2^4\int_0^1\int_1^3 \frac{v^2w}{u}du\ dv\ dw = 2\ln 3$$

38. $u = xy, v = yz, w = xz, 1 \le u \le 2, 1 \le v \le 3, 1 \le w \le 4$,

$x = \sqrt{uw/v}, y = \sqrt{uv/w}, z = \sqrt{vw/u}, \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \dfrac{1}{2\sqrt{uvw}}$

$$V = \iiint\limits_G dV = \int_1^2\int_1^3\int_1^4 \frac{1}{2\sqrt{uvw}}dw\,dv\,du = 4(\sqrt{2} - 1)(\sqrt{3} - 1)$$

39. **(b)** If $x = x(u,v), y = y(u,v)$ where $u = u(x,y), v = v(x,y)$, then by the chain rule

$$\frac{\partial x}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial x}{\partial x} = 1, \frac{\partial x}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial x}{\partial y} = 0$$

$$\frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial y}{\partial x} = 0, \frac{\partial y}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial y}{\partial y} = 1$$

40. **(a)** $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u;\quad u = x + y, v = \dfrac{y}{x+y}$,

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ -y/(x+y)^2 & x/(x+y)^2 \end{vmatrix} = \frac{x}{(x+y)^2} + \frac{y}{(x+y)^2} = \frac{1}{x+y} = \frac{1}{u};$$

$$\frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)} = 1$$

(b) $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} v & u \\ 0 & 2v \end{vmatrix} = 2v^2;\quad u = x/\sqrt{y}, v = \sqrt{y}$,

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1/\sqrt{y} & -x/(2y^{3/2}) \\ 0 & 1/(2\sqrt{y}) \end{vmatrix} = \frac{1}{2y} = \frac{1}{2v^2}; \frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)} = 1$$

(c) $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} u & v \\ u & -v \end{vmatrix} = -2uv;\quad u = \sqrt{x+y}, v = \sqrt{x-y}$,

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1/(2\sqrt{x+y}) & 1/(2\sqrt{x+y}) \\ 1/(2\sqrt{x-y}) & -1/(2\sqrt{x-y}) \end{vmatrix} = -\frac{1}{2\sqrt{x^2-y^2}} = -\frac{1}{2uv}; \frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)} = 1$$

41. $\dfrac{\partial(u,v)}{\partial(x,y)} = 3xy^4 = 3v$ so $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{3v}$; $\dfrac{1}{3}\displaystyle\iint\limits_S \frac{\sin u}{v}dA_{uv} = \frac{1}{3}\int_1^2\int_\pi^{2\pi} \frac{\sin u}{v}du\ dv = -\frac{2}{3}\ln 2$

42. $\dfrac{\partial(u,v)}{\partial(x,y)} = 8xy$ so $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{8xy}$; $xy\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = xy\left(\dfrac{1}{8xy}\right) = \dfrac{1}{8}$ so

$$\frac{1}{8}\iint_S dA_{uv} = \frac{1}{8}\int_9^{16}\int_1^4 du\,dv = 21/8$$

43. $\dfrac{\partial(u,v)}{\partial(x,y)} = -2(x^2+y^2)$ so $\dfrac{\partial(x,y)}{\partial(u,v)} = -\dfrac{1}{2(x^2+y^2)}$;

$$(x^4-y^4)e^{xy}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{x^4-y^4}{2(x^2+y^2)}e^{xy} = \frac{1}{2}(x^2-y^2)e^{xy} = \frac{1}{2}ve^u \text{ so}$$

$$\frac{1}{2}\iint_S ve^u dA_{uv} = \frac{1}{2}\int_3^4\int_1^3 ve^u du\,dv = \frac{7}{4}(e^3-e)$$

44. Set $u = x+y+2z, v = x-2y+z, w = 4x+y+z$, then $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 1 & 2 \\ 1 & -2 & 1 \\ 4 & 1 & 1 \end{vmatrix} = 18$, and

$$V = \iiint_R dx\,dy\,dz = \int_{-6}^{6}\int_{-2}^{2}\int_{-3}^{3}\frac{\partial(x,y,z)}{\partial(u,v,w)}du\,dv\,dw = 6(4)(12)\frac{1}{18} = 16$$

45. **(a)** Let $u = x+y, v = y$, then the triangle R with vertices $(0,0),(1,0)$ and $(0,1)$ becomes the triangle in the uv-plane with vertices $(0,0),(1,0),(1,1)$, and

$$\iint_R f(x+y)dA = \int_0^1\int_0^u f(u)\frac{\partial(x,y)}{\partial(u,v)}dv\,du = \int_0^1 uf(u)\,du$$

(b) $\displaystyle\int_0^1 ue^u\,du = (u-1)e^u\Big]_0^1 = 1$

46. **(a)** $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$, $\left|\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = r$

(b) $\dfrac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} = \rho^2\sin\phi$; $\left|\dfrac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}\right| = \rho^2\sin\phi$

CHAPTER 15 SUPPLEMENTARY EXERCISES

3. **(a)** $\displaystyle\iint_R dA$ **(b)** $\displaystyle\iiint_G dV$ **(c)** $\displaystyle\iint_R \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$

4. **(a)** $x = a\sin\phi\cos\theta, y = a\sin\phi\sin\theta, z = a\cos\phi, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi$
(b) $x = a\cos\theta, y = a\sin\theta, z = z, 0 \le \theta \le 2\pi, 0 \le z \le h$

7. $\displaystyle\int_0^1\int_{1-\sqrt{1-y^2}}^{1+\sqrt{1-y^2}} f(x,y)\,dx\,dy$

8. $\displaystyle\int_0^2\int_x^{2x} f(x,y)\,dy\,dx + \int_2^3\int_x^{6-x} f(x,y)\,dy\,dx$

9. **(a)** $(1,2)=(b,d),(2,1)=(a,c)$, so $a=2,b=1,c=1,d=2$

(b) $\iint\limits_R dA = \int_0^1\int_0^1 \frac{\partial(x,y)}{\partial(u,v)}du\,dv = \int_0^1\int_0^1 3du\,dv = 3$

10. If $0<x,y<\pi$ then $0<\sin\sqrt{xy}\le 1$, with equality only on the hyperbola $xy=\pi^2/4$, so

$$0=\int_0^\pi\int_0^\pi 0\,dy\,dx < \int_0^\pi\int_0^\pi \sin\sqrt{xy}\,dy\,dx < \int_0^\pi\int_0^\pi 1\,dy\,dx = \pi^2$$

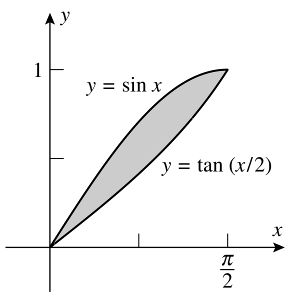
11. $\int_{1/2}^1 2x\cos(\pi x^2)\,dx = \frac{1}{\pi}\sin(\pi x^2)\Big]_{1/2}^1 = -1/(\sqrt{2}\pi)$

12. $\int_0^2 \frac{x^2}{2}e^{y^3}\Big]_{x=-y}^{x=2y} dy = \frac{3}{2}\int_0^2 y^2e^{y^3}\,dy = \frac{1}{2}e^{y^3}\Big]_0^2 = \frac{1}{2}\left(e^8-1\right)$

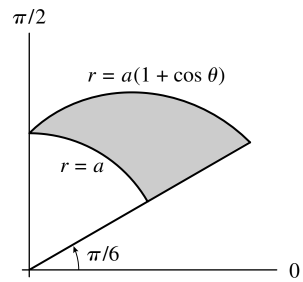
13. $\int_0^1\int_{2y}^2 e^xe^y\,dx\,dy$

14. $\int_0^\pi\int_0^x \frac{\sin x}{x}\,dy\,dx$

15.



16.



17. $2\int_0^8\int_0^{y^{1/3}} x^2\sin y^2dx\,dy = \frac{2}{3}\int_0^8 y\sin y^2\,dy = -\frac{1}{3}\cos y^2\Big]_0^8 = \frac{1}{3}(1-\cos 64)\approx 0.20271$

18. $\int_0^{\pi/2}\int_0^2 (4-r^2)r\,dr\,d\theta = 2\pi$

19. $\sin 2\theta = 2\sin\theta\cos\theta = \frac{2xy}{x^2+y^2}$, and $r=2a\sin\theta$ is the circle $x^2+(y-a)^2=a^2$, so

$$\int_0^a\int_{a-\sqrt{a^2-x^2}}^{a+\sqrt{a^2-x^2}} \frac{2xy}{x^2+y^2}dy\,dx = \int_0^a x\left[\ln\left(a+\sqrt{a^2-x^2}\right)-\ln\left(a-\sqrt{a^2-x^2}\right)\right]\,dx = a^2$$

20. $\int_{\pi/4}^{\pi/2}\int_0^2 4r^2(\cos\theta\sin\theta)\,r\,dr\,d\theta = -4\cos 2\theta\Big]_{\pi/4}^{\pi/2} = 4$

21. $\int_0^{2\pi}\int_0^2\int_{r^4}^{16} r^2\cos^2\theta\,r\,dz\,dr\,d\theta = \int_0^{2\pi}\cos^2\theta\,d\theta\int_0^2 r^3(16-r^4)\,dr = 32\pi$

22. $\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \frac{1}{1+\rho^2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \left(1-\frac{\pi}{4}\right)\frac{\pi}{2}\int_0^{\pi/2}\sin\phi\,d\phi$

$$= \left(1-\frac{\pi}{4}\right)\frac{\pi}{2}\left(-\cos\phi\right)\Big]_0^{\pi/2} = \left(1-\frac{\pi}{4}\right)\frac{\pi}{2}$$

23. **(a)** $\displaystyle\int_0^{2\pi}\int_0^{\pi/3}\int_0^{a}(\rho^2\sin^2\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\int_0^{2\pi}\int_0^{\pi/3}\int_0^{a}\rho^4\sin^3\phi\,d\rho\,d\phi\,d\theta$

(b) $\displaystyle\int_0^{2\pi}\int_0^{\sqrt{3}a/2}\int_{r/\sqrt{3}}^{\sqrt{a^2-r^2}}r^2\,dz\,r dr\,d\theta=\int_0^{2\pi}\int_0^{\sqrt{3}a/2}\int_{r/\sqrt{3}}^{\sqrt{a^2-r^2}}r^3\,dz\,dr\,d\theta$

(c) $\displaystyle\int_{-\sqrt{3}a/2}^{\sqrt{3}a/2}\int_{-\sqrt{(3a^2/4)-x^2}}^{\sqrt{(3a^2/4)-x^2}}\int_{\sqrt{x^2+y^2}/\sqrt{3}}^{\sqrt{a^2-x^2-y^2}}(x^2+y^2)\,dz\,dy\,dx$

24. **(a)** $\displaystyle\int_0^4\int_{-\sqrt{4x-x^2}}^{\sqrt{4x-x^2}}\int_{x^2+y^2}^{4x}dz\,dy\,dx$

(b) $\displaystyle\int_{-\pi/2}^{\pi/2}\int_0^{4\cos\theta}\int_{r^2}^{4r\cos\theta}r\,dz\,dr\,d\theta$

25. $\displaystyle\int_0^2\int_{(y/2)^{1/3}}^{2-y/2}dx\,dy=\int_0^2\left(2-\frac{y}{2}-\left(\frac{y}{2}\right)^{1/3}\right)dy=\left(2y-\frac{y^2}{4}-\frac{3}{2}\left(\frac{y}{2}\right)^{4/3}\right)\Big]_0^2=\frac{3}{2}$

26. $\displaystyle A=6\int_0^{\pi/6}\int_0^{\cos 3\theta}r\,dr\,d\theta=3\int_0^{\pi/6}\cos^2 3\theta=\pi/4$

27. $\displaystyle V=\int_0^{2\pi}\int_0^{a/\sqrt{3}}\int_{\sqrt{3}r}^{a}r\,dz\,dr\,d\theta=2\pi\int_0^{a/\sqrt{3}}r(a-\sqrt{3}r)\,dr=\frac{\pi a^3}{9}$

28. The intersection of the two surfaces projects onto the yz-plane as $2y^2+z^2=1$, so

$$V=4\int_0^{1/\sqrt{2}}\int_0^{\sqrt{1-2y^2}}\int_{y^2+z^2}^{1-y^2}dx\,dz\,dy$$
$$=4\int_0^{1/\sqrt{2}}\int_0^{\sqrt{1-2y^2}}(1-2y^2-z^2)\,dz\,dy=4\int_0^{1/\sqrt{2}}\frac{2}{3}(1-2y^2)^{3/2}\,dy=\frac{\sqrt{2}\pi}{4}$$

29. $\|\mathbf{r}_u\times\mathbf{r}_v\|=\sqrt{2u^2+2v^2+4}$,

$$S=\iint\limits_{u^2+v^2\le 4}\sqrt{2u^2+2v^2+4}\,dA=\int_0^{2\pi}\int_0^2\sqrt{2}\sqrt{r^2+2}\,r\,dr\,d\theta=\frac{8\pi}{3}(3\sqrt{3}-1)$$

30. $\displaystyle\|\mathbf{r}_u\times\mathbf{r}_v\|=\sqrt{1+u^2},\ S=\int_0^2\int_0^{3u}\sqrt{1+u^2}dv\,du=\int_0^2 3u\sqrt{1+u^2}du=5^{3/2}-1$

31. $(\mathbf{r}_u\times\mathbf{r}_v)\Big]_{\substack{u=1\\v=2}}=\langle-2,-4,1\rangle$, tangent plane $2x+4y-z=5$

32. $u=-3,v=0,\ (\mathbf{r}_u\times\mathbf{r}_v)\Big]_{\substack{u=-3\\v=0}}=\langle-18,0,-3\rangle$, tangent plane $6x+z=-9$

33. $\displaystyle A=\int_{-4}^4\int_{y^2/4}^{2+y^2/8}dx\,dy=\int_{-4}^4\left(2-\frac{y^2}{8}\right)dy=\frac{32}{3};\bar{y}=0$ by symmetry;

$$\int_{-4}^4\int_{y^2/4}^{2+y^2/8}x\,dx\,dy=\int_{-4}^4\left(2+\frac{1}{4}y^2-\frac{3}{128}y^4\right)dy=\frac{256}{15},\ \bar{x}=\frac{3}{32}\frac{256}{15}=\frac{8}{5};\text{ centroid }\left(\frac{8}{5},0\right)$$

34. $A = \pi ab/2, \bar{x} = 0$ by symmetry,

$$\int_{-a}^{a}\int_{0}^{b\sqrt{1-x^2/a^2}} y\,dy\,dx = \frac{1}{2}\int_{-a}^{a} b^2(1 - x^2/a^2)dx = 2ab^2/3, \text{ centroid } \left(0, \frac{4b}{3\pi}\right)$$

35. $V = \dfrac{1}{3}\pi a^2 h, \bar{x} = \bar{y} = 0$ by symmetry,

$$\int_0^{2\pi}\int_0^a\int_0^{h-rh/a} rz\,dz\,dr\,d\theta = \pi\int_0^a rh^2\left(1 - \frac{r}{a}\right)^2 dr = \pi a^2h^2/12, \text{ centroid } (0,0,h/4)$$

36. $V = \displaystyle\int_{-2}^{2}\int_{x^2}^{4}\int_0^{4-y} dz\,dy\,dx = \int_{-2}^{2}\int_{x^2}^{4}(4-y)dy\,dx = \int_{-2}^{2}\left(8 - 4x^2 + \frac{1}{2}x^4\right)dx = \frac{256}{15},$

$$\int_{-2}^{2}\int_{x^2}^{4}\int_0^{4-y} y\,dz\,dy\,dx = \int_{-2}^{2}\int_{x^2}^{4}(4y - y^2)\,dy\,dx = \int_{-2}^{2}\left(\frac{1}{3}x^6 - 2x^4 + \frac{32}{3}\right)dx = \frac{1024}{35}$$

$$\int_{-2}^{2}\int_{x^2}^{4}\int_0^{4-y} z\,dz\,dy\,dx = \int_{-2}^{2}\int_{x^2}^{4}\frac{1}{2}(4-y)^2dy\,dx = \int_{-2}^{2}\left(-\frac{x^6}{6} + 2x^4 - 8x^2 + \frac{32}{3}\right)dx = \frac{2048}{105}$$

$\bar{x} = 0$ by symmetry, centroid $\left(0, \dfrac{12}{7}, \dfrac{8}{7}\right)$

37. The two quarter-circles with center at the origin and of radius A and $\sqrt{2}A$ lie inside and outside of the square with corners $(0,0), (A,0), (A,A), (0,A)$, so the following inequalities hold:

$$\int_0^{\pi/2}\int_0^A \frac{1}{(1+r^2)^2}r\,dr\,d\theta \le \int_0^A\int_0^A \frac{1}{(1+x^2+y^2)^2}dx\,dy \le \int_0^{\pi/2}\int_0^{\sqrt{2}A}\frac{1}{(1+r^2)^2}r\,dr\,d\theta$$

The integral on the left can be evaluated as $\dfrac{\pi A^2}{4(1+A^2)}$ and the integral on the right equals $\dfrac{2\pi A^2}{4(1+2A^2)}$. Since both of these quantities tend to $\dfrac{\pi}{4}$ as $A \to +\infty$, it follows by sandwiching that $\displaystyle\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(1+x^2+y^2)^2}dx\,dy = \frac{\pi}{4}.$

38. The centroid of the circle which generates the tube travels a distance

$$s = \int_0^{4\pi}\sqrt{\sin^2 t + \cos^2 t + 1/16}\,dt = \sqrt{17}\pi, \text{ so } V = \pi(1/2)^2\sqrt{17}\pi = \sqrt{17}\pi^2/4.$$

39. **(a)** Let S_1 be the set of points (x,y,z) which satisfy the equation $x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$, and let S_2 be the set of points (x,y,z) where $x = a(\sin\phi\cos\theta)^3, y = a(\sin\phi\sin\theta)^3, z = a\cos^3\phi$, $0 \le \phi \le \pi, 0 \le \theta < 2\pi$.

If (x,y,z) is a point of S_2 then

$x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}[(\sin\phi\cos\theta)^3 + (\sin\phi\sin\theta)^3 + \cos^3\phi] = a^{2/3}$

so (x,y,z) belongs to S_1.

If (x,y,z) is a point of S_1 then $x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$. Let

$x_1 = x^{1/3}, y_1 = y^{1/3}, z_1 = z^{1/3}, a_1 = a^{1/3}$. Then $x_1^2 + y_1^2 + z_1^2 = a_1^2$, so in spherical coordinates $x_1 = a_1\sin\phi\cos\theta, y_1 = a_1\sin\phi\sin\theta, z_1 = a_1\cos\phi$, with

$\theta = \tan^{-1}\left(\dfrac{y_1}{x_1}\right) = \tan^{-1}\left(\dfrac{y}{x}\right)^{1/3}, \phi = \cos^{-1}\dfrac{z_1}{a_1} = \cos^{-1}\left(\dfrac{z}{a}\right)^{1/3}$. Then

$x = x_1^3 = a_1^3(\sin\phi\cos\theta)^3 = a(\sin\phi\cos\theta)^3$, similarly $y = a(\sin\phi\sin\theta)^3, z = a\cos\phi$ so (x,y,z) belongs to S_2. Thus $S_1 = S_2$

(b) Let $a = 1$ and $\mathbf{r} = (\cos\theta\sin\phi)^3\mathbf{i} + (\sin\theta\sin\phi)^3\mathbf{j} + \cos^3\phi\mathbf{k}$, then

$$S = 8\int_0^{\pi/2}\int_0^{\pi/2} \|\mathbf{r}_\theta \times \mathbf{r}_\phi\| d\phi\, d\theta$$

$$= 72\int_0^{\pi/2}\int_0^{\pi/2} \sin\theta\cos\theta\sin^4\phi\cos\phi\sqrt{\cos^2\phi + \sin^2\phi\sin^2\theta\cos^2\theta}\, d\theta\, d\phi \approx 4.4506$$

(c) $$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \left\| \begin{array}{ccc} \sin^3\phi\cos^3\theta & 3\rho\sin^2\phi\cos\phi\cos^3\theta & -3\rho\sin^3\phi\cos^2\theta\sin\theta \\ \sin^3\phi\sin^3\theta & 3\rho\sin^2\phi\cos\phi\sin^3\theta & 3\rho\sin^3\phi\sin^2\theta\cos\theta \\ \cos^3\phi & -3\rho\cos^2\phi\sin\phi & 0 \end{array} \right\|$$

$$= 9\rho^2\cos^2\theta\sin^2\theta\cos^2\phi\sin^5\phi,$$

$$V = 9\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^2\cos^2\theta\sin^2\theta\cos^2\phi\sin^5\phi\, d\rho\, d\phi\, d\theta = \frac{4}{35}\pi a^3$$

40. $$V = \frac{4}{3}\pi a^3, \bar{d} = \frac{3}{4\pi a^3}\iiint_{\rho\le a} \rho dV = \frac{3}{4\pi a^3}\int_0^{\pi}\int_0^{2\pi}\int_0^{a} \rho^3\sin\phi\, d\rho\, d\theta\, d\phi = \frac{3}{4\pi a^3}2\pi(2)\frac{a^4}{4} = \frac{3}{4}a$$

41. **(a)** $(x/a)^2 + (y/b)^2 + (z/c)^2 = \sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \cos^2\phi = \sin^2\phi + \cos^2\phi = 1$, an ellipsoid

(b) $\mathbf{r}(\phi,\theta) = \langle 2\sin\phi\cos\theta, 3\sin\phi\sin\theta, 4\cos\phi\rangle$; $\mathbf{r}_\phi \times \mathbf{r}_\theta = 2\langle 6\sin^2\phi\cos\theta, 4\sin^2\phi\sin\theta, 3\cos\phi\sin\phi\rangle$,

$$\|\mathbf{r}_\phi \times \mathbf{r}_\theta\| = 2\sqrt{16\sin^4\phi + 20\sin^4\phi\cos^2\theta + 9\sin^2\phi\cos^2\phi},$$

$$S = \int_0^{2\pi}\int_0^{\pi} 2\sqrt{16\sin^4\phi + 20\sin^4\phi\cos^2\theta + 9\sin^2\phi\cos^2\phi}\, d\phi\, d\theta \approx 111.5457699$$