CHAPTER 14
Partial Derivatives

EXERCISE SET 14.1

1. **(a)** $f(2,1) = (2)^2(1) + 1 = 5$ **(b)** $f(1,2) = (1)^2(2) + 1 = 3$
(c) $f(0,0) = (0)^2(0) + 1 = 1$ **(d)** $f(1,-3) = (1)^2(-3) + 1 = -2$
(e) $f(3a, a) = (3a)^2(a) + 1 = 9a^3 + 1$ **(f)** $f(ab,\, a-b) = (ab)^2(a-b) + 1 = a^3b^2 - a^2b^3 + 1$

2. **(a)** $2t$ **(b)** $2x$ **(c)** $2y^2 + 2y$

3. **(a)** $f(x+y, x-y) = (x+y)(x-y) + 3 = x^2 - y^2 + 3$
(b) $f\left(xy, 3x^2y^3\right) = (xy)\left(3x^2y^3\right) + 3 = 3x^3y^4 + 3$

4. **(a)** $(x/y)\sin(x/y)$ **(b)** $xy\sin(xy)$ **(c)** $(x-y)\sin(x-y)$

5. $F(g(x), h(y)) = F\left(x^3, 3y+1\right) = x^3e^{x^3(3y+1)}$

6. $g(u(x,y), v(x,y)) = g\left(x^2y^3, \pi xy\right) = \pi xy \sin\left[\left(x^2y^3\right)^2(\pi xy)\right] = \pi xy\sin\left(\pi x^5y^7\right)$

7. **(a)** $t^2 + 3t^{10}$ **(b)** 0 **(c)** 3076

8. $\sqrt{t}e^{-3\ln\left(t^2+1\right)} = \dfrac{\sqrt{t}}{\left(t^2+1\right)^3}$

9. **(a)** At $T = 25$ there is a drop in temperature of 12 degrees when v changes from 5 to 10, thus $WCI \approx (2/5)(-12) + 22 = 22 - 24/5 = 17.2^\circ$ F.
(b) At $v = 5$ there is an increase in temperature of 5 degrees as T changes from 25 to 30 degrees, thus $WCI \approx (3/5)5 + 22 = 25^\circ$ F.

10. **(a)** $T \approx (4/5)(-7) + 22 = 22 - 5.6 = 16.4^\circ$ F
(b) $T \approx (2/5)6 + 16 = 16 + 2.4 = 18.4^\circ$ F

11. **(a)** The depression is $20 - 16 = 4$, so the relative humidity is 66%.
(b) The relative humidity $\approx 77 - (1/2)7 = 73.5\%$.
(c) The relative humidity $\approx 59 + (2/5)4 = 60.6\%$.

12. **(a)** 4° C
(b) The relative humidity $\approx 62 - (1/4)9 = 59.75\%$.
(c) The relative humidity $\approx 77 + (1/5)(79 - 77) = 77.4\%$.

13. **(a)** 19 **(b)** -9 **(c)** 3
(d) $a^6 + 3$ **(e)** $-t^8 + 3$ **(f)** $(a+b)(a-b)^2b^3 + 3$

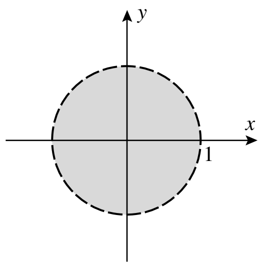
14. **(a)** $x^2(x+y)(x-y) + (x+y) = x^2\left(x^2 - y^2\right) + (x+y) = x^4 - x^2y^2 + x + y$
(b) $(xz)(xy)(y/x) + xy = xy^2z + xy$

15. $F\left(x^2, y+1, z^2\right) = (y+1)e^{x^2(y+1)z^2}$ **16.** $g\left(x^2z^3, \pi xyz, xy/z\right) = (xy/z)\sin\left(\pi x^3yz^4\right)$

17. **(a)** $f(\sqrt{5}, 2, \pi, -3\pi) = 80\sqrt{\pi}$ **(b)** $f(1, 1, \ldots, 1) = \displaystyle\sum_{k=1}^{n} k = n(n+1)/2$

18. **(a)** $f(-2,2,0,\pi/4)=1$
(b) $f(1,2,\ldots,n)=n(n+1)(2n+1)/6$, see Theorem 2(b), Section 5.4

19.



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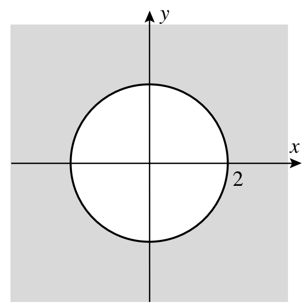

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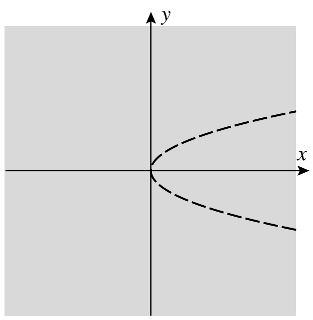

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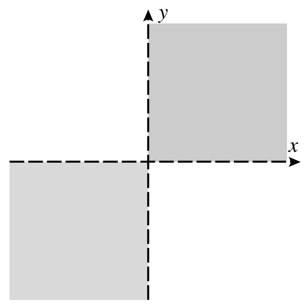

23. **(a)** all points above or on the line $y=-2$
(b) all points on or within the sphere $x^2+y^2+z^2=25$
(c) all points in 3-space

24. **(a)** all points on or between the vertical lines $x=\pm 2$.
(b) all points above the line $y=2x$
(c) all points not on the plane $x+y+z=0$

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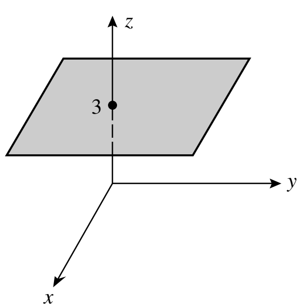

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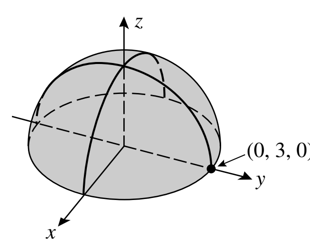

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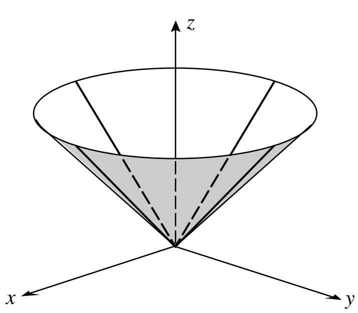

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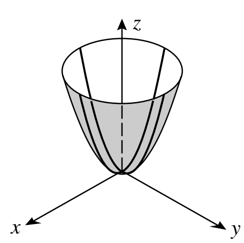

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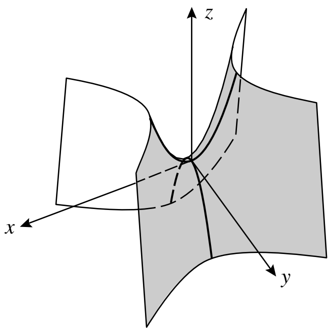

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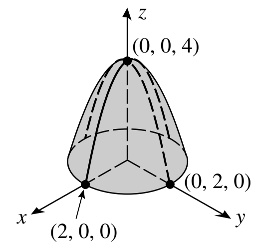

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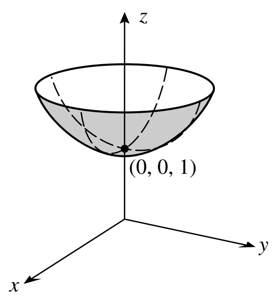

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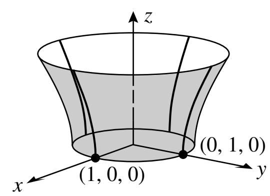

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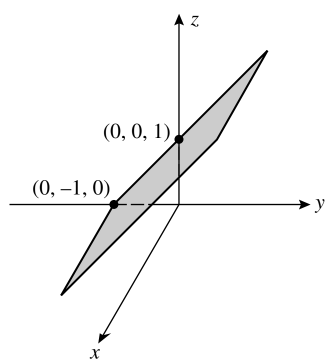

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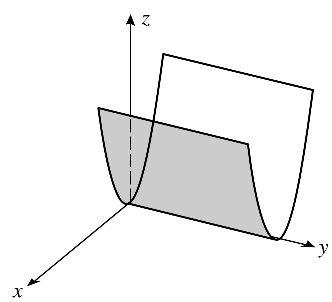

35. **(a)** $f(x, y) = 1 - x^2 - y^2$, because $f = c$ is a circle of radius $\sqrt{1-c}$ (provided $c \le 1$), and the radii in (a) decrease as c increases.

(b) $f(x, y) = \sqrt{x^2 + y^2}$ because $f = c$ is a circle of radius c, and the radii increase uniformly.

(c) $f(x, y) = x^2 + y^2$ because $f = c$ is a circle of radius $\sqrt{c}$ and the radii in the plot grow like the square root function.

36. **(a)** III, because the surface has 9 peaks along the edges, three peaks to each edge

(b) IV, because the center is relatively flat and the deep valley in the first quadrant points in the direction of the positive x-axis

(c) I, because the deep valley in the first quadrant points in the direction of the positive y-axis

(d) II, because the surface has four peaks

37. **(a)** A **(b)** B **(c)** increase
(d) decrease **(e)** increase **(f)** decrease

38. **(a)** Medicine Hat, since the contour lines are closer together near Medicine Hat than they are near Chicago.

(b) The change in atmospheric pressure is about $\Delta p \approx 999 - 1010 = -11$, so the average rate of change is $\Delta p/1400 \approx -0.0079$.

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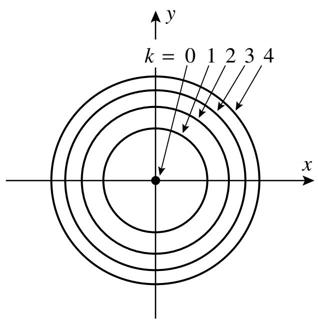

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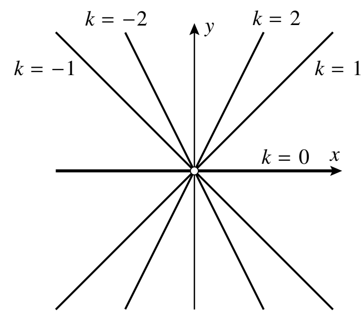

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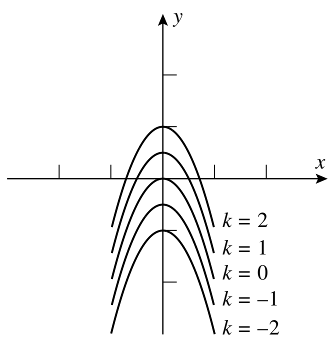

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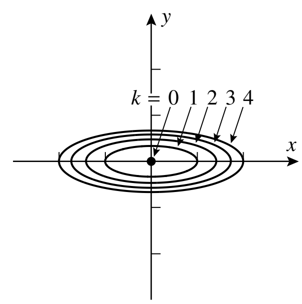

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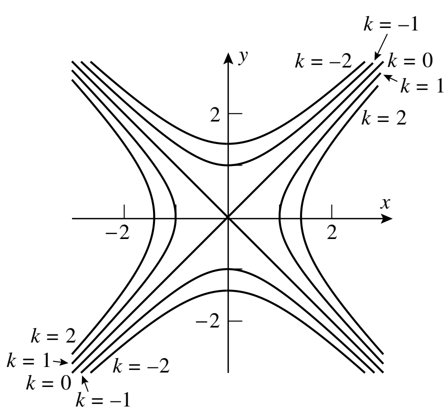

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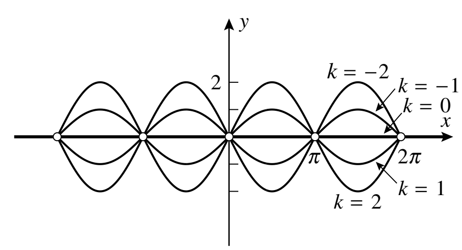

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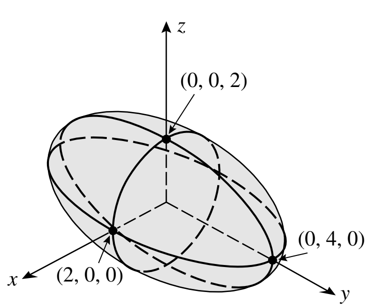

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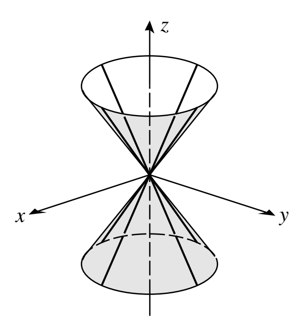

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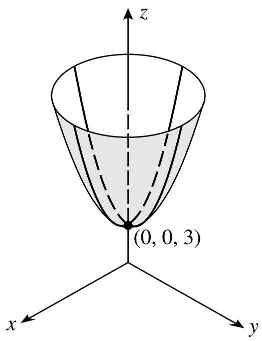

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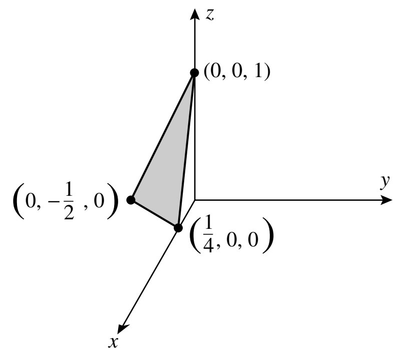


49. concentric spheres, common center at (2,0,0)

50. parallel planes, common normal $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

51. concentric cylinders, common axis the y-axis

52. circular paraboloids, common axis the z-axis, all the same shape but with different vertices along z-axis.

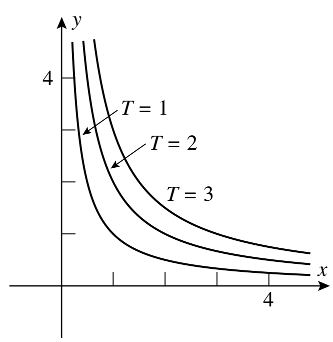
53. **(a)** $f(-1, 1) = 0$; $x^2 - 2x^3 + 3xy = 0$ **(b)** $f(0, 0) = 0$; $x^2 - 2x^3 + 3xy = 0$
(c) $f(2, -1) = -18$; $x^2 - 2x^3 + 3xy = -18$

54. **(a)** $f(\ln 2,\, 1) = 2$; $ye^x = 2$ **(b)** $f(0, 3) = 3$; $ye^x = 3$
(c) $f(1, -2) = -2e$; $ye^x = -2e$

55. **(a)** $f(1, -2, 0) = 5$; $x^2 + y^2 - z = 5$ **(b)** $f(1, 0, 3) = -2$; $x^2 + y^2 - z = -2$
(c) $f(0, 0, 0) = 0$; $x^2 + y^2 - z = 0$

56. **(a)** $f(1, 0, 2) = 3$; $xyz + 3 = 3$, $xyz = 0$ **(b)** $f(-2, 4, 1) = -5$; $xyz + 3 = -5$, $xyz = -8$
(c) $f(0, 0, 0) = 3$; $xyz = 0$

57. **(a)**

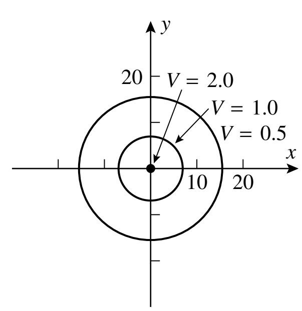


(b) At $(1, 4)$ the temperature is $T(1, 4) = 4$ so the temperature will remain constant along the path $xy = 4$.

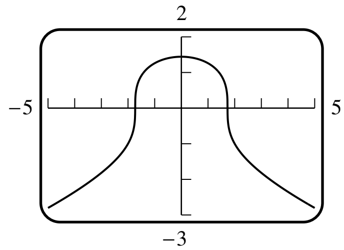
58. $V = \dfrac{8}{\sqrt{16 + x^2 + y^2}}$

$x^2 + y^2 = \dfrac{64}{V^2} - 16$

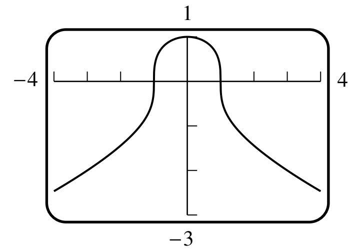
the equipotential curves are circles.



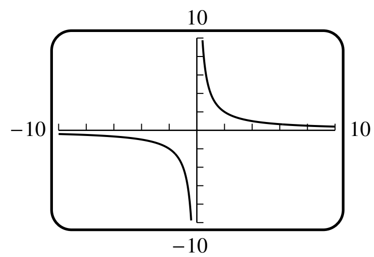
59. **(a)**



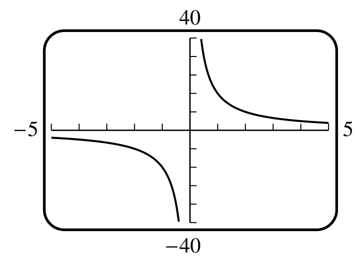
(b)



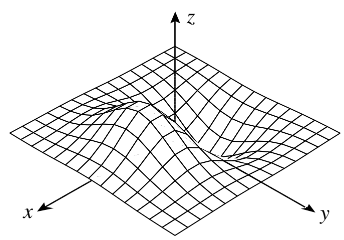
60. **(a)**



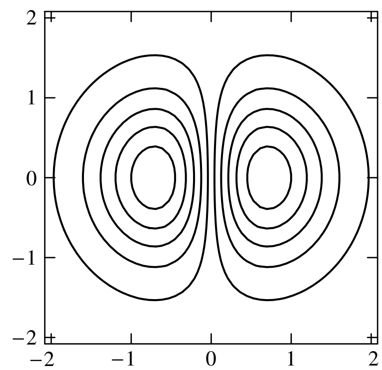
(b)



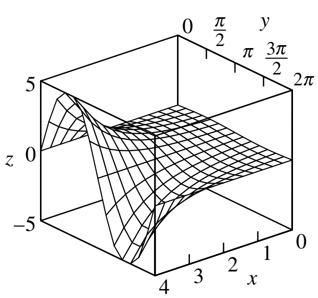
61. **(a)**



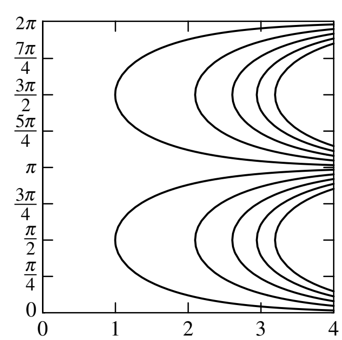
(b)



62. **(a)**



(b)

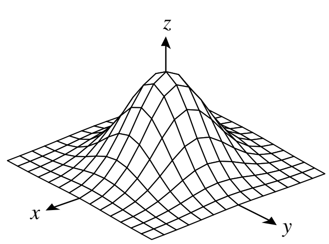


63. **(a)** The graph of g is the graph of f shifted one unit in the positive x-direction.

(b) The graph of g is the graph of f shifted one unit up the z-axis.

(c) The graph of g is the graph of f shifted one unit down the y-axis and then inverted with respect to the plane $z = 0$.

64. **(a)**



(b) If a is positive and increasing then the graph of g is more pointed, and in the limit as $a \to +\infty$ the graph approaches a 'spike' on the z-axis of height 1. As a decreases to zero the graph of g gets flatter until it finally approaches the plane $z = 1$.

EXERCISE SET 14.2

1. 35 **2.** $\pi^2/2$ **3.** -8 **4.** e^{-7} **5.** 0 **6.** 0

7. **(a)** Along $x = 0$ $\lim\limits_{(x,y)\to(0,0)} \dfrac{3}{x^2+2y^2} = \lim\limits_{y\to 0} \dfrac{3}{2y^2}$ does not exist.

(b) Along $x = 0$, $\lim\limits_{(x,y)\to(0,0)} \dfrac{x+y}{x+y^2} = \lim\limits_{y\to 0} \dfrac{1}{y}$ does not exist.

8. **(a)** Along $y = 0 : \lim\limits_{x\to 0} \dfrac{x}{x^2} = \lim\limits_{x\to 0} \dfrac{1}{x}$ does not exist because $\left|\dfrac{1}{x}\right| \to +\infty$ as $x \to 0$ so the original limit does not exist.

(b) Along $y = 0 : \lim\limits_{x\to 0} \dfrac{1}{x}$ does not exist, so the original limit does not exist.

9. Let $z = x^2 + y^2$, then $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin\left(x^2+y^2\right)}{x^2+y^2} = \lim\limits_{z\to 0^+} \dfrac{\sin z}{z} = 1$

10. Let $z = x^2 + y^2$, then $\lim\limits_{(x,y)\to(0,0)} \dfrac{1-\cos\left(x^2+y^2\right)}{x^2+y^2} = \lim\limits_{z\to 0^+} \dfrac{1-\cos z}{z} = \lim\limits_{z\to 0^+} \dfrac{\sin z}{1} = 0$

11. Let $z = x^2 + y^2$, then $\lim\limits_{(x,y)\to(0,0)} e^{-1/\left(x^2+y^2\right)} = \lim\limits_{z\to 0^+} e^{-1/z} = 0$

12. With $z = x^2 + y^2$, $\lim\limits_{z\to+\infty} \dfrac{1}{\sqrt{z}} e^{-1/\sqrt{z}}$; let $w = \dfrac{1}{\sqrt{z}}$, $\lim\limits_{w\to+\infty} \dfrac{w}{e^w} = 0$

13. $\lim\limits_{(x,y)\to(0,0)} \dfrac{\left(x^2+y^2\right)\left(x^2-y^2\right)}{x^2+y^2} = \lim\limits_{(x,y)\to(0,0)} \left(x^2-y^2\right) = 0$

14. $\lim\limits_{(x,y)\to(0,0)} \dfrac{\left(x^2+4y^2\right)\left(x^2-4y^2\right)}{x^2+4y^2} = \lim\limits_{(x,y)\to(0,0)} \left(x^2-4y^2\right) = 0$

15. along $y = 0 : \lim\limits_{x\to 0} \dfrac{0}{3x^2} = \lim\limits_{x\to 0} 0 = 0$; along $y = x : \lim\limits_{x\to 0} \dfrac{x^2}{5x^2} = \lim\limits_{x\to 0} 1/5 = 1/5$
so the limit does not exist.

16. Let $z = x^2 + y^2$, then $\lim\limits_{(x,y)\to(0,0)} \dfrac{1-x^2-y^2}{x^2+y^2} = \lim\limits_{z\to 0^+} \dfrac{1-z}{z} = +\infty$ so the limit does not exist.

17. 8/3 **18.** $\ln 5$

19. Let $t = \sqrt{x^2+y^2+z^2}$, then $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{\sin\left(x^2+y^2+z^2\right)}{\sqrt{x^2+y^2+z^2}} = \lim\limits_{t\to 0^+} \dfrac{\sin\left(t^2\right)}{t} = 0$

20. With $t=\sqrt{x^2+y^2+z^2}$, $\lim_{t\to 0^+}\dfrac{\sin t}{t^2}=\lim_{t\to 0^+}\dfrac{\cos t}{2t}=+\infty$ so the limit does not exist.

21. $y\ln(x^2+y^2)=r\sin\theta\ln r^2=2r(\ln r)\sin\theta$, so $\lim_{(x,y)\to(0,0)} y\ln(x^2+y^2)=\lim_{r\to 0^+}2r(\ln r)\sin\theta=0$

22. $\dfrac{x^2y^2}{\sqrt{x^2+y^2}}=\dfrac{(r^2\cos^2\theta)(r^2\sin^2\theta)}{r}=r^3\cos^2\theta\sin^2\theta$, so $\lim_{(x,y)\to(0,0)}\dfrac{x^2y^2}{\sqrt{x^2+y^2}}=0$

23. $\dfrac{e^{\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}=\dfrac{e^\rho}{\rho}$, so $\lim_{(x,y,z)\to(0,0,0)}\dfrac{e^{\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}=\lim_{\rho\to 0^+}\dfrac{e^\rho}{\rho}$ does not exist.

24. $\lim_{(x,y,z)\to(0,0,0)}\tan^{-1}\left[\dfrac{1}{x^2+y^2+z^2}\right]=\lim_{\rho\to 0^+}\tan^{-1}\dfrac{1}{\rho^2}=\dfrac{\pi}{2}$

25. **(a)** No, since there seem to be points near $(0,0)$ with $z=0$ and other points near $(0,0)$ with $z\approx 1/2$.

(b) $\lim_{x\to 0}\dfrac{mx^3}{x^4+m^2x^2}=\lim_{x\to 0}\dfrac{mx}{x^2+m^2}=0$ **(c)** $\lim_{x\to 0}\dfrac{x^4}{2x^4}=\lim_{x\to 0}1/2=1/2$

(d) A limit must be unique if it exists, so $f(x,y)$ cannot have a limit as $(x,y)\to(0,0)$.

26. **(a)** Along $y=mx$: $\lim_{x\to 0}\dfrac{mx^4}{2x^6+m^2x^2}=\lim_{x\to 0}\dfrac{mx^2}{2x^4+m^2}=0$;

along $y=kx^2$: $\lim_{x\to 0}\dfrac{kx^5}{2x^6+k^2x^4}=\lim_{x\to 0}\dfrac{kx}{2x^2+k^2}=0$.

(b) $\lim_{x\to 0}\dfrac{x^6}{2x^6+x^6}=\lim_{x\to 0}\dfrac{1}{3}=\dfrac{1}{3}\neq 0$

27. **(a)** $\lim_{t\to 0}\dfrac{abct^3}{a^2t^2+b^4t^4+c^4t^4}=\lim_{t\to 0}\dfrac{abct}{a^2+b^4t^2+c^4t^2}=0$

(b) $\lim_{t\to 0}\dfrac{t^4}{t^4+t^4+t^4}=\lim_{t\to 0}1/3=1/3$

28. $\pi/2$ because $\dfrac{x^2+1}{x^2+(y-1)^2}\to+\infty$ as $(x,y)\to(0,1)$

29. $-\pi/2$ because $\dfrac{x^2-1}{x^2+(y-1)^2}\to-\infty$ as $(x,y)\to(0,1)$

30. with $z=x^2+y^2$, $\lim_{z\to 0^+}\dfrac{\sin z}{z}=1=f(0,0)$

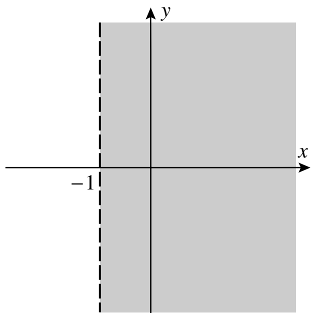
31. No, because $\lim_{(x,y)\to(0,0)}\dfrac{x^2}{x^2+y^2}$ does not exist.

Along $x=0$: $\lim_{y\to 0}\left(0/y^2\right)=\lim_{y\to 0}0=0$; along $y=0$: $\lim_{x\to 0}\left(x^2/x^2\right)=\lim_{x\to 0}1=1$.

32. Using polar coordinates with $r>0$, $xy=r^2\sin\theta\cos\theta$ and $x^2+y^2=r^2$ so

$|xy\ln\left(x^2+y^2\right)|=|r^2\sin\theta\cos\theta\ln r^2|\le|2r^2\ln r|$, but $\lim_{r\to 0^+}2r^2\ln r=0$ thus

$\lim_{(x,y)\to(0,0)}xy\ln\left(x^2+y^2\right)=0$; $f(x,y)$ will be continuous at (0,0) if we define $f(0,0)=0$.

33.


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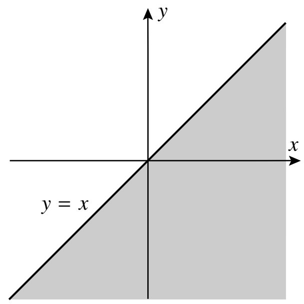

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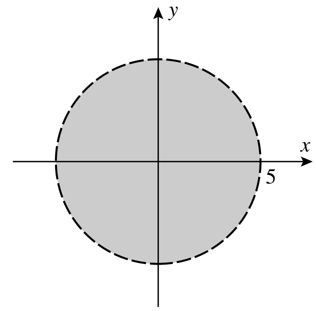

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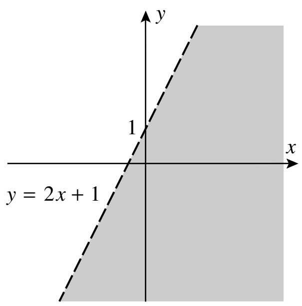

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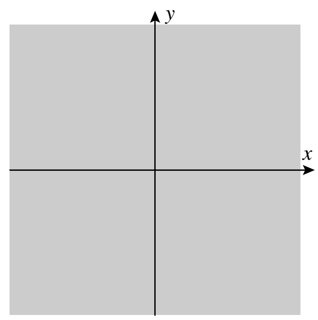

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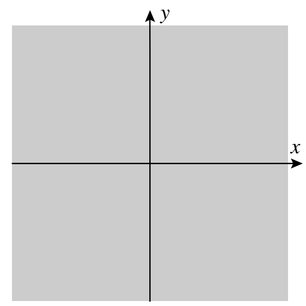

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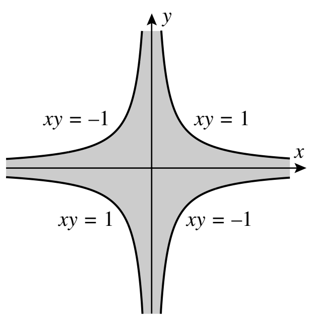

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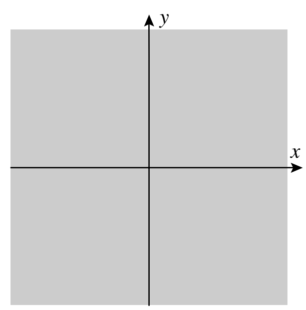


41. all of 3-space

42. all points inside the sphere with radius 2 and center at the origin

43. all points not on the cylinder $x^2 + z^2 = 1$

44. all of 3-space

EXERCISE SET 14.3

1. **(a)** $9x^2y^2$ **(b)** $6x^3y$ **(c)** $9y^2$ **(d)** $9x^2$
(e) $6y$ **(f)** $6x^3$ **(g)** 36 **(h)** 12

2. **(a)** $2e^{2x}\sin y$ **(b)** $e^{2x}\cos y$ **(c)** $2\sin y$ **(d)** 0
(e) $\cos y$ **(f)** e^{2x} **(g)** 0 **(h)** 4

3. **(a)** $\dfrac{\partial z}{\partial x} = \dfrac{3}{2\sqrt{3x+2y}}$; slope $= \dfrac{3}{8}$ **(b)** $\dfrac{\partial z}{\partial y} = \dfrac{1}{\sqrt{3x+2y}}$; slope $= \dfrac{1}{4}$

4. **(a)** $\dfrac{\partial z}{\partial x} = e^{-y}$; slope $= 1$ **(b)** $\dfrac{\partial z}{\partial y} = -xe^{-y} + 5$; slope $= 2$

5. **(a)** $\dfrac{\partial z}{\partial x} = -4\cos(y^2 - 4x)$; rate of change $= -4\cos 7$

(b) $\dfrac{\partial z}{\partial y} = 2y\cos(y^2 - 4x)$; rate of change $= 2\cos 7$

6. **(a)** $\dfrac{\partial z}{\partial x} = -\dfrac{1}{(x+y)^2}$; rate of change $= -\dfrac{1}{4}$ **(b)** $\dfrac{\partial z}{\partial y} = -\dfrac{1}{(x+y)^2}$; rate of change $= -\dfrac{1}{4}$

7. $\partial z/\partial x =$ slope of line parallel to xz-plane $= -4$; $\partial z/\partial y =$ slope of line parallel to yz-plane $= 1/2$

8. Moving to the right from (x_0, y_0) decreases $f(x, y)$, so $f_x < 0$; moving up increases f, so $f_y > 0$.

9. **(a)** The right-hand estimate is $\partial r/\partial v \approx (222 - 197)/(85 - 80) = 5$; the left-hand estimate is $\partial r/\partial v \approx (197 - 173)/(80 - 75) = 4.8$; the average is $\partial r/\partial v \approx 4.9$.

(b) The right-hand estimate is $\partial r/\partial \theta \approx (200 - 197)/(45 - 40) = 0.6$; the left-hand estimate is $\partial r/\partial \theta \approx (197 - 188)/(40 - 35) = 1.8$; the average is $\partial r/\partial \theta \approx 1.2$.

10. **(a)** The right-hand estimate is $\partial r/\partial v \approx (253 - 226)/(90 - 85) = 5.4$; the left-hand estimate is $(226 - 200)/(85 - 80) = 5.2$; the average is $\partial r/\partial v \approx 5.3$.

(b) The right-hand estimate is $\partial r/\partial \theta \approx (222 - 226)/(50 - 45) = -0.8$; the left-hand estimate is $(226 - 222)/(45 - 40) = 0.8$; the average is $\partial r/\partial v \approx 0$.

11. $\partial z/\partial x = 8xy^3 e^{x^2y^3}$, $\partial z/\partial y = 12x^2y^2 e^{x^2y^3}$

12. $\partial z/\partial x = -5x^4y^4 \sin(x^5y^4)$, $\partial z/\partial y = -4x^5y^3 \sin\left(x^5y^4\right)$

13. $\partial z/\partial x = x^3/(y^{3/5} + x) + 3x^2 \ln(1 + xy^{-3/5})$, $\partial z/\partial y = -(3/5)x^4/(y^{8/5} + xy)$

14. $\partial z/\partial x = ye^{xy}\sin(4y^2)$, $\partial z/\partial y = 8ye^{xy}\cos(4y^2) + xe^{xy}\sin(4y^2)$

15. $\dfrac{\partial z}{\partial x} = -\dfrac{y(x^2 - y^2)}{(x^2 + y^2)^2}$, $\dfrac{\partial z}{\partial y} = \dfrac{x(x^2 - y^2)}{(x^2 + y^2)^2}$

16. $\dfrac{\partial z}{\partial x} = \dfrac{xy^3(3x + 4y)}{2(x + y)^{3/2}}$, $\dfrac{\partial z}{\partial y} = \dfrac{x^2y^2(6x + 5y)}{2(x + y)^{3/2}}$

17. $f_x(x, y) = (3/2)x^2y\left(5x^2 - 7\right)\left(3x^5y - 7x^3y\right)^{-1/2}$

$f_y(x, y) = (1/2)x^3\left(3x^2 - 7\right)\left(3x^5y - 7x^3y\right)^{-1/2}$

18. $f_x(x, y) = -2y/(x - y)^2$, $f_y(x, y) = 2x/(x - y)^2$

19. $f_x(x, y) = \dfrac{y^{-1/2}}{y^2 + x^2}$, $f_y(x, y) = -\dfrac{xy^{-3/2}}{y^2 + x^2} - \dfrac{3}{2}y^{-5/2}\tan^{-1}(x/y)$

20. $f_x(x, y) = 3x^2e^{-y} + (1/2)x^{-1/2}y^3 \sec\sqrt{x}\tan\sqrt{x}$, $f_y(x, y) = -x^3e^{-y} + 3y^2\sec\sqrt{x}$

21. $f_x(x, y) = -(4/3)y^2\sec^2 x\left(y^2\tan x\right)^{-7/3}$, $f_y(x, y) = -(8/3)y\tan x\left(y^2\tan x\right)^{-7/3}$

22. $f_x(x, y) = 2y^2\cosh\sqrt{x}\sinh\left(xy^2\right)\cosh\left(xy^2\right) + \dfrac{1}{2}x^{-1/2}\sinh\sqrt{x}\sinh^2\left(xy^2\right)$

$f_y(x, y) = 4xy\cosh\sqrt{x}\sinh\left(xy^2\right)\cosh\left(xy^2\right)$

23. $f_x(x,y) = -2x,\ f_x(3,1) = -6;\ f_y(x,y) = -21y^2,\ f_y(3,1) = -21$

24. $\partial f/\partial x = x^2y^2e^{xy} + 2xye^{xy},\ \partial f/\partial x\,\big|_{(1,1)} = 3e;\ \partial f/\partial y = x^3ye^{xy} + x^2e^{xy},\ \partial f/\partial y\,\big|_{(1,1)} = 2e$

25. $\partial z/\partial x = x(x^2+4y^2)^{-1/2},\ \partial z/\partial x\,\big|_{(1,2)} = 1/\sqrt{17}\ ;\ \partial z/\partial y = 4y(x^2+4y^2)^{-1/2},\ \partial z/\partial y\,\big|_{(1,2)} = 8/\sqrt{17}$

26. $\partial w/\partial x = -x^2y\sin xy + 2x\cos xy,\ \dfrac{\partial w}{\partial x}(1/2,\pi) = -\pi/4;\ \partial w/\partial y = -x^3\sin xy,\ \dfrac{\partial w}{\partial y}(1/2,\pi) = -1/8$

27. **(a)** $2xy^4z^3 + y$ **(b)** $4x^2y^3z^3 + x$ **(c)** $3x^2y^4z^2 + 2z$
(d) $2y^4z^3 + y$ **(e)** $32z^3 + 1$ **(f)** 438

28. **(a)** $2xy\cos z$ **(b)** $x^2\cos z$ **(c)** $-x^2y\sin z$
(d) $4y\cos z$ **(e)** $4\cos z$ **(f)** 0

29. $f_x = 2z/x,\ f_y = z/y,\ f_z = \ln(x^2y\cos z) - z\tan z$

30. $f_x = y^{-5/2}z\sec(xz/y)\tan(xz/y),\ f_y = -xy^{-7/2}z\sec(xz/y)\tan(xz/y) - (3/2)y^{-5/2}\sec(xz/y),$
$f_z = xy^{-5/2}\sec(xz/y)\tan(xz/y)$

31. $f_x = -y^2z^3/\left(1 + x^2y^4z^6\right),\ f_y = -2xyz^3/\left(1 + x^2y^4z^6\right),\ f_z = -3xy^2z^2/\left(1 + x^2y^4z^6\right)$

32. $f_x = 4xyz\cosh\sqrt{z}\sinh\left(x^2yz\right)\cosh\left(x^2yz\right),\ f_y = 2x^2z\cosh\sqrt{z}\sinh\left(x^2yz\right)\cosh\left(x^2yz\right),$
$f_z = 2x^2y\cosh\sqrt{z}\sinh\left(x^2yz\right)\cosh\left(x^2yz\right) + (1/2)z^{-1/2}\sinh\sqrt{z}\sinh^2\left(x^2yz\right)$

33. $\partial w/\partial x = yze^z\cos xz,\ \partial w/\partial y = e^z\sin xz,\ \partial w/\partial z = ye^z(\sin xz + x\cos xz)$

34. $\partial w/\partial x = 2x/\left(y^2+z^2\right),\ \partial w/\partial y = -2y\left(x^2+z^2\right)/\left(y^2+z^2\right)^2,\ \partial w/\partial z = 2z\left(y^2-x^2\right)/\left(y^2+z^2\right)^2$

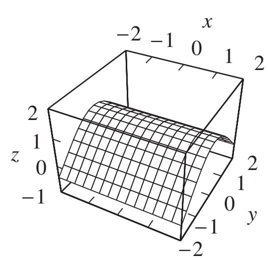
35. $\partial w/\partial x = x/\sqrt{x^2+y^2+z^2},\ \partial w/\partial y = y/\sqrt{x^2+y^2+z^2},\ \partial w/\partial z = z/\sqrt{x^2+y^2+z^2}$

36. $\partial w/\partial x = 2y^3e^{2x+3z},\ \partial w/\partial y = 3y^2e^{2x+3z},\ \partial w/\partial z = 3y^3e^{2x+3z}$

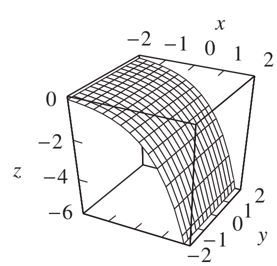
37. **(a)** e **(b)** $2e$ **(c)** e

38. **(a)** $2/\sqrt{7}$ **(b)** $4/\sqrt{7}$ **(c)** $1/\sqrt{7}$

39. **(a)**



(b)



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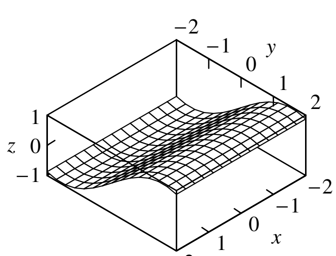
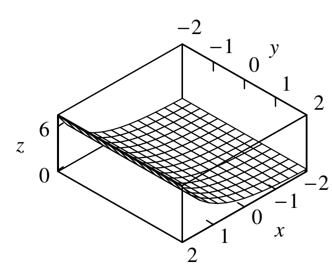




41. $\partial z/\partial x = 2x + 6y(\partial y/\partial x) = 2x,\ \partial z/\partial x\big]_{(2,1)} = 4$

42. $\partial z/\partial y = 6y,\ \partial z/\partial y\,\big|_{(2,1)} = 6$

43. $\partial z/\partial x = -x\left(29 - x^2 - y^2\right)^{-1/2},\ \partial z/\partial x]_{(4,3)} = -2$

44. **(a)** $\partial z/\partial y = 8y,\ \partial z/\partial y]_{(-1,1)} = 8$ **(b)** $\partial z/\partial x = 2x,\ \partial z/\partial x]_{(-1,1)} = -2$

45. **(a)** $\partial V/\partial r = 2\pi rh$ **(b)** $\partial V/\partial h = \pi r^2$
(c) $\partial V/\partial r]_{r=6,\ h=4} = 48\pi$ **(d)** $\partial V/\partial h]_{r=8,\ h=10} = 64\pi$

46. **(a)** $\partial V/\partial s = \dfrac{\pi s d^2}{6\sqrt{4s^2 - d^2}}$ **(b)** $\partial V/\partial d = \dfrac{\pi d(8s^2 - 3d^2)}{24\sqrt{4s^2 - d^2}}$
(c) $\partial V/\partial s]_{s=10,\ d=16} = 320\pi/9$ **(d)** $\partial V/\partial d]_{s=10,\ d=16} = 16\pi/9$

47. **(a)** $P = 10T/V,\ \partial P/\partial T = 10/V,\ \partial P/\partial T]_{T=80,\ V=50} = 1/5\ \text{lb}/(\text{in}^2\text{K})$
(b) $V = 10T/P, \partial V/\partial P = -10T/P^2,$ if $V = 50$ and $T = 80$ then
$P = 10(80)/(50) = 16,\ \partial V/\partial P]_{T=80,\ P=16} = -25/8(\text{in}^5/\text{lb})$

48. **(a)** $\partial T/\partial x = 3x^2 + 1,\ \partial T/\partial x]_{(1,2)} = 4$ **(b)** $\partial T/\partial y = 4y,\ \partial T/\partial y]_{(1,2)} = 8$

49. **(a)** $V = lwh, \partial V/\partial l = wh = 6$
(b) $\partial V/\partial w = lh = 15$
(c) $\partial V/\partial h = lw = 10$

50. **(a)** $\partial A/\partial a = (1/2)b\sin\theta = (1/2)(10)\left(\sqrt{3}/2\right) = 5\sqrt{3}/2$
(b) $\partial A/\partial \theta = (1/2)ab\cos\theta = (1/2)(5)(10)(1/2) = 25/2$
(c) $b = (2A\csc\theta)/a,\ \partial b/\partial a = -(2A\csc\theta)/a^2 = -b/a = -2$

51. $\partial V/\partial r = \dfrac{2}{3}\pi rh = \dfrac{2}{r}(\dfrac{1}{3}\pi r^2 h) = 2V/r$

52. **(a)** $\partial z/\partial y = x^2,\ \partial z/\partial y]_{(1,3)} = 1$, $\mathbf{j} + \mathbf{k}$ is parallel to the tangent line so $x = 1,\ y = 3 + t,$
$z = 3 + t$
(b) $\partial z/\partial x = 2xy,\ \partial z/\partial x]_{(1,3)} = 6$, $\mathbf{i} + 6\mathbf{k}$ is parallel to the tangent line so $x = 1 + t,\ y = 3,$
$z = 3 + 6t$

53. **(a)** $2x - 2z(\partial z/\partial x) = 0,\ \partial z/\partial x = x/z = \pm 3/(2\sqrt{6}) = \pm\sqrt{6}/4$
(b) $z = \pm\sqrt{x^2 + y^2 - 1},\ \partial z/\partial x = \pm x/\sqrt{x^2 + y^2 - 1} = \pm\sqrt{6}/4$

54. **(a)** $2y - 2z(\partial z/\partial y) = 0,\ \partial z/\partial y = y/z = \pm 4/(2\sqrt{6}) = \pm\sqrt{6}/3$
(b) $z = \pm\sqrt{x^2 + y^2 - 1},\ \partial z/\partial y = \pm y/\sqrt{x^2 + y^2 - 1} = \pm\sqrt{6}/3$

55. $\dfrac{3}{2}\left(x^2 + y^2 + z^2\right)^{1/2}\left(2x + 2z\dfrac{\partial z}{\partial x}\right) = 0,\ \partial z/\partial x = -x/z$; similarly, $\partial z/\partial y = -y/z$

56. $\dfrac{4x - 3z^2(\partial z/\partial x)}{2x^2 + y - z^3} = 1,\ \dfrac{\partial z}{\partial x} = \dfrac{4x - 2x^2 - y + z^3}{3z^2};\ \dfrac{1 - 3z^2(\partial z/\partial y)}{2x^2 + y - z^3} = 0,\ \dfrac{\partial z}{\partial y} = \dfrac{1}{3z^2}$

57. $2x + z\left(xy\dfrac{\partial z}{\partial x} + yz\right)\cos xyz + \dfrac{\partial z}{\partial x}\sin xyz = 0,\ \dfrac{\partial z}{\partial x} = -\dfrac{2x + yz^2\cos xyz}{xyz\cos xyz + \sin xyz}$;

$z\left(xy\dfrac{\partial z}{\partial y} + xz\right)\cos xyz + \dfrac{\partial z}{\partial y}\sin xyz = 0,\ \dfrac{\partial z}{\partial y} = -\dfrac{xz^2\cos xyz}{xyz\cos xyz + \sin xyz}$

58. $e^{xy}(\cosh z)\dfrac{\partial z}{\partial x} + ye^{xy}\sinh z - z^2 - 2xz\dfrac{\partial z}{\partial x} = 0,\ \dfrac{\partial z}{\partial x} = \dfrac{z^2 - ye^{xy}\sinh z}{e^{xy}\cosh z - 2xz}$;

$e^{xy}(\cosh z)\dfrac{\partial z}{\partial y} + xe^{xy}\sinh z - 2xz\dfrac{\partial z}{\partial y} = 0,\ \dfrac{\partial z}{\partial y} = -\dfrac{xe^{xy}\sinh z}{e^{xy}\cosh z - 2xz}$

59. $(3/2)\left(x^2+y^2+z^2+w^2\right)^{1/2}\left(2x + 2w\dfrac{\partial w}{\partial x}\right) = 0$, $\partial w/\partial x = -x/w$; similarly, $\partial w/\partial y = -y/w$ and $\partial w/\partial z = -z/w$

60. $\partial w/\partial x = -4x/3$, $\partial w/\partial y = -1/3$, $\partial w/\partial z = (2x^2 + y - z^3 + 3z^2 + 3w)/3$

61. $\dfrac{\partial w}{\partial x} = -\dfrac{yzw\cos xyz}{2w + \sin xyz},\ \dfrac{\partial w}{\partial y} = -\dfrac{xzw\cos xyz}{2w + \sin xyz},\ \dfrac{\partial w}{\partial z} = -\dfrac{xyw\cos xyz}{2w + \sin xyz}$

62. $\dfrac{\partial w}{\partial x} = \dfrac{ye^{xy}\sinh w}{z^2 - e^{xy}\cosh w},\ \dfrac{\partial w}{\partial y} = \dfrac{xe^{xy}\sinh w}{z^2 - e^{xy}\cosh w},\ \dfrac{\partial w}{\partial z} = \dfrac{2zw}{e^{xy}\cosh w - z^2}$

63. $f_x = e^{x^2},\ f_y = -e^{y^2}$

64. $f_x = ye^{x^2y^2},\ f_y = xe^{x^2y^2}$

65. **(a)** $-\dfrac{1}{4x^{3/2}}\cos y$ **(b)** $-\sqrt{x}\cos y$ **(c)** $-\dfrac{\sin y}{2\sqrt{x}}$ **(d)** $-\dfrac{\sin y}{2\sqrt{x}}$

66. **(a)** $8 + 84x^2y^5$ **(b)** $140x^4y^3$ **(c)** $140x^3y^4$ **(d)** $140x^3y^4$

67. $f_x = 8x - 8y^4,\ f_y = -32xy^3 + 35y^4,\ f_{xy} = f_{yx} = -32y^3$

68. $f_x = x/\sqrt{x^2+y^2},\ f_y = y/\sqrt{x^2+y^2},\ f_{xy} = f_{yx} = -xy(x^2+y^2)^{-3/2}$

69. $f_x = e^x\cos y,\ f_y = -e^x\sin y,\ f_{xy} = f_{yx} = -e^x\sin y$

70. $f_x = e^{x-y^2},\ f_y = -2ye^{x-y^2},\ f_{xy} = f_{yx} = -2ye^{x-y^2}$

71. $f_x = 4/(4x-5y),\ f_y = -5/(4x-5y),\ f_{xy} = f_{yx} = 20/(4x-5y)^2$

72. $f_x = 2x/(x^2+y^2), f_y = 2y/(x^2+y^2),\ f_{xy} = -4xy/(x^2+y^2)^2$

73. $f_x = 2y/(x+y)^2, f_y = -2x/(x+y)^2, f_{xy} = f_{yx} = 2(x-y)/(x+y)^3$

74. $f_x = 4xy^2/(x^2+y^2)^2,\ f_y = -4x^2y/(x^2+y^2)^2,\ f_{xy} = f_{yx} = 8xy(x^2-y^2)/(x^2+y^2)^3$

75. III is a plane, and its partial derivatives are constants, so III cannot be $f(x, y)$. If I is the graph of $z = f(x, y)$ then (by inspection) f_y is constant as y varies, but neither II nor III is constant as y varies. Hence $z = f(x, y)$ has II as its graph, and as II seems to be an odd function of x and an even function of y, f_x has I as its graph and f_y has III as its graph.

76. The slope at P in the positive x-direction is negative, the slope in the positive y-direction is negative, thus $\partial z/\partial x < 0, \partial z/\partial y < 0$; the curve through P which is parallel to the x-axis is concave down, so $\partial^2 z/\partial x^2 < 0$; the curve parallel to the y-axis is concave down, so $\partial^2 z/\partial y^2 < 0$.

77. **(a)** $\dfrac{\partial^3 f}{\partial x^3}$ **(b)** $\dfrac{\partial^3 f}{\partial y^2 \partial x}$ **(c)** $\dfrac{\partial^4 f}{\partial x^2 \partial y^2}$ **(d)** $\dfrac{\partial^4 f}{\partial y^3 \partial x}$

78. **(a)** f_{xyy} **(b)** f_{xxxx} **(c)** f_{xxyy} **(d)** f_{yyyxx}

79. **(a)** $30xy^4 - 4$ **(b)** $60x^2y^3$ **(c)** $60x^3y^2$

80. **(a)** $120(2x - y)^2$ **(b)** $-240(2x - y)^2$ **(c)** $480(2x - y)$

81. **(a)** $f_{xyy}(0, 1) = -30$ **(b)** $f_{xxx}(0, 1) = -125$ **(c)** $f_{yyxx}(0, 1) = 150$

82. **(a)** $\dfrac{\partial^3 w}{\partial y^2 \partial x} = -e^y \sin\ x,\ \left.\dfrac{\partial^3 w}{\partial y^2 \partial x}\right|_{(\pi/4,0)} = -1/\sqrt{2}$

(b) $\dfrac{\partial^3 w}{\partial x^2 \partial y} = -e^y \cos\ x,\ \left.\dfrac{\partial^3 w}{\partial x^2 \partial y}\right|_{(\pi/4,0)} = -1/\sqrt{2}$

83. **(a)** $f_{xy} = 15x^2y^4z^7 + 2y$ **(b)** $f_{yz} = 35x^3y^4z^6 + 3y^2$
(c) $f_{xz} = 21x^2y^5z^6$ **(d)** $f_{zz} = 42x^3y^5z^5$
(e) $f_{zyy} = 140x^3y^3z^6 + 6y$ **(f)** $f_{xxy} = 30xy^4z^7$
(g) $f_{zyx} = 105x^2y^4z^6$ **(h)** $f_{xxyz} = 210xy^4z^6$

84. **(a)** $160(4x - 3y + 2z)^3$ **(b)** $-1440(4x - 3y + 2z)^2$ **(c)** $-5760(4x - 3y + 2z)$

85. **(a)** $f_x = 2x + 2y,\ f_{xx} = 2,\ f_y = -2y + 2x,\ f_{yy} = -2;\ f_{xx} + f_{yy} = 2 - 2 = 0$

(b) $z_x = e^x \sin y - e^y \sin x,\ z_{xx} = e^x \sin y - e^y \cos x,\ z_y = e^x \cos y + e^y \cos x,$
$z_{yy} = -e^x \sin y + e^y \cos x;\ z_{xx} + z_{yy} = e^x \sin y - e^y \cos x - e^x \sin y + e^y \cos x = 0$

(c) $z_x = \dfrac{2x}{x^2 + y^2} - 2\dfrac{y}{x^2}\dfrac{1}{1 + (y/x)^2} = \dfrac{2x - 2y}{x^2 + y^2},\ z_{xx} = -2\dfrac{x^2 - y^2 - 2xy}{(x^2 + y^2)^2},$

$z_y = \dfrac{2y}{x^2 + y^2} + 2\dfrac{1}{x}\dfrac{1}{1 + (y/x)^2} = \dfrac{2y + 2x}{x^2 + y^2},\ z_{yy} = -2\dfrac{y^2 - x^2 + 2xy}{(x^2 + y^2)^2};$

$z_{xx} + z_{yy} = -2\dfrac{x^2 - y^2 - 2xy}{(x^2 + y^2)^2} - 2\dfrac{y^2 - x^2 + 2xy}{(x^2 + y^2)^2} = 0$

86. **(a)** $z_t = -e^{-t} \sin(x/c),\ z_x = (1/c)e^{-t} \cos(x/c),\ z_{xx} = -(1/c^2)e^{-t} \sin(x/c);$
$z_t - c^2 z_{xx} = -e^{-t} \sin(x/c) - c^2(-(1/c^2)e^{-t} \sin(x/c)) = 0$

(b) $z_t = -e^{-t} \cos(x/c),\ z_x = -(1/c)e^{-t} \sin(x/c),\ z_{xx} = -(1/c^2)e^{-t} \cos(x/c);$
$z_t - c^2 z_{xx} = -e^{-t} \cos(x/c) - c^2(-(1/c^2)e^{-t} \cos(x/c)) = 0$

87. $u_x = \omega \sin c\omega t \cos \omega x,\ u_{xx} = -\omega^2 \sin c\omega t \sin \omega x,\ u_t = c\omega \cos c\omega t \sin \omega x,\ u_{tt} = -c^2\omega^2 \sin c\omega t \sin \omega x;$
$u_{xx} - \dfrac{1}{c^2} u_{tt} = -\omega^2 \sin c\omega t \sin \omega x - \dfrac{1}{c^2}(-c^2)\omega^2 \sin c\omega t \sin \omega x = 0$

88. **(a)** $\partial u/\partial x = \partial v/\partial y = 2x,\ \partial u/\partial y = -\partial v/\partial x = -2y$

(b) $\partial u/\partial x = \partial v/\partial y = e^x \cos y,\ \partial u/\partial y = -\partial v/\partial x = -e^x \sin y$

(c) $\partial u/\partial x = \partial v/\partial y = 2x/(x^2 + y^2),\ \partial u/\partial y = -\partial v/\partial x = 2y/(x^2 + y^2)$

89. $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$ so $\partial^2 u/\partial x^2 = \partial^2 v/\partial x\partial y$, and $\partial^2 u/\partial y^2 = -\partial^2 v/\partial y\partial x$, $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = \partial^2 v/\partial x\partial y - \partial^2 v/\partial y\partial x$, if $\partial^2 v/\partial x\partial y = \partial^2 v/\partial y\partial x$ then $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$; thus u satisfies Laplace's equation. The proof that v satisfies Laplace's equation is similar. Adding Laplace's equations for u and v gives Laplaces' equation for $u + v$.

90. $\partial^2 R/\partial R_1^2 = -2R_2^2/(R_1+R_2)^3,\ \partial^2 R/\partial R_2^2 = -2R_1^2/(R_1+R_2)^3,$

$$\left(\partial^2 R/\partial R_1^2\right)\left(\partial^2 R/\partial R_2^2\right) = 4R_1^2R_2^2/\left(R_1+R_2\right)^6 = \left[4/\left(R_1+R_2\right)^4\right]\left[R_1R_2/\left(R_1+R_2\right)\right]^2 = 4R^2/\left(R_1+R_2\right)^4$$

91. $\partial f/\partial v = 8vw^3x^4y^5,\ \partial f/\partial w = 12v^2w^2x^4y^5,\ \partial f/\partial x = 16v^2w^3x^3y^5,\ \partial f/\partial y = 20v^2w^3x^4y^4$

92. $\partial w/\partial r = \cos st + ue^u \cos ur,\ \partial w/\partial s = -rt\sin st,$
$\partial w/\partial t = -rs\sin st,\ \partial w/\partial u = re^u\cos ur + e^u \sin ur$

93. $\partial f/\partial v_1 = 2v_1/\left(v_3^2+v_4^2\right),\ \partial f/\partial v_2 = -2v_2/\left(v_3^2+v_4^2\right),\ \partial f/\partial v_3 = -2v_3\left(v_1^2-v_2^2\right)/\left(v_3^2+v_4^2\right)^2,$
$\partial f/\partial v_4 = -2v_4\left(v_1^2-v_2^2\right)/\left(v_3^2+v_4^2\right)^2$

94. $\dfrac{\partial V}{\partial x} = 2xe^{2x-y} + e^{2x-y},\ \dfrac{\partial V}{\partial y} = -xe^{2x-y} + w,\ \dfrac{\partial V}{\partial z} = w^2e^{zw},\ \dfrac{\partial V}{\partial w} = wze^{zw} + e^{zw} + y$

95. **(a)** 0 **(b)** 0 **(c)** 0 **(d)** 0
(e) $2(1+yw)e^{yw}\sin z\cos z$ **(f)** $2xw(2+yw)e^{yw}\sin z\cos z$

96. 128, −512, 32, 64/3

97. $\partial w/\partial x_i = -i\sin(x_1+2x_2+\ldots+nx_n)$

98. $\partial w/\partial x_i = \dfrac{1}{n}\left(\displaystyle\sum_{k=1}^{n} x_k\right)^{(1/n)-1}$

99. **(a)** xy-plane, $f_x = 12x^2y + 6xy,\ f_y = 4x^3 + 3x^2,\ f_{xy} = f_{yx} = 12x^2 + 6x$
(b) $y \neq 0, f_x = 3x^2/y,\ f_y = -x^3/y^2,\ f_{xy} = f_{yx} = -3x^2/y^2$

100. **(a)** $x^2 + y^2 > 1$, (the exterior of the circle of radius 1 about the origin);
$f_x = x/\sqrt{x^2+y^2-1},\ f_y = y/\sqrt{x^2+y^2-1},\ f_{xy} = f_{yx} = -xy\left(x^2+y^2-1\right)^{-3/2}$
(b) xy-plane, $f_x = 2x\cos(x^2+y^3),\ f_y = 3y^2\cos(x^2+y^3),\ f_{xy} = f_{yx} = -6xy^2\sin\left(x^2+y^3\right)$

101. $f_x(2,-1) = \displaystyle\lim_{x\to 2}\frac{f(x,-1)-f(2,-1)}{x-2} = \lim_{x\to 2}\frac{2x^2+3x+1-15}{x-2} = \lim_{x\to 2}(2x+7) = 11$ and

$f_y(2,-1) = \displaystyle\lim_{y\to -1}\frac{f(2,y)-f(2,-1)}{y+1} = \lim_{y\to -1}\frac{8-6y+y^2-15}{y+1} = \lim_{y\to -1} y - 7 = -8$

102. $f_x(x,y) = \dfrac{2}{3}(x^2+y^2)^{-1/3}(2x) = \dfrac{4x}{3(x^2+y^2)^{1/3}},\ (x,y) \neq (0,0);$

$f_x(0,0) = \dfrac{d}{dx}[f(x,0)]\Big]_{x=0} = \dfrac{d}{dx}[x^{4/3}]\Big]_{x=0} = \dfrac{4}{3}x^{1/3}\Big]_{x=0} = 0.$

103. **(a)** $f_y(0,0) = \left.\dfrac{d}{dy}[f(0,y)]\right]_{y=0} = \left.\dfrac{d}{dy}[y]\right]_{y=0} = 1$

(b) If $(x,y) \neq (0,0)$, then $f_y(x,y) = \dfrac{1}{3}(x^3+y^3)^{-2/3}(3y^2) = \dfrac{y^2}{(x^3+y^3)^{2/3}}$;
$f_y(x,y)$ does not exist when $y \neq 0$ and $y = -x$

EXERCISE SET 14.4

1. **(a)** Let $f(x,y) = e^x \sin y$; $f(0,0) = 0$, $f_x(0,0) = 0$, $f_y(0,0) = 1$, so $e^x \sin y \approx y$

(b) Let $f(x,y) = \dfrac{2x+1}{y+1}$; $f(0,0) = 1$, $f_x(0,0) = 2$, $f_y(0,0) = -1$, so $\dfrac{2x+1}{y+1} \approx 1 + 2x - y$

2. $f(1,1) = 1$, $f_x(x,y) = \alpha x^{\alpha-1}y^\beta$, $f_x(1,1) = \alpha$, $f_y(x,y) = \beta x^\alpha y^{\beta-1}$, $f_y(1,1) = \beta$, so
$x^\alpha y^\beta \approx 1 + \alpha(x-1) + \beta(y-1)$

3. **(a)** Let $f(x,y,z) = xyz + 2$, then $f_x = f_y = f_z = 1$ at $x = y = z = 1$, and
$L(x,y,z) = f(1,1,1) + f_x(x-1) + f_y(y-1) + f_z(z-1) = 3 + x - 1 + y - 1 + z - 1 = x + y + z$

(b) Let $f(x,y,z) = \dfrac{4x}{y+z}$, then $f_x = 2$, $f_y = -1$, $f_z = -1$ at $x = y = z = 1$, and

$$\begin{aligned} L(x,y,z) &= f(1,1,1) + f_x(x-1) + f_y(y-1) + f_z(z-1) \\ &= 2 + 2(x-1) - (y-1) - (z-1) = 2x - y - z + 2 \end{aligned}$$

4. Let $f(x,y,z) = x^\alpha y^\beta z^\gamma$, then $f_x = \alpha$, $f_y = \beta$, $f_z = \gamma$ at $x = y = z = 1$, and
$f(x,y,z) \approx f(1,1,1) + f_x(x-1) + f_y(y-1) + f_z(z-1) = 1 + \alpha(x-1) + \beta(y-1) + \gamma(z-1)$

5. $f(x,y) \approx f(3,4) + f_x(x-3) + f_y(y-4) = 5 + 2(x-3) - (y-4)$ and
$f(3.01, 3.98) \approx 5 + 2(0.01) - (-0.02) = 5.04$

6. $f(x,y) \approx f(-1,2) + f_x(x+1) + f_y(y-2) = 2 + (x+1) + 3(y-2)$ and
$f(-0.99, 2.02) \approx 2 + 0.01 + 3(0.02) = 2.07$

7. $L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$ and
$L(1.1, 0.9) = 3.15 = 3 + 2(0.1) + f_y(1,1)(-0.1)$ so $f_y(1,1) = -0.05/(-0.1) = 0.5$

8. $L(x,y) = 3 + f_x(0,-1)x - 2(y+1)$, $3.3 = 3 + f_x(0,-1)(0.1) - 2(-0.1)$, so $f_x(0,-1) = 0.1/0.1 = 1$

9. $L(x,y,z) = f(1,2,3) + (x-1) + 2(y-2) + 3(z-3)$,
$f(1.01, 2.02, 3.03) \approx 4 + 0.01 + 2(0.02) + 3(0.03) = 4.14$

10. $L(x,y,z) = f(2,1,-2) - (x-2) + (y-1) - 2(z+2)$,
$f(1.98, 0.99, -1.97) \approx 0.02 - 0.01 - 2(0.03) = -0.05$

11. $x - y + 2z - 2 = L(x,y,z) = f(3,2,1) + f_x(3,2,1)(x-3) + f_y(3,2,1)(y-2) + f_z(3,2,1)(z-1)$, so
$f_x(3,2,1) = 1$, $f_y(3,2,1) = -1$, $f_z(3,2,1) = 2$ and $f(3,2,1) = L(3,2,1) = 1$

12. $L(x,y,z) = x + 2y + 3z + 4 = (x-0) + 2(y+1) + 3(z+2) - 4$,
$f(0,-1,-2) = -4$, $f_x(0,-1,-2) = 1$, $f_y(0,-1,-2) = 2$, $f_z(0,-1,-2) = 3$

13. $L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$,
$2y - 2x - 2 = x_0^2 + y_0^2 + 2x_0(x-x_0) + 2y_0(y-y_0)$, from which it follows that $x_0 = -1, y_0 = 1$.

14. $f(x,y) = x^2y$, so $f_x(x_0,y_0) = 2x_0y_0,\ f_y(x_0,y_0) = x_0^2$, and
$L(x,y) = f(x_0,y_0) + 2x_0y_0(x-x_0) + x_0^2(y-y_0)$. But $L(x,y) = 8 - 4x + 4y$, hence
$-4 = 2x_0y_0,\ 4 = x_0^2$ and $8 = f(x_0,y_0) - 2x_0^2y_0 - x_0^2y_0 = -2x_0^2y_0$. Thus either $x_0 = -2, y_0 = 1$ from which it follows that $8 = -8$, a contradiction, or $x_0 = 2, y_0 = -1$, which is a solution since then $8 = -2x_0^2y_0 = 8$ is true.

15. $L(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0)$,
$y + 2z - 1 = x_0y_0 + z_0^2 + y_0(x-x_0) + x_0(y-y_0) + 2z_0(z-z_0)$, so that $x_0 = 1, y_0 = 0, z_0 = 1$.

16. $L(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0)$.
Then $x - y - z - 2 = x_0y_0z_0 + y_0z_0(x-x_0) + x_0z_0(y-y_0) + x_0y_0(z-z_0)$, hence
$y_0z_0 = 1, x_0z_0 = -1, x_0y_0 = -1$, and $-2 = x_0y_0z_0 - 3x_0y_0z_0$, or $x_0y_0z_0 = 1$. Since now
$x_0 = -y_0 = -z_0$, we must have $|x_0| = |y_0| = |z_0| = 1$ or else $|x_0y_0z_0| \neq 1$, impossible. Thus $x_0 = 1, y_0 = z_0 = -1$ (note that $(-1,1,1)$ is not a solution).

17. **(a)** $f(P) = 1/5, f_x(P) = -x/(x^2+y^2)^{-3/2}\Big]_{(x,y)=(4,3)} = -4/125$,

$f_y(P) = -y/(x^2+y^2)^{-3/2}\Big]_{(x,y)=(4,3)} = -3/125, L(x,y) = \dfrac{1}{5} - \dfrac{4}{125}(x-4) - \dfrac{3}{125}(y-3)$

(b) $L(Q) - f(Q) = \dfrac{1}{5} - \dfrac{4}{125}(-0.08) - \dfrac{3}{125}(0.01) - 0.2023342382 \approx -0.0000142382$,
$|PQ| = \sqrt{0.08^2 + 0.01^2} \approx 0.0008062257748, |L(Q) - f(Q)|/|PQ| \approx 0.000176603$

18. **(a)** $f(P) = 1, f_x(P) = 0.5, f_y(P) = 0.3, L(x,y) = 1 + 0.5(x-1) + 0.3(y-1)$
(b) $L(Q) - f(Q) = 1 + 0.5(0.05) + 0.3(-0.03) - 1.05^{0.5}0.97^{0.3} \approx 0.00063$,
$|PQ| = \sqrt{0.05^2 + 0.03^2} \approx 0.05831, |L(Q) - f(Q)|/|PQ| \approx 0.0107$

19. **(a)** $f(P) = 0, f_x(P) = 0, f_y(P) = 0, L(x,y) = 0$
(b) $L(Q) - f(Q) = -0.003\sin(0.004) \approx -0.000012, |PQ| = \sqrt{0.003^2 + 0.004^2} = 0.005$,
$|L(Q) - f(Q)|/|PQ| \approx 0.0024$

20. **(a)** $f(P) = \ln 2, f_x(P) = 1, f_y(P) = 1/2, L(x,y) = \ln 2 + (x-1) + \dfrac{1}{2}(y-2)$
(b) $L(Q) - f(Q) = \ln 2 + 0.01 + (1/2)(0.02) - \ln 2.0402 \approx 0.0000993383$,
$|PQ| = \sqrt{0.01^2 + 0.02^2} \approx 0.02236067978,\ |L(Q) - f(Q)|/|PQ| \approx 0.0044425$

21. **(a)** $f(P) = 6, f_x(P) = 6, f_y(P) = 3, f_z(P) = 2, L(x,y) = 6 + 6(x-1) + 3(y-2) + 2(z-3)$
(b) $L(Q) - f(Q) = 6 + 6(0.001) + 3(0.002) + 2(0.003) - 6.018018006 = -.000018006$,
$|PQ| = \sqrt{0.001^2 + 0.002^2 + 0.003^2} \approx .0003741657387;\quad |L(Q) - f(Q)|/|PQ| \approx -0.000481$

22. **(a)** $f(P) = 0, f_x(P) = 1/2, f_y(P) = 1/2, f_z(P) = 0, L(x,y) = \dfrac{1}{2}(x+1) + \dfrac{1}{2}(y-1)$
(b) $L(Q) - f(Q) = 0, |L(Q) - f(Q)|/|PQ| = 0$

23. **(a)** $f(P) = e, f_x(P) = e, f_y(P) = -e, f_z(P) = -e, L(x, y) = e + e(x-1) - e(y+1) - e(z+1)$
(b) $L(Q) - f(Q) = e - 0.01e + 0.01e - 0.01e - 0.99e^{0.9999} = 0.99(e - e^{0.9999})$,
$|PQ| = \sqrt{0.01^2 + 0.01^2 + 0.01^2} \approx 0.01732, |L(Q) - f(Q)|/|PQ| \approx 0.01554$

24. **(a)** $f(P) = 0, f_x(P) = 1, f_y(P) = -1, f_z(P) = 1,\ L(x, y, z) = (x-2) - (y-1) + (z+1)$
(b) $L(Q) - f(Q) = 0.02 + 0.03 - 0.01 - \ln 1.0403 \approx 0.00049086691$,
$|PQ| = \sqrt{0.02^2 + 0.03^2 + 0.01^2} \approx 0.03742,\ |L(Q) - f(Q)|/|PQ| \approx 0.01312$

25. $dz = 7dx - 2dy$

26. $dz = ye^{xy}dx + xe^{xy}dy$

27. $dz = 3x^2y^2dx + 2x^3ydy$

28. $dz = (10xy^5 - 2)dx + (25x^2y^4 + 4)dy$

29. $dz = \left[y/\left(1 + x^2y^2\right)\right] dx + \left[x/\left(1 + x^2y^2\right)\right] dy$

30. $dz = 2\sec^2(x - 3y)\tan(x - 3y)dx - 6\sec^2(x - 3y)\tan(x - 3y)dy$

31. $dw = 8dx - 3dy + 4dz$

32. $dw = yze^{xyz}dx + xze^{xyz}dy + xye^{xyz}dz$

33. $dw = 3x^2y^2zdx + 2x^3yzdy + x^3y^2dz$

34. $dw = \left(8xy^3z^7 - 3y\right) dx + \left(12x^2y^2z^7 - 3x\right) dy + \left(28x^2y^3z^6 + 1\right) dz$

35. $dw = \dfrac{yz}{1 + x^2y^2z^2}dx + \dfrac{xz}{1 + x^2y^2z^2}dy + \dfrac{xy}{1 + x^2y^2z^2}dz$

36. $dw = \dfrac{1}{2\sqrt{x}}dx + \dfrac{1}{2\sqrt{y}}dy + \dfrac{1}{2\sqrt{z}}dz$

37. $df = (2x + 2y - 4)dx + 2xdy$; $x = 1,\ y = 2,\ dx = 0.01,\ dy = 0.04$ so
$df = 0.10$ and $\Delta f = 0.1009$

38. $df = (1/3)x^{-2/3}y^{1/2}dx + (1/2)x^{1/3}y^{-1/2}dy$; $x = 8,\ y = 9,\ dx = -0.22,\ dy = 0.03$ so $df = -0.045$ and $\Delta f \approx -0.045613$

39. $df = -x^{-2}dx - y^{-2}dy$; $x = -1,\ y = -2,\ dx = -0.02,\ dy = -0.04$ so
$df = 0.03$ and $\Delta f \approx 0.029412$

40. $df = \dfrac{y}{2(1 + xy)}dx + \dfrac{x}{2(1 + xy)}dy$; $x = 0,\ y = 2,\ dx = -0.09,\ dy = -0.02$ so
$df = -0.09$ and $\Delta f \approx -0.098129$

41. $df = 2y^2z^3dx + 4xyz^3dy + 6xy^2z^2dz, x = 1, y = -1, z = 2, dx = -0.01, dy = -0.02, dz = 0.02$ so $df = 0.96$ and $\Delta f \approx 0.97929$

42. $df = \dfrac{yz(y + z)}{(x + y + z)^2}dx + \dfrac{xz(x + z)}{(x + y + z)^2}dy + \dfrac{xy(x + y)}{(x + y + z)^2}dz, x = -1, y = -2, z = 4, dx = -0.04,$
$dy = 0.02, dz = -0.03$ so $df = 0.58$ and $\Delta f \approx 0.60529$

43. Label the four smaller rectangles A, B, C, D starting with the lower left and going clockwise. Then the increase in the area of the rectangle is represented by B, C and D; and the portions B and D represent the approximation of the increase in area given by the total differential.

44. $V + \Delta V = (\pi/3)4.05^2(19.95) \approx 109.0766250\pi, V = 320\pi/3, \Delta V \approx 2.40996\pi$;
$dV = (2/3)\pi r h dr + (1/3)\pi r^2 dh$; $r = 4$, $h = 20$, $dr = 0.05$, $dh = -0.05$ so $dV = 2.4\pi$, and $\Delta V/dV \approx 1.00415$.

45. $A = xy$, $dA = ydx + xdy$, $dA/A = dx/x + dy/y$, $|dx/x| \le 0.03$ and $|dy/y| \le 0.05$,
$|dA/A| \le |dx/x| + |dy/y| \le 0.08 = 8\%$

46. $V = (1/3)\pi r^2 h$, $dV = (2/3)\pi r h dr + (1/3)\pi r^2 dh$, $dV/V = 2(dr/r) + dh/h$, $|dr/r| \le 0.01$ and $|dh/h| \le 0.04$, $|dV/V| \le 2|dr/r| + |dh/h| \le 0.06 = 6\%$.

47. $z = \sqrt{x^2+y^2}$, $dz = \dfrac{x}{\sqrt{x^2+y^2}}dx + \dfrac{y}{\sqrt{x^2+y^2}}dy$,

$$\frac{dz}{z} = \frac{x}{x^2+y^2}dx + \frac{y}{x^2+y^2}dy = \frac{x^2}{x^2+y^2}\left(\frac{dx}{x}\right) + \frac{y^2}{x^2+y^2}\left(\frac{dy}{y}\right),$$

$\left|\dfrac{dz}{z}\right| \le \dfrac{x^2}{x^2+y^2}\left|\dfrac{dx}{x}\right| + \dfrac{y^2}{x^2+y^2}\left|\dfrac{dy}{y}\right|$, if $\left|\dfrac{dx}{x}\right| \le r/100$ and $\left|\dfrac{dy}{y}\right| \le r/100$ then

$\left|\dfrac{dz}{z}\right| \le \dfrac{x^2}{x^2+y^2}(r/100) + \dfrac{y^2}{x^2+y^2}(r/100) = \dfrac{r}{100}$ so the percentage error in z is at most about $r\%$.

48. **(a)** $z = \sqrt{x^2+y^2}$, $dz = x\left(x^2+y^2\right)^{-1/2}dx + y\left(x^2+y^2\right)^{-1/2}dy$,
$|dz| \le x\left(x^2+y^2\right)^{-1/2}|dx| + y\left(x^2+y^2\right)^{-1/2}|dy|$; if $x = 3$, $y = 4$, $|dx| \le 0.05$, and $|dy| \le 0.05$ then $|dz| \le (3/5)(0.05) + (4/5)(0.05) = 0.07$ cm

(b) $A = (1/2)xy$, $dA = (1/2)ydx + (1/2)xdy$,
$|dA| \le (1/2)y|dx| + (1/2)x|dy| \le 2(0.05) + (3/2)(0.05) = 0.175\ \text{cm}^2$.

49. $dT = \dfrac{\pi}{g\sqrt{L/g}}dL - \dfrac{\pi L}{g^2\sqrt{L/g}}dg$, $\dfrac{dT}{T} = \dfrac{1}{2}\dfrac{dL}{L} - \dfrac{1}{2}\dfrac{dg}{g}$; $|dL/L| \le 0.005$ and $|dg/g| \le 0.001$ so
$|dT/T| \le (1/2)(0.005) + (1/2)(0.001) = 0.003 = 0.3\%$

50. $dP = (k/V)dT - (kT/V^2)dV$, $dP/P = dT/T - dV/V$; if $dT/T = 0.03$ and $dV/V = 0.05$ then $dP/P = -0.02$ so there is about a 2% decrease in pressure.

51. **(a)** $\left|\dfrac{d(xy)}{xy}\right| = \left|\dfrac{y\,dx + x\,dy}{xy}\right| = \left|\dfrac{dx}{x} + \dfrac{dy}{y}\right| \le \left|\dfrac{dx}{x}\right| + \left|\dfrac{dy}{y}\right| \le \dfrac{r}{100} + \dfrac{s}{100}$; $(r+s)\%$

(b) $\left|\dfrac{d(x/y)}{x/y}\right| = \left|\dfrac{y\,dx - x\,dy}{xy}\right| = \left|\dfrac{dx}{x} - \dfrac{dy}{y}\right| \le \left|\dfrac{dx}{x}\right| + \left|\dfrac{dy}{y}\right| \le \dfrac{r}{100} + \dfrac{s}{100}$; $(r+s)\%$

(c) $\left|\dfrac{d(x^2y^3)}{x^2y^3}\right| = \left|\dfrac{2xy^3\,dx + 3x^2y^2\,dy}{x^2y^3}\right| = \left|2\dfrac{dx}{x} + 3\dfrac{dy}{y}\right| \le 2\left|\dfrac{dx}{x}\right| + 3\left|\dfrac{dy}{y}\right|$
$\le 2\dfrac{r}{100} + 3\dfrac{s}{100}$; $(2r+3s)\%$

(d) $\left|\dfrac{d(x^3y^{1/2})}{x^3y^{1/2}}\right| = \left|\dfrac{3x^2y^{1/2}\,dx + (1/2)x^3y^{-1/2}\,dy}{x^3y^{1/2}}\right| = \left|3\dfrac{dx}{x} + \dfrac{1}{2}\dfrac{dy}{y}\right| \le 3\left|\dfrac{dx}{x}\right| + \dfrac{1}{2}\left|\dfrac{dy}{y}\right|$
$\le 3\dfrac{r}{100} + \dfrac{1}{2}\dfrac{s}{100}$; $(3r + \frac{1}{2}s)\%$

52. $R = 1/(1/R_1 + 1/R_2 + 1/R_3)$, $\partial R/\partial R_1 = \dfrac{1}{R_1^2(1/R_1 + 1/R_2 + 1/R_3)^2} = R^2/R_1^2$, similarly $\partial R/\partial R_2 = R^2/R_2^2$ and $\partial R/\partial R_3 = R^2/R_3^2$ so $\dfrac{dR}{R} = (R/R_1)\dfrac{dR_1}{R_1} + (R/R_2)\dfrac{dR_2}{R_2} + (R/R_3)\dfrac{dR_3}{R_3}$,

$$\begin{aligned}\left|\frac{dR}{R}\right| &\le (R/R_1)\left|\frac{dR_1}{R_1}\right| + (R/R_2)\left|\frac{dR_2}{R_2}\right| + (R/R_3)\left|\frac{dR_3}{R_3}\right| \\ &\le (R/R_1)(0.10) + (R/R_2)(0.10) + (R/R_3)(0.10) \\ &= R(1/R_1 + 1/R_2 + 1/R_3)(0.10) = (1)(0.10) = 0.10 = 10\%\end{aligned}$$

53. $dA = \dfrac{1}{2}b\sin\theta da + \dfrac{1}{2}a\sin\theta db + \dfrac{1}{2}ab\cos\theta d\theta$,

$$\begin{aligned}|dA| &\le \frac{1}{2}b\sin\theta|da| + \frac{1}{2}a\sin\theta|db| + \frac{1}{2}ab\cos\theta|d\theta| \\ &\le \frac{1}{2}(50)(1/2)(1/2) + \frac{1}{2}(40)(1/2)(1/4) + \frac{1}{2}(40)(50)\left(\sqrt{3}/2\right)(\pi/90) \\ &= 35/4 + 50\pi\sqrt{3}/9 \approx 39 \text{ ft}^2\end{aligned}$$

54. $V = \ell wh$, $dV = whd\ell + \ell hdw + \ell wdh$, $|dV/V| \le |d\ell/\ell| + |dw/w| + |dh/h| \le 3(r/100) = 3r\%$

55. If $f(x,y) = f(x_0,y_0)$ for all (x,y) then $L(x,y) = f(x_0,y_0)$ since the first partial derivatives of f are zero. Thus the error E is zero and f is differentiable. The proof for three variables is analogous.

56. Let $f(x,y) = ax + by + c$. Then $L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0) = ax_0 + by_0 + c + a(x - x_0) + b(y - y_0) = ax + by + c$, so $L = f$ and thus E is zero. For three variables the proof is analogous.

57. $f_x = 2x\sin y$, $f_y = x^2\cos y$ are both continuous everywhere, so f is differentiable everywhere.

58. $f_x = y\sin z$, $f_y = x\sin z$, $f_z = xy\cos z$ are all continuous everywhere, so f is differentiable everywhere.

59. $f_x = 2x$, $f_y = 2y$, $f_z = 2z$ so $L(x,y,z) = 0$, $E = f - L = x^2 + y^2 + z^2$, and $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{E(x,y,z)}{\sqrt{x^2+y^2+z^2}} = \lim_{(x,y,z)\to(0,0,0)} \sqrt{x^2+y^2+z^2} = 0$, so f is differentiable at $(0,0,0)$.

60. $f_x = 2xr(x^2+y^2+z^2)^{r-1}$, $f_y = 2yr(x^2+y^2+z^2)^{r-1}$, $f_z = 2zr(x^2+y^2+z^2)^{r-1}$, so the partials of f exist only if $r \ge 1$. If so then $L(x,y,z) = 0$, $E(x,y,z) = f(x,y,z)$ and $\dfrac{E(x,y,z)}{\sqrt{x^2+y^2+z^2}} = (x^2+y^2+z^2)^{r-1/2}$, so f is differentiable at $(0,0,0)$ if and only if $r > 1/2$.

61. Let $\epsilon > 0$. Then $\displaystyle\lim_{x\to x_0}\frac{f(x)}{g(x)} = 0$ if and only if there exists $\delta > 0$ such that $\left|\dfrac{f(x)}{g(x)}\right| < \epsilon$ whenever $|x - x_0| < \delta$. But this condition is equivalent to $\left|\dfrac{f(x)}{|g(x)|}\right| < \epsilon$, and thus the two limits both exist or neither exists.

62. f is continuous at (x_0,y_0) if and only if $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$. Since the limit of M is clearly $f(x_0,y_0)$, the limit of f will be $f(x_0,y_0)$ if and only if $\displaystyle\lim_{(x,y)\to(x_0,y_0)} E(x,y) = 0$.

63. If f is differentiable at (x_0,y_0) then $L(x,y)$ exists and is a linear function and thus differentiable, and thus the difference $E = f - L$ is also differentiable.

64. That f is differentiable means that $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \frac{E_f(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$, where $E_f(x,y) = f(x,y) - L_f(x,y)$; here $L_f(x,y)$ is the linear approximation to f at (x_0,y_0). Let f_x and f_y denote $f_x(x_0,y_0), f_y(x_0,y_0)$ respectively. Then $g(x,y,z) = z - f(x,y)$,

$$L_f(x,y) = f(x_0,y_0) + f_x(x-x_0) + f_y(y-y_0),$$

$$\begin{aligned} L_g(x,y,z) &= g(x_0,y_0,z_0) + g_x(x-x_0) + g_y(y-y_0) + g_z(z-z_0)\ , \\ &= 0 - f_x(x-x_0) - f_y(y-y_0) + (z-z_0) \end{aligned}$$

and

$$\begin{aligned} E_g(x,y,z) &= g(x,y,z) - L_g(x,y,z) = (z - f(x,y)) + f_x(x-x_0) + f_y(y-y_0) - (z-z_0) \\ &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - f(x,y) = -E_f(x,y) \end{aligned}$$

Thus $\displaystyle\frac{|E_g(x,y,z)|}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} \le \frac{|E_f(x,y)|}{\sqrt{(x-x_0)^2+(y-y_0)^2}}$

so $\displaystyle\lim_{(x,y,z)\to(x_0,y_0,z_0)} \frac{E_g(x,y,z)}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} = 0$

and g is differentiable at (x_0,y_0,z_0).

65. Let $x > 0$. Then $\dfrac{f(x,y)-f(x,0)}{y-0}$ can be $-1/y$ or 0 depending on whether $y > 0$ or $y < 0$. Thus the partial derivative $f_y(x,0)$ cannot exist. A similar argument works for $f_x(0,y)$ if $y > 0$.

66. The condition $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$ is equivalent to $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \epsilon(x,y) = 0$ which is equivalent to ϵ being continuous at (x_0,y_0) with $\epsilon(0,0) = 0$. Since ϵ is continuous, f is differentiable.

EXERCISE SET 14.5

1. $42t^{13}$

2. $\dfrac{2(3+t^{-1/3})}{3(2t+t^{2/3})}$

3. $3t^{-2}\sin(1/t)$

4. $\dfrac{1-2t^4-8t^4\ln t}{2t\sqrt{1+\ln t-2t^4\ln t}}$

5. $-\dfrac{10}{3}t^{7/3}e^{1-t^{10/3}}$

6. $(1+t)e^t\cosh(te^t/2)\sinh(te^t/2)$

7. $165t^{32}$

8. $\dfrac{3-(4/3)t^{-1/3}-24t^{-7}}{3t-2t^{2/3}+4t^{-6}}$

9. $-2t\cos\left(t^2\right)$

10. $\dfrac{1-512t^5-2560t^5\ln t}{2t\sqrt{1+\ln t-512t^5\ln t}}$

11. 3264

12. 0

13. $\partial z/\partial u = 24u^2v^2 - 16uv^3 - 2v + 3,\ \partial z/\partial v = 16u^3v - 24u^2v^2 - 2u - 3$

14. $\partial z/\partial u = 2u/v^2 - u^2v\sec^2(u/v) - 2uv^2\tan(u/v)$
$\partial z/\partial v = -2u^2/v^3 + u^3\sec^2(u/v) - 2u^2v\tan(u/v)$

15. $\partial z/\partial u = -\dfrac{2\sin u}{3\sin v},\ \partial z/\partial v = -\dfrac{2\cos u\cos v}{3\sin^2 v}$

16. $\partial z/\partial u = 3 + 3v/u - 4u,\ \partial z/\partial v = 2 + 3\ln u + 2\ln v$

17. $\partial z/\partial u = e^u,\ \partial z/\partial v = 0$

18. $\partial z/\partial u = -\sin(u-v)\sin\left(u^2+v^2\right) + 2u\cos(u-v)\cos\left(u^2+v^2\right)$
$\partial z/\partial v = \sin(u-v)\sin\left(u^2+v^2\right) + 2v\cos(u-v)\cos\left(u^2+v^2\right)$

19. $\partial T/\partial r = 3r^2\sin\theta\cos^2\theta - 4r^3\sin^3\theta\cos\theta$
$\partial T/\partial\theta = -2r^3\sin^2\theta\cos\theta + r^4\sin^4\theta + r^3\cos^3\theta - 3r^4\sin^2\theta\cos^2\theta$

20. $dR/d\phi = 5e^{5\phi}$

21. $\partial t/\partial x = \left(x^2+y^2\right)/\left(4x^2y^3\right)$, $\partial t/\partial y = \left(y^2-3x^2\right)/\left(4xy^4\right)$

22. $\partial w/\partial u = \dfrac{2v^2\left[u^2v^2-(u-2v)^2\right]}{\left[u^2v^2+(u-2v)^2\right]^2}$, $\partial w/\partial v = \dfrac{u^2\left[(u-2v)^2-u^2v^2\right]}{\left[u^2v^2+(u-2v)^2\right]^2}$

23. $\partial z/\partial r = (dz/dx)(\partial x/\partial r) = 2r\cos^2\theta/\left(r^2\cos^2\theta+1\right)$,
$\partial z/\partial\theta = (dz/dx)(\partial x/\partial\theta) = -2r^2\sin\theta\cos\theta/\left(r^2\cos^2\theta+1\right)$

24.
$$\begin{aligned}\partial u/\partial x &= (\partial u/\partial r)(dr/dx) + (\partial u/\partial t)(\partial t/\partial x)\\ &= \left(s^2\ln t\right)(2x) + \left(rs^2/t\right)\left(y^3\right) = x(4y+1)^2\left(1+2\ln xy^3\right)\\ \partial u/\partial y &= (\partial u/\partial s)(ds/dy) + (\partial u/\partial t)(\partial t/\partial y)\\ &= (2rs\ln t)(4) + \left(rs^2/t\right)\left(3xy^2\right) = 8x^2(4y+1)\ln xy^3 + 3x^2(4y+1)^2/y\end{aligned}$$

25. $\partial w/\partial\rho = 2\rho\left(4\sin^2\phi+\cos^2\phi\right)$, $\partial w/\partial\phi = 6\rho^2\sin\phi\cos\phi$, $\partial w/\partial\theta = 0$

26.
$$\begin{aligned}\frac{dw}{dx} &= \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\frac{dy}{dx} + \frac{\partial w}{\partial z}\frac{dz}{dx} = 3y^2z^3 + (6xyz^3)(6x) + 9xy^2z^2\frac{1}{2\sqrt{x-1}}\\ &= 3(3x^2+2)^2(x-1)^{3/2} + 36x^2(3x^2+2)(x-1)^{3/2} + \frac{9}{2}x(3x^2+2)^2\sqrt{x-1}\\ &= \frac{3}{2}(3x^2+2)(39x^3-30x^2+10x-4)\sqrt{x-1}\end{aligned}$$

27. $-\pi$

28. $351/2$, -168

29. $\sqrt{3}e^{\sqrt{3}}$, $\left(2-4\sqrt{3}\right)e^{\sqrt{3}}$

30. 1161

31. $F(x,y) = x^2y^3 + \cos y$, $\dfrac{dy}{dx} = -\dfrac{2xy^3}{3x^2y^2-\sin y}$

32. $F(x,y) = x^3 - 3xy^2 + y^3 - 5$, $\dfrac{dy}{dx} = -\dfrac{3x^2-3y^2}{-6xy+3y^2} = \dfrac{x^2-y^2}{2xy-y^2}$

33. $F(x,y) = e^{xy} + ye^y - 1$, $\dfrac{dy}{dx} = -\dfrac{ye^{xy}}{xe^{xy}+ye^y+e^y}$

34. $F(x,y) = x - (xy)^{1/2} + 3y - 4$, $\dfrac{dy}{dx} = -\dfrac{1-(1/2)(xy)^{-1/2}y}{-(1/2)(xy)^{-1/2}x+3} = \dfrac{2\sqrt{xy}-y}{x-6\sqrt{xy}}$

35. $\dfrac{\partial F}{\partial x} + \dfrac{\partial F}{\partial z}\dfrac{\partial z}{\partial x} = 0$ so $\dfrac{\partial z}{\partial x} = -\dfrac{\partial F/\partial x}{\partial F/\partial z}$.

36. $\dfrac{\partial F}{\partial y} + \dfrac{\partial F}{\partial z}\dfrac{\partial z}{\partial y} = 0$ so $\dfrac{\partial z}{\partial y} = -\dfrac{\partial F/\partial y}{\partial F/\partial z}$.

37. $\dfrac{\partial z}{\partial x} = \dfrac{2x+yz}{6yz-xy}$, $\dfrac{\partial z}{\partial y} = \dfrac{xz-3z^2}{6yz-xy}$

38. $\ln(1+z)+xy^2+z-1=0;\ \dfrac{\partial z}{\partial x}=-\dfrac{y^2(1+z)}{2+z},\ \dfrac{\partial z}{\partial y}=-\dfrac{2xy(1+z)}{2+z}$

39. $ye^x-5\sin 3z-3z=0;\ \dfrac{\partial z}{\partial x}=-\dfrac{ye^x}{-15\cos 3z-3}=\dfrac{ye^x}{15\cos 3z+3},\ \dfrac{\partial z}{\partial y}=\dfrac{e^x}{15\cos 3z+3}$

40. $\dfrac{\partial z}{\partial x}=-\dfrac{ze^{yz}\cos xz-ye^{xy}\cos yz}{ye^{xy}\sin yz+xe^{yz}\cos xz+ye^{yz}\sin xz},\ \dfrac{\partial z}{\partial y}=-\dfrac{ze^{xy}\sin yz-xe^{xy}\cos yz+ze^{yz}\sin xz}{ye^{xy}\sin yz+xe^{yz}\cos xz+ye^{yz}\sin xz}$

41. $D=\left(x^2+y^2\right)^{1/2}$ where x and y are the distances of cars A and B, respectively, from the intersection and D is the distance between them.
$dD/dt=\left[x/\left(x^2+y^2\right)^{1/2}\right](dx/dt)+\left[y/\left(x^2+y^2\right)^{1/2}\right](dy/dt)$, $dx/dt=-25$ and $dy/dt=-30$ when $x=0.3$ and $y=0.4$ so $dD/dt=(0.3/0.5)(-25)+(0.4/0.5)(-30)=-39$ mph.

42. $T=(1/10)PV$, $dT/dt=(V/10)(dP/dt)+(P/10)(dV/dt)$, $dV/dt=4$ and $dP/dt=-1$ when $V=200$ and $P=5$ so $dT/dt=(20)(-1)+(1/2)(4)=-18$ K/s.

43. $A=\dfrac{1}{2}ab\sin\theta$ but $\theta=\pi/6$ when $a=4$ and $b=3$ so $A=\dfrac{1}{2}(4)(3)\sin(\pi/6)=3$.

Solve $\dfrac{1}{2}ab\sin\theta=3$ for θ to get $\theta=\sin^{-1}\left(\dfrac{6}{ab}\right)$, $0\le\theta\le\pi/2$.

$$\frac{d\theta}{dt}=\frac{\partial\theta}{\partial a}\frac{da}{dt}+\frac{\partial\theta}{\partial b}\frac{db}{dt}=\frac{1}{\sqrt{1-\dfrac{36}{a^2b^2}}}\left(-\frac{6}{a^2b}\right)\frac{da}{dt}+\frac{1}{\sqrt{1-\dfrac{36}{a^2b^2}}}\left(-\frac{6}{ab^2}\right)\frac{db}{dt}$$

$$=-\frac{6}{\sqrt{a^2b^2-36}}\left(\frac{1}{a}\frac{da}{dt}+\frac{1}{b}\frac{db}{dt}\right),\ \frac{da}{dt}=1\ \text{and}\ \frac{db}{dt}=1$$

when $a=4$ and $b=3$ so $\dfrac{d\theta}{dt}=-\dfrac{6}{\sqrt{144-36}}\left(\dfrac{1}{4}+\dfrac{1}{3}\right)=-\dfrac{7}{12\sqrt{3}}=-\dfrac{7}{36}\sqrt{3}$ radians/s

44. From the law of cosines, $c=\sqrt{a^2+b^2-2ab\cos\theta}$ where c is the length of the third side.

$\theta=\pi/3$ so $c=\sqrt{a^2+b^2-ab}$,

$$\frac{dc}{dt}=\frac{\partial c}{\partial a}\frac{da}{dt}+\frac{\partial c}{\partial b}\frac{db}{dt}=\frac{1}{2}(a^2+b^2-ab)^{-1/2}(2a-b)\frac{da}{dt}+\frac{1}{2}\left(a^2+b^2-ab\right)^{-1/2}(2b-a)\frac{db}{dt}$$

$$=\frac{1}{2\sqrt{a^2+b^2-ab}}\left[(2a-b)\frac{da}{dt}+(2b-a)\frac{db}{dt}\right],\ \frac{da}{dt}=2\ \text{and}\ \frac{db}{dt}=1\ \text{when}\ a=5\ \text{and}\ b=10$$

so $\dfrac{dc}{dt}=\dfrac{1}{2\sqrt{75}}[(0)(2)+(15)(1)]=\sqrt{3}/2$ cm/s. The third side is increasing.

45. $V=(\pi/4)D^2h$ where D is the diameter and h is the height, both measured in inches, $dV/dt=(\pi/2)Dh(dD/dt)+(\pi/4)D^2(dh/dt)$, $dD/dt=3$ and $dh/dt=24$ when $D=30$ and $h=240$, so $dV/dt=(\pi/2)(30)(240)(3)+(\pi/4)(30)^2(24)=16{,}200\pi$ in^3/year.

46. $\dfrac{dT}{dt}=\dfrac{\partial T}{\partial x}\dfrac{dx}{dt}+\dfrac{\partial T}{\partial y}\dfrac{dy}{dt}=\dfrac{y^2}{x}\dfrac{dx}{dt}+2y\ln x\dfrac{dy}{dt}$, $dx/dt=1$ and $dy/dt=-4$ at (3,2) so

$dT/dt=(4/3)(1)+(4\ln 3)(-4)=4/3-16\ln 3^\circ$ C/s.

47. **(a)** $V = \ell wh, \dfrac{dV}{dt} = \dfrac{\partial V}{\partial \ell}\dfrac{d\ell}{dt} + \dfrac{\partial V}{\partial w}\dfrac{dw}{dt} + \dfrac{\partial V}{\partial h}\dfrac{dh}{dt} = wh\dfrac{d\ell}{dt} + \ell h\dfrac{dw}{dt} + \ell w\dfrac{dh}{dt}$

$= (3)(6)(1) + (2)(6)(2) + (2)(3)(3) = 60 \text{ in}^3/\text{s}$

(b) $D = \sqrt{\ell^2 + w^2 + h^2}$; $dD/dt = (\ell/D)d\ell/dt + (w/D)dw/dt + (h/D)dh/dt$

$= (2/7)(1) + (3/7)(2) + (6/7)(3) = 26/7 \text{ in/s}$

48. $S = 2(lw + wh + lh), \dfrac{dS}{dt} = \dfrac{\partial S}{\partial w}\dfrac{dw}{dt} + \dfrac{\partial S}{\partial l}\dfrac{dl}{dt} + \dfrac{\partial S}{\partial h}\dfrac{dh}{dt}$

$= 2(l + h)\dfrac{dw}{dt} + 2(w + h)\dfrac{dl}{dt} + 2(w + l)\dfrac{dh}{dt} = 80 \text{ in}^2/\text{s}$

49. **(a)** $f(tx, ty) = 3t^2x^2 + t^2y^2 = t^2 f(x, y)$; $n = 2$

(b) $f(tx, ty) = \sqrt{t^2x^2 + t^2y^2} = tf(x, y)$; $n = 1$

(c) $f(tx, ty) = t^3x^2y - 2t^3y^3 = t^3 f(x, y)$; $n = 3$

(d) $f(tx, ty) = 5/\left(t^2x^2 + 2t^2y^2\right)^2 = t^{-4} f(x, y)$; $n = -4$

50. **(a)** If $f(u, v) = t^n f(x, y)$, then $\dfrac{\partial f}{\partial u}\dfrac{du}{dt} + \dfrac{\partial f}{\partial v}\dfrac{dv}{dt} = nt^{n-1}f(x, y)$, $x\dfrac{\partial f}{\partial u} + y\dfrac{\partial f}{\partial v} = nt^{n-1}f(x, y)$;

let $t = 1$ to get $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = nf(x, y)$.

(b) If $f(x, y) = 3x^2 + y^2$ then $xf_x + yf_y = 6x^2 + 2y^2 = 2f(x, y)$;
If $f(x, y) = \sqrt{x^2 + y^2}$ then $xf_x + yf_y = x^2/\sqrt{x^2 + y^2} + y^2/\sqrt{x^2 + y^2} = \sqrt{x^2 + y^2} = f(x, y)$;
If $f(x, y) = x^2y - 2y^3$ then $xf_x + yf_y = 3x^2y - 6y^3 = 3f(x, y)$;
If $f(x, y) = \dfrac{5}{(x^2 + 2y^2)^2}$ then $xf_x + yf_y = x\dfrac{5(-2)2x}{(x^2 + 2y^2)^3} + y\dfrac{5(-2)4y}{(x^2 + 2y^2)^3} = -4f(x, y)$

51. **(a)** $\dfrac{\partial z}{\partial x} = \dfrac{dz}{du}\dfrac{\partial u}{\partial x}, \dfrac{\partial z}{\partial y} = \dfrac{dz}{du}\dfrac{\partial u}{\partial y}$

(b) $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{dz}{du}\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial}{\partial x}\left(\dfrac{dz}{du}\right)\dfrac{\partial u}{\partial x} = \dfrac{dz}{du}\dfrac{\partial^2 u}{\partial x^2} + \dfrac{d^2 z}{du^2}\left(\dfrac{\partial u}{\partial x}\right)^2$;

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{dz}{du}\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial}{\partial y}\left(\frac{dz}{du}\right)\frac{\partial u}{\partial x} = \frac{dz}{du}\frac{\partial^2 u}{\partial y \partial x} + \frac{d^2 z}{du^2}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{dz}{du}\frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial y}\left(\frac{dz}{du}\right)\frac{\partial u}{\partial y} = \frac{dz}{du}\frac{\partial^2 u}{\partial y^2} + \frac{d^2 z}{du^2}\left(\frac{\partial u}{\partial y}\right)^2$$

52. **(a)** $z = f(u)$, $u = x^2 - y^2$; $\partial z/\partial x = (dz/du)(\partial u/\partial x) = 2x dz/du$
$\partial z/\partial y = (dz/du)(\partial u/\partial y) = -2y dz/du$, $y\partial z/\partial x + x\partial z/\partial y = 2xy dz/du - 2xy dz/du = 0$

(b) $z = f(u)$, $u = xy$; $\dfrac{\partial z}{\partial x} = \dfrac{dz}{du}\dfrac{\partial u}{\partial x} = y\dfrac{dz}{du}$, $\dfrac{\partial z}{\partial y} = \dfrac{dz}{du}\dfrac{\partial u}{\partial y} = x\dfrac{dz}{du}$,

$x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y} = xy\dfrac{dz}{du} - xy\dfrac{dz}{du} = 0$.

(c) $yz_x + xz_y = y(2x\cos(x^2 - y^2)) - x(2y\cos(x^2 - y^2)) = 0$

(d) $xz_x - yz_y = xye^{xy} - yxe^{xy} = 0$

53. Let $z = f(u)$ where $u = x + 2y$; then $\partial z/\partial x = (dz/du)(\partial u/\partial x) = dz/du$,
$\partial z/\partial y = (dz/du)(\partial u/\partial y) = 2dz/du$ so $2\partial z/\partial x - \partial z/\partial y = 2dz/du - 2dz/du = 0$

54. Let $z = f(u)$ where $u = x^2 + y^2$; then $\partial z/\partial x = (dz/du)(\partial u/\partial x) = 2x\ dz/du$,
$\partial z/\partial y = (dz/du)(\partial u/\partial y) = 2ydz/du$ so $y\ \partial z/\partial x - x\partial z/\partial y = 2xydz/du - 2xydz/du = 0$

55. $\dfrac{\partial w}{\partial x} = \dfrac{dw}{du}\dfrac{\partial u}{\partial x} = \dfrac{dw}{du}$, $\dfrac{\partial w}{\partial y} = \dfrac{dw}{du}\dfrac{\partial u}{\partial y} = 2\dfrac{dw}{du}$, $\dfrac{\partial w}{\partial z} = \dfrac{dw}{du}\dfrac{\partial u}{\partial z} = 3\dfrac{dw}{du}$, so $\dfrac{\partial w}{\partial x} + \dfrac{\partial w}{\partial y} + \dfrac{\partial w}{\partial z} = 6\dfrac{dw}{du}$

56. $\partial w/\partial x = (dw/d\rho)(\partial\rho/\partial x) = (x/\rho)dw/d\rho$, similarly $\partial w/\partial y = (y/\rho)dw/d\rho$ and
$\partial w/\partial z = (z/\rho)dw/d\rho$ so $(\partial w/\partial x)^2 + (\partial w/\partial y)^2 + (\partial w/\partial z)^2 = (dw/d\rho)^2$

57. $z = f(u, v)$ where $u = x - y$ and $v = y - x$,
$\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x} = \dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v}$ and $\dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial y} = -\dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v}$ so $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0$

58. Let $w = f(r, s, t)$ where $r = x - y$, $s = y - z$, $t = z - x$;
$\partial w/\partial x = (\partial w/\partial r)(\partial r/\partial x) + (\partial w/\partial t)(\partial t/\partial x) = \partial w/\partial r - \partial w/\partial t$, similarly
$\partial w/\partial y = -\partial w/\partial r + \partial w/\partial s$ and $\partial w/\partial z = -\partial w/\partial s + \partial w/\partial t$ so $\partial w/\partial x + \partial w/\partial y + \partial w/\partial z = 0$

59. **(a)** $1 = -r\sin\theta\dfrac{\partial\theta}{\partial x} + \cos\theta\dfrac{\partial r}{\partial x}$ and $0 = r\cos\theta\dfrac{\partial\theta}{\partial x} + \sin\theta\dfrac{\partial r}{\partial x}$; solve for $\partial r/\partial x$ and $\partial\theta/\partial x$.

(b) $0 = -r\sin\theta\dfrac{\partial\theta}{\partial y} + \cos\theta\dfrac{\partial r}{\partial y}$ and $1 = r\cos\theta\dfrac{\partial\theta}{\partial y} + \sin\theta\dfrac{\partial r}{\partial y}$; solve for $\partial r/\partial y$ and $\partial\theta/\partial y$.

(c)
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial\theta}\frac{\partial\theta}{\partial x} = \frac{\partial z}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial z}{\partial\theta}\sin\theta.$$
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial z}{\partial\theta}\frac{\partial\theta}{\partial y} = \frac{\partial z}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial z}{\partial\theta}\cos\theta.$$

(d) Square and add the results of Parts (a) and (b).

(e) From Part (c),
$$\begin{aligned}\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial z}{\partial\theta}\sin\theta\right)\frac{\partial r}{\partial x} + \frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial z}{\partial\theta}\sin\theta\right)\frac{\partial\theta}{\partial x}\\ &= \left(\frac{\partial^2 z}{\partial r^2}\cos\theta + \frac{1}{r^2}\frac{\partial z}{\partial\theta}\sin\theta - \frac{1}{r}\frac{\partial^2 z}{\partial r\partial\theta}\sin\theta\right)\cos\theta\\ &\quad + \left(\frac{\partial^2 z}{\partial\theta\partial r}\cos\theta - \frac{\partial z}{\partial r}\sin\theta - \frac{1}{r}\frac{\partial^2 z}{\partial\theta^2}\sin\theta - \frac{1}{r}\frac{\partial z}{\partial\theta}\cos\theta\right)\left(-\frac{\sin\theta}{r}\right)\\ &= \frac{\partial^2 z}{\partial r^2}\cos^2\theta + \frac{2}{r^2}\frac{\partial z}{\partial\theta}\sin\theta\cos\theta - \frac{2}{r}\frac{\partial^2 z}{\partial\theta\partial r}\sin\theta\cos\theta + \frac{1}{r^2}\frac{\partial^2 z}{\partial\theta^2}\sin^2\theta + \frac{1}{r}\frac{\partial z}{\partial r}\sin^2\theta.\end{aligned}$$

Similarly, from Part (c),
$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2}\sin^2\theta - \frac{2}{r^2}\frac{\partial z}{\partial\theta}\sin\theta\cos\theta + \frac{2}{r}\frac{\partial^2 z}{\partial\theta\partial r}\sin\theta\cos\theta + \frac{1}{r^2}\frac{\partial^2 z}{\partial\theta^2}\cos^2\theta + \frac{1}{r}\frac{\partial z}{\partial r}\cos^2\theta.$$

Add to get $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial^2 z}{\partial r^2} + \dfrac{1}{r^2}\dfrac{\partial^2 z}{\partial\theta^2} + \dfrac{1}{r}\dfrac{\partial z}{\partial r}$.

60. $z_x = \dfrac{-2y}{x^2+y^2}, z_{xx} = \dfrac{4xy}{(x^2+y^2)^2}, z_y = \dfrac{2x}{x^2+y^2}, z_{yy} = -\dfrac{4xy}{(x^2+y^2)^2}, z_{xx} + z_{yy} = 0;$

$z = \tan^{-1}\dfrac{2r^2\cos\theta\sin\theta}{r^2(\cos^2\theta - \sin^2\theta)} = \tan^{-1}\tan 2\theta = 2\theta + k\pi$ for some fixed k; $z_r = 0, z_{\theta\theta} = 0$

61. **(a)** By the chain rule, $\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\cos\theta + \dfrac{\partial u}{\partial y}\sin\theta$ and $\dfrac{\partial v}{\partial \theta} = -\dfrac{\partial v}{\partial x}r\sin\theta + \dfrac{\partial v}{\partial y}r\cos\theta$, use the Cauchy-Riemann conditions $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ in the equation for $\dfrac{\partial u}{\partial r}$ to get $\dfrac{\partial u}{\partial r} = \dfrac{\partial v}{\partial y}\cos\theta - \dfrac{\partial v}{\partial x}\sin\theta$ and compare to $\dfrac{\partial v}{\partial \theta}$ to see that $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$. The result $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$ can be obtained by considering $\dfrac{\partial v}{\partial r}$ and $\dfrac{\partial u}{\partial \theta}$.

(b) $u_x = \dfrac{2x}{x^2+y^2},\ v_y = 2\dfrac{1}{x}\dfrac{1}{1+(y/x)^2} = \dfrac{2x}{x^2+y^2} = u_x;$

$u_y = \dfrac{2y}{x^2+y^2},\ v_x = -2\dfrac{y}{x^2}\dfrac{1}{1+(y/x)^2} = -\dfrac{2y}{x^2+y^2} = -u_y;$

$u = \ln r^2, v = 2\theta, u_r = 2/r, v_\theta = 2$, so $u_r = \dfrac{1}{r}v_\theta,\ u_\theta = 0, v_r = 0$, so $v_r = -\dfrac{1}{r}u_\theta$

62. **(a)** $u_x = f'(x+ct),\ u_{xx} = f''(x+ct),\ u_t = cf'(x+ct),\ u_{tt} = c^2 f''(x+ct); u_{tt} = c^2 u_{xx}$

(b) Substitute g for f and $-c$ for c in Part (a).

(c) Since the sum of derivatives equals the derivative of the sum, the result follows from Parts (a) and (b).

(d) $\sin t \sin x = \dfrac{1}{2}(-\cos(x+t) + \cos(x-t))$

63. $\partial w/\partial\rho = (\sin\phi\cos\theta)\partial w/\partial x + (\sin\phi\sin\theta)\partial w/\partial y + (\cos\phi)\,\partial w/\partial z$
$\partial w/\partial\phi = (\rho\cos\phi\cos\theta)\partial w/\partial x + (\rho\cos\phi\sin\theta)\partial w/\partial y - (\rho\sin\phi)\partial w/\partial z$
$\partial w/\partial\theta = -(\rho\sin\phi\sin\theta)\partial w/\partial x + (\rho\sin\phi\cos\theta)\partial w/\partial y$

64. **(a)** $\dfrac{\partial w}{\partial x} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial x}$ **(b)** $\dfrac{\partial w}{\partial y} = \dfrac{\partial f}{\partial y} + \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial y}$

65. $w_r = e^r/\left(e^r+e^s+e^t+e^u\right),\ w_{rs} = -e^r e^s/\left(e^r+e^s+e^t+e^u\right)^2,$
$w_{rst} = 2e^r e^s e^t/\left(e^r+e^s+e^t+e^u\right)^3,$
$w_{rstu} = -6e^r e^s e^t e^u/\left(e^r+e^s+e^t+e^u\right)^4 = -6e^{r+s+t+u}/e^{4w} = -6e^{r+s+t+u-4w}$

66. $\partial w/\partial y_1 = a_1\partial w/\partial x_1 + a_2\partial w/\partial x_2 + a_3\partial w/\partial x_3,$
$\partial w/\partial y_2 = b_1\partial w/\partial x_1 + b_2\partial w/\partial x_2 + b_3\partial w/\partial x_3$

67. **(a)** $dw/dt = \displaystyle\sum_{i=1}^{4}(\partial w/\partial x_i)(dx_i/dt)$

(b) $\partial w/\partial v_j = \displaystyle\sum_{i=1}^{4}(\partial w/\partial x_i)(\partial x_i/\partial v_j)$ for $j = 1,2,3$

68. Let $u = x_1^2 + x_2^2 + ... + x_n^2$; then $w = u^k$, $\partial w/\partial x_i = ku^{k-1}(2x_i) = 2k\ x_i u^{k-1}$,

$\partial^2 w/\partial x_i^2 = 2k(k-1)x_i u^{k-2}(2x_i) + 2ku^{k-1} = 4k(k-1)x_i^2 u^{k-2} + 2ku^{k-1}$ for $i = 1, 2, \ldots, n$

so $\sum_{i=1}^{n} \partial^2 w/\partial x_i^2 = 4k(k-1)u^{k-2}\sum_{i=1}^{n} x_i^2 + 2kn\,u^{k-1}$

$= 4k(k-1)u^{k-2}u + 2kn\,u^{k-1} = 2ku^{k-1}[2(k-1)+n]$

which is 0 if $k = 0$ or if $2(k-1) + n = 0$, $k = 1 - n/2$.

69. $dF/dx = (\partial F/\partial u)(du/dx) + (\partial F/\partial v)(dv/dx)$

$= f(u)g'(x) - f(v)h'(x) = f(g(x))g'(x) - f(h(x))h'(x)$

70. Represent the line segment C that joins A and B by $x = x_0 + (x_1 - x_0)t$, $y = y_0 + (y_1 - y_0)t$ for $0 \le t \le 1$. Let $F(t) = f(x_0 + (x_1 - x_0)t, y_0 + (y_1 - y_0)t)$ for $0 \le t \le 1$; then $f(x_1, y_1) - f(x_0, y_0) = F(1) - F(0)$. Apply the Mean Value Theorem to $F(t)$ on the interval [0,1] to get $[F(1) - F(0)]/(1 - 0) = F'(t^*)$, $F(1) - F(0) = F'(t^*)$ for some t^* in (0,1) so $f(x_1, y_1) - f(x_0, y_0) = F'(t^*)$. By the chain rule, $F'(t) = f_x(x, y)(dx/dt) + f_y(x, y)(dy/dt) = f_x(x, y)(x_1 - x_0) + f_y(x, y)(y_1 - y_0)$. Let (x^*, y^*) be the point on C for $t = t^*$ then $f(x_1, y_1) - f(x_0, y_0) = F'(t^*) = f_x(x^*, y^*)(x_1 - x_0) + f_y(x^*, y^*)(y_1 - y_0)$.

71. Let (a, b) be any point in the region, if (x, y) is in the region then by the result of Exercise 70 $f(x, y) - f(a, b) = f_x(x^*, y^*)(x - a) + f_y(x^*, y^*)(y - b)$ where (x^*, y^*) is on the line segment joining (a, b) and (x, y). If $f_x(x, y) = f_y(x, y) = 0$ throughout the region then $f(x, y) - f(a, b) = (0)(x - a) + (0)(y - b) = 0$, $f(x, y) = f(a, b)$ so $f(x, y)$ is constant on the region.

EXERCISE SET 14.6

1. $\nabla f(x, y) = (3y/2)(1 + xy)^{1/2}\mathbf{i} + (3x/2)(1 + xy)^{1/2}\mathbf{j}$, $\nabla f(3, 1) = 3\mathbf{i} + 9\mathbf{j}$, $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = 12/\sqrt{2} = 6\sqrt{2}$

2. $\nabla f(x, y) = 2ye^{2xy}\mathbf{i} + 2xe^{2xy}\mathbf{j}$, $\nabla f(4, 0) = 8\mathbf{j}$, $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = 32/5$

3. $\nabla f(x, y) = \left[2x/\left(1 + x^2 + y\right)\right]\mathbf{i} + \left[1/\left(1 + x^2 + y\right)\right]\mathbf{j}$, $\nabla f(0, 0) = \mathbf{j}$, $D_{\mathbf{u}}f = -3/\sqrt{10}$

4. $\nabla f(x, y) = -\left[(c + d)y/(x - y)^2\right]\mathbf{i} + \left[(c + d)x/(x - y)^2\right]\mathbf{j}$,
$\nabla f(3, 4) = -4(c + d)\mathbf{i} + 3(c + d)\mathbf{j}$, $D_{\mathbf{u}}f = -(7/5)(c + d)$

5. $\nabla f(x, y, z) = 20x^4y^2z^3\mathbf{i} + 8x^5yz^3\mathbf{j} + 12x^5y^2z^2\mathbf{k}$, $\nabla f(2, -1, 1) = 320\mathbf{i} - 256\mathbf{j} + 384\mathbf{k}$, $D_{\mathbf{u}}f = -320$

6. $\nabla f(x, y, z) = yze^{xz}\mathbf{i} + e^{xz}\mathbf{j} + (xye^{xz} + 2z)\mathbf{k}$, $\nabla f(0, 2, 3) = 6\mathbf{i} + \mathbf{j} + 6\mathbf{k}$, $D_{\mathbf{u}}f = 45/7$

7. $\nabla f(x, y, z) = \dfrac{2x}{x^2 + 2y^2 + 3z^2}\mathbf{i} + \dfrac{4y}{x^2 + 2y^2 + 3z^2}\mathbf{j} + \dfrac{6z}{x^2 + 2y^2 + 3z^2}\mathbf{k}$,
$\nabla f(-1, 2, 4) = (-2/57)\mathbf{i} + (8/57)\mathbf{j} + (24/57)\mathbf{k}$, $D_{\mathbf{u}}f = -314/741$

8. $\nabla f(x, y, z) = yz\cos xyz\mathbf{i} + xz\cos xyz\mathbf{j} + xy\cos xyz\mathbf{k}$,
$\nabla f(1/2, 1/3, \pi) = (\pi\sqrt{3}/6)\mathbf{i} + (\pi\sqrt{3}/4)\mathbf{j} + (\sqrt{3}/12)\mathbf{k}$, $D_{\mathbf{u}}f = (1 - \pi)/12$

9. $\nabla f(x, y) = 12x^2y^2\mathbf{i} + 8x^3y\mathbf{j}$, $\nabla f(2, 1) = 48\mathbf{i} + 64\mathbf{j}$, $\mathbf{u} = (4/5)\mathbf{i} - (3/5)\mathbf{j}$, $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = 0$

10. $\nabla f(x,y) = (2x-3y)\mathbf{i} + \left(-3x+12y^2\right)\mathbf{j}$, $\nabla f(-2,0) = -4\mathbf{i}+6\mathbf{j}$, $\mathbf{u} = (\mathbf{i}+2\mathbf{j})/\sqrt{5}$, $D_{\mathbf{u}}f = 8/\sqrt{5}$

11. $\nabla f(x,y) = \left(y^2/x\right)\mathbf{i} + 2y\ln x\mathbf{j}$, $\nabla f(1,4) = 16\mathbf{i}$, $\mathbf{u} = (-\mathbf{i}+\mathbf{j})/\sqrt{2}$, $D_{\mathbf{u}}f = -8\sqrt{2}$

12. $\nabla f(x,y) = e^x\cos y\mathbf{i} - e^x\sin y\mathbf{j}$, $\nabla f(0,\pi/4) = (\mathbf{i}-\mathbf{j})/\sqrt{2}$, $\mathbf{u} = (5\mathbf{i}-2\mathbf{j})/\sqrt{29}$, $D_{\mathbf{u}}f = 7/\sqrt{58}$

13. $\nabla f(x,y) = -\left[y/\left(x^2+y^2\right)\right]\mathbf{i} + \left[x/\left(x^2+y^2\right)\right]\mathbf{j}$,
$\nabla f(-2,2) = -(\mathbf{i}+\mathbf{j})/4$, $\mathbf{u} = -(\mathbf{i}+\mathbf{j})/\sqrt{2}$, $D_{\mathbf{u}}f = \sqrt{2}/4$

14. $\nabla f(x,y) = (e^y - ye^x)\mathbf{i} + (xe^y - e^x)\mathbf{j}$, $\nabla f(0,0) = \mathbf{i}-\mathbf{j}$, $\mathbf{u} = (5\mathbf{i}-2\mathbf{j})/\sqrt{29}$, $D_{\mathbf{u}}f = 7/\sqrt{29}$

15. $\nabla f(x,y,z) = \left(3x^2z-2xy\right)\mathbf{i} - x^2\mathbf{j} + \left(x^3+2z\right)\mathbf{k}$, $\nabla f(2,-1,1) = 16\mathbf{i}-4\mathbf{j}+10\mathbf{k}$,
$\mathbf{u} = (3\mathbf{i}-\mathbf{j}+2\mathbf{k})/\sqrt{14}$, $D_{\mathbf{u}}f = 72/\sqrt{14}$

16. $\nabla f(x,y,z) = -x\left(x^2+z^2\right)^{-1/2}\mathbf{i} + \mathbf{j} - z\left(x^2+z^2\right)^{-1/2}\mathbf{k}$, $\nabla f(-3,1,4) = (3/5)\mathbf{i}+\mathbf{j}-(4/5)\mathbf{k}$,
$\mathbf{u} = (2\mathbf{i}-2\mathbf{j}-\mathbf{k})/3$, $D_{\mathbf{u}}f = 0$

17. $\nabla f(x,y,z) = -\dfrac{1}{z+y}\mathbf{i} - \dfrac{z-x}{(z+y)^2}\mathbf{j} + \dfrac{y+x}{(z+y)^2}\mathbf{k}$, $\nabla f(1,0,-3) = (1/3)\mathbf{i} + (4/9)\mathbf{j} + (1/9)\mathbf{k}$,
$\mathbf{u} = (-6\mathbf{i}+3\mathbf{j}-2\mathbf{k})/7$, $D_{\mathbf{u}}f = -8/63$

18. $\nabla f(x,y,z) = e^{x+y+3z}(\mathbf{i}+\mathbf{j}+3\mathbf{k})$, $\nabla f(-2,2,-1) = e^{-3}(\mathbf{i}+\mathbf{j}+3\mathbf{k})$, $\mathbf{u} = (20\mathbf{i}-4\mathbf{j}+5\mathbf{k})/21$,
$D_{\mathbf{u}}f = (31/21)e^{-3}$

19. $\nabla f(x,y) = (y/2)(xy)^{-1/2}\mathbf{i} + (x/2)(xy)^{-1/2}\mathbf{j}$, $\nabla f(1,4) = \mathbf{i} + (1/4)\mathbf{j}$,
$\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} = (1/2)\mathbf{i} + \left(\sqrt{3}/2\right)\mathbf{j}$, $D_{\mathbf{u}}f = 1/2 + \sqrt{3}/8$

20. $\nabla f(x,y) = [2y/(x+y)^2]\mathbf{i} - [2x/(x+y)^2]\mathbf{j}$, $\nabla f(-1,-2) = -(4/9)\mathbf{i} + (2/9)\mathbf{j}$, $\mathbf{u} = \mathbf{j}$, $D_{\mathbf{u}}f = 2/9$

21. $\nabla f(x,y) = 2\sec^2(2x+y)\mathbf{i} + \sec^2(2x+y)\mathbf{j}$, $\nabla f(\pi/6,\pi/3) = 8\mathbf{i}+4\mathbf{j}$, $\mathbf{u} = (\mathbf{i}-\mathbf{j})/\sqrt{2}$, $D_{\mathbf{u}}f = 2\sqrt{2}$

22. $\nabla f(x,y) = \cosh x\cosh y\mathbf{i} + \sinh x\sinh y\mathbf{j}$, $\nabla f(0,0) = \mathbf{i}$, $\mathbf{u} = -\mathbf{i}$, $D_{\mathbf{u}}f = -1$

23. $\nabla f(x,y) = y(x+y)^{-2}\mathbf{i} - x(x+y)^{-2}\mathbf{j}$, $\nabla f(1,0) = -\mathbf{j}$, $\overrightarrow{PQ} = -2\mathbf{i}-\mathbf{j}$, $\mathbf{u} = (-2\mathbf{i}-\mathbf{j})/\sqrt{5}$,
$D_{\mathbf{u}}f = 1/\sqrt{5}$

24. $\nabla f(x,y) = -e^{-x}\sec y\mathbf{i} + e^{-x}\sec y\tan y\mathbf{j}$,
$\nabla f(0,\pi/4) = \sqrt{2}(-\mathbf{i}+\mathbf{j})$, $\overrightarrow{PO} = -(\pi/4)\mathbf{j}$, $\mathbf{u} = -\mathbf{j}$, $D_{\mathbf{u}}f = -\sqrt{2}$

25. $\nabla f(x,y) = \dfrac{ye^y}{2\sqrt{xy}}\mathbf{i} + \left(\sqrt{xy}e^y + \dfrac{xe^y}{2\sqrt{xy}}\right)\mathbf{j}$, $\nabla f(1,1) = (e/2)(\mathbf{i}+3\mathbf{j})$, $\mathbf{u} = -\mathbf{j}$, $D_{\mathbf{u}}f = -3e/2$

26. $\nabla f(x,y) = -y(x+y)^{-2}\mathbf{i} + x(x+y)^{-2}\mathbf{j}$, $\nabla f(2,3) = (-3\mathbf{i}+2\mathbf{j})/25$, if $D_{\mathbf{u}}f = 0$ then $\mathbf{u}$ and ∇f are orthogonal, by inspection $2\mathbf{i}+3\mathbf{j}$ is orthogonal to $\nabla f(2,3)$ so $\mathbf{u} = \pm(2\mathbf{i}+3\mathbf{j})/\sqrt{13}$.

27. $\nabla f(2,1,-1) = -\mathbf{i}+\mathbf{j}-\mathbf{k}$. $\overrightarrow{PQ} = -3\mathbf{i}+\mathbf{j}+\mathbf{k}$, $\mathbf{u} = (-3\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{11}$, $D_{\mathbf{u}}f = 3/\sqrt{11}$

28. $\nabla f(-1,-2,1) = 13\mathbf{i}+5\mathbf{j}-20\mathbf{k}$, $\mathbf{u} = -\mathbf{k}$, $D_{\mathbf{u}}f = 20$

29. Solve the system $(3/5)f_x(1,2) - (4/5)f_y(1,2) = -5$, $(4/5)f_x(1,2) + (3/5)f_y(1,2) = 10$ for

(a) $f_x(1,2) = 5$ **(b)** $f_y(1,2) = 10$

(c) $\nabla f(1,2) = 5\mathbf{i}+10\mathbf{j}$, $\mathbf{u} = (-\mathbf{i}-2\mathbf{j})/\sqrt{5}$, $D_{\mathbf{u}}f = -5\sqrt{5}$.

30. $\nabla f(-5,1) = -3\mathbf{i}+2\mathbf{j}$, $\overrightarrow{PQ} = \mathbf{i}+2\mathbf{j}$, $\mathbf{u} = (\mathbf{i}+2\mathbf{j})/\sqrt{5}$, $D_{\mathbf{u}}f = 1/\sqrt{5}$

31. f increases the most in the direction of III.

32. The contour lines are closer at P, so the function is increasing more rapidly there, hence ∇f is larger at P.

33. $\nabla z = 4\mathbf{i}-8\mathbf{j}$

34. $\nabla z = -4e^{-3y}\sin 4x\mathbf{i} - 3e^{-3y}\cos 4x\mathbf{j}$

35. $\nabla w = \dfrac{x}{x^2+y^2+z^2}\mathbf{i} + \dfrac{y}{x^2+y^2+z^2}\mathbf{j} + \dfrac{z}{x^2+y^2+z^2}\mathbf{k}$

36. $\nabla w = e^{-5x}\sec(x^2yz)\left[\left(2xyz\tan(x^2yz)-5\right)\mathbf{i} + x^2z\tan(x^2yz)\,\mathbf{j} + x^2y\tan(x^2yz)\,\mathbf{k}\right]$

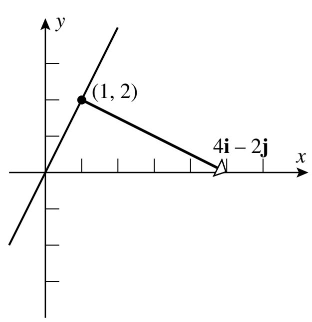
37. $\nabla f(x,y) = 3(2x+y)\left(x^2+xy\right)^2\mathbf{i} + 3x\left(x^2+xy\right)^2\mathbf{j}$, $\nabla f(-1,-1) = -36\mathbf{i}-12\mathbf{j}$

38. $\nabla f(x,y) = -x\left(x^2+y^2\right)^{-3/2}\mathbf{i} - y\left(x^2+y^2\right)^{-3/2}\mathbf{j}$, $\nabla f(3,4) = -(3/125)\mathbf{i} - (4/125)\mathbf{j}$

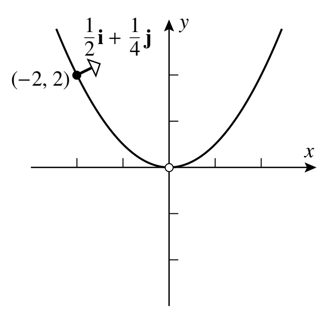
39. $\nabla f(x,y,z) = [y/(x+y+z)]\mathbf{i} + [y/(x+y+z) + \ln(x+y+z)]\mathbf{j} + [y/(x+y+z)]\mathbf{k}$,
$\nabla f(-3,4,0) = 4\mathbf{i}+4\mathbf{j}+4\mathbf{k}$

40. $\nabla f(x,y,z) = 3y^2z\tan^2 x\sec^2 x\,\mathbf{i} + 2yz\tan^3 x\,\mathbf{j} + y^2\tan^3 x\,\mathbf{k}$, $\nabla f(\pi/4,-3) = 54\,\mathbf{i} - 6\,\mathbf{j} + 9\,\mathbf{k}$

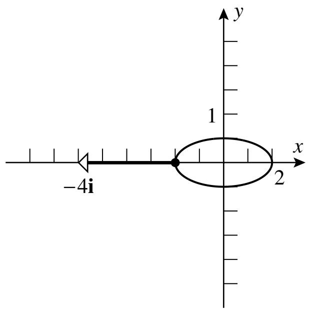
41. $f(1,2) = 3$,
level curve $4x-2y+3 = 3$,
$2x - y = 0$;
$\nabla f(x,y) = 4\mathbf{i}-2\mathbf{j}$
$\nabla f(1,2) = 4\mathbf{i}-2\mathbf{j}$



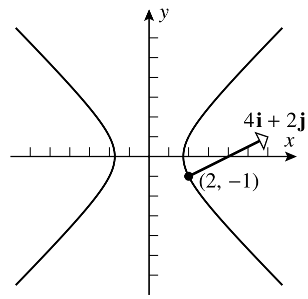
42. $f(-2,2) = 1/2$,
level curve $y/x^2 = 1/2$,
$y = x^2/2$ for $x \neq 0$.
$\nabla f(x,y) = -\left(2y/x^3\right)\mathbf{i} + \left(1/x^2\right)\mathbf{j}$
$\nabla f(-2,2) = (1/2)\mathbf{i} + (1/4)\mathbf{j}$



43. $f(-2,0)=4$,
level curve $x^2+4y^2=4$,
$x^2/4+y^2=1$.
$\nabla f(x,y)=2x\mathbf{i}+8y\mathbf{j}$
$\nabla f(-2,0)=-4\mathbf{i}$



44. $f(2,-1)=3$,
level curve $x^2-y^2=3$.
$\nabla f(x,y)=2x\mathbf{i}-2y\mathbf{j}$
$\nabla f(2,-1)=4\mathbf{i}+2\mathbf{j}$



45. $\nabla f(x,y)=8xy\mathbf{i}+4x^2\mathbf{j}$, $\nabla f(1,-2)=-16\mathbf{i}+4\mathbf{j}$ is normal to the level curve through P so $\mathbf{u}=\pm(-4\mathbf{i}+\mathbf{j})/\sqrt{17}$.

46. $\nabla f(x,y)=(6xy-y)\mathbf{i}+\left(3x^2-x\right)\mathbf{j}$, $\nabla f(2,-3)=-33\mathbf{i}+10\mathbf{j}$ is normal to the level curve through P so $\mathbf{u}=\pm(-33\mathbf{i}+10\mathbf{j})/\sqrt{1189}$.

47. $\nabla f(x,y)=12x^2y^2\mathbf{i}+8x^3y\mathbf{j}$, $\nabla f(-1,1)=12\mathbf{i}-8\mathbf{j}$, $\mathbf{u}=(3\mathbf{i}-2\mathbf{j})/\sqrt{13}$, $\|\nabla f(-1,1)\|=4\sqrt{13}$

48. $\nabla f(x,y)=3\mathbf{i}-(1/y)\mathbf{j}$, $\nabla f(2,4)=3\mathbf{i}-(1/4)\mathbf{j}$, $\mathbf{u}=(12\mathbf{i}-\mathbf{j})/\sqrt{145}$, $\|\nabla f(2,4)\|=\sqrt{145}/4$

49. $\nabla f(x,y)=x\left(x^2+y^2\right)^{-1/2}\mathbf{i}+y\left(x^2+y^2\right)^{-1/2}\mathbf{j}$,
$\nabla f(4,-3)=(4\mathbf{i}-3\mathbf{j})/5$, $\mathbf{u}=(4\mathbf{i}-3\mathbf{j})/5$, $\|\nabla f(4,-3)\|=1$

50. $\nabla f(x,y)=y(x+y)^{-2}\mathbf{i}-x(x+y)^{-2}\mathbf{j}$, $\nabla f(0,2)=(1/2)\mathbf{i}$, $\mathbf{u}=\mathbf{i}$, $\|\nabla f(0,2)\|=1/2$

51. $\nabla f(1,1,-1)=3\mathbf{i}-3\mathbf{j}$, $\mathbf{u}=(\mathbf{i}-\mathbf{j})/\sqrt{2}$, $\|\nabla f(1,1,-1)\|=3\sqrt{2}$

52. $\nabla f(0,-3,0)=(\mathbf{i}-3\mathbf{j}+4\mathbf{k})/6$, $\mathbf{u}=(\mathbf{i}-3\mathbf{j}+4\mathbf{k})/\sqrt{26}$, $\|\nabla f(0,-3,0)\|=\sqrt{26}/6$

53. $\nabla f(1,2,-2)=(-\mathbf{i}+\mathbf{j})/2$, $\mathbf{u}=(-\mathbf{i}+\mathbf{j})/\sqrt{2}$, $\|\nabla f(1,2,-2)\|=1/\sqrt{2}$

54. $\nabla f(4,2,2)=(\mathbf{i}-\mathbf{j}-\mathbf{k})/8$, $\mathbf{u}=(\mathbf{i}-\mathbf{j}-\mathbf{k})/\sqrt{3}$, $\|\nabla f(4,2,2)\|=\sqrt{3}/8$

55. $\nabla f(x,y)=-2x\mathbf{i}-2y\mathbf{j}$, $\nabla f(-1,-3)=2\mathbf{i}+6\mathbf{j}$, $\mathbf{u}=-(\mathbf{i}+3\mathbf{j})/\sqrt{10}$, $-\|\nabla f(-1,-3)\|=-2\sqrt{10}$

56. $\nabla f(x,y)=ye^{xy}\mathbf{i}+xe^{xy}\mathbf{j}$; $\nabla f(2,3)=e^6(3\mathbf{i}+2\mathbf{j})$, $\mathbf{u}=-(3\mathbf{i}+2\mathbf{j})/\sqrt{13}$, $-\|\nabla f(2,3)\|=-\sqrt{13}e^6$

57. $\nabla f(x,y)=-3\sin(3x-y)\mathbf{i}+\sin(3x-y)\mathbf{j}$,
$\nabla f(\pi/6,\pi/4)=(-3\mathbf{i}+\mathbf{j})/\sqrt{2}$, $\mathbf{u}=(3\mathbf{i}-\mathbf{j})/\sqrt{10}$, $-\|\nabla f(\pi/6,\pi/4)\|=-\sqrt{5}$

58. $\nabla f(x,y)=\dfrac{y}{(x+y)^2}\sqrt{\dfrac{x+y}{x-y}}\mathbf{i}-\dfrac{x}{(x+y)^2}\sqrt{\dfrac{x+y}{x-y}}\mathbf{j}$, $\nabla f(3,1)=(\sqrt{2}/16)(\mathbf{i}-3\mathbf{j})$,
$\mathbf{u}=-(\mathbf{i}-3\mathbf{j})/\sqrt{10}$, $-\|\nabla f(3,1)\|=-\sqrt{5}/8$

59. $\nabla f(5,7,6)=-\mathbf{i}+11\mathbf{j}-12\mathbf{k}$, $\mathbf{u}=(\mathbf{i}-11\mathbf{j}+12\mathbf{k})/\sqrt{266}$, $-\|\nabla f(5,7,6)\|=-\sqrt{266}$

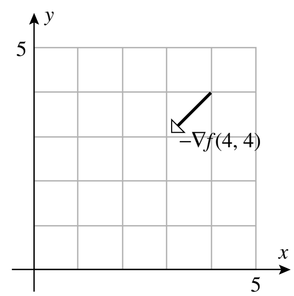
60. $\nabla f(0,1,\pi/4) = 2\sqrt{2}(\mathbf{i}-\mathbf{k})$, $\mathbf{u} = -(\mathbf{i}-\mathbf{k})/\sqrt{2}$, $-\|\nabla f(0,1,\pi/4)\| = -4$

61. $\nabla f(4,-5) = 2\mathbf{i}-\mathbf{j}$, $\mathbf{u} = (5\mathbf{i}+2\mathbf{j})/\sqrt{29}$, $D_{\mathbf{u}}f = 8/\sqrt{29}$

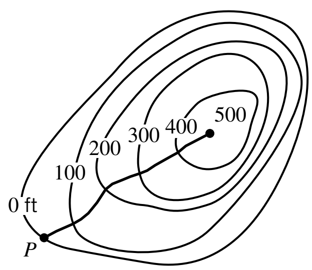
62. Let $\mathbf{u} = u_1\mathbf{i}+u_2\mathbf{j}$ where $u_1^2+u_2^2=1$, but $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = u_1 - 2u_2 = -2$ so $u_1 = 2u_2-2$, $(2u_2-2)^2+u_2^2=1$, $5u_2^2-8u_2+3=0$, $u_2=1$ or $u_2=3/5$ thus $u_1=0$ or $u_1=-4/5$; $\mathbf{u}=\mathbf{j}$ or $\mathbf{u} = -\frac{4}{5}\mathbf{i}+\frac{3}{5}\mathbf{j}$.

63. **(a)** At $(1,2)$ the steepest ascent seems to be in the direction $\mathbf{i}+\mathbf{j}$ and the slope in that direction seems to be $0.5/(\sqrt{2}/2) = 1/\sqrt{2}$, so $\nabla f \approx \frac{1}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}$, which has the required direction and magnitude.

(b) The direction of $-\nabla f(4,4)$ appears to be $-\mathbf{i}-\mathbf{j}$ and its magnitude appears to be $1/0.8 = 5/4$.

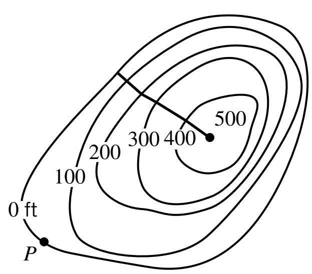


64. **(a)**



Depart from each contour line in a direction orthogonal to that contour line, as an approximation to the optimal path.

(b)



At the top there is no contour line, so head for the nearest contour line. From then on depart from each contour line in a direction orthogonal to that contour line, as in Part (a).

65. $\nabla z = 6x\mathbf{i}-2y\mathbf{j}$, $\|\nabla z\| = \sqrt{36x^2+4y^2} = 6$ if $36x^2+4y^2=36$; all points on the ellipse $9x^2+y^2=9$.

66. $\nabla z = 3\mathbf{i}+2y\mathbf{j}$, $\|\nabla z\| = \sqrt{9+4y^2}$, so $\nabla\|\nabla z\| = \dfrac{4y}{\sqrt{9+4y^2}}\mathbf{j}$, and $\nabla\|\nabla z\|\Big|_{(x,y)=(5,2)} = \dfrac{8}{5}\mathbf{j}$

67. $\mathbf{r} = t\mathbf{i}-t^2\mathbf{j}$, $d\mathbf{r}/dt = \mathbf{i}-2t\mathbf{j} = \mathbf{i}-4\mathbf{j}$ at the point $(2,-4)$, $\mathbf{u} = (\mathbf{i}-4\mathbf{j})/\sqrt{17}$;
$\nabla z = 2x\mathbf{i}+2y\mathbf{j} = 4\mathbf{i}-8\mathbf{j}$ at $(2,-4)$, hence $dz/ds = D_{\mathbf{u}}z = \nabla z \cdot \mathbf{u} = 36/\sqrt{17}$.

68. **(a)** $\nabla T(x,y)=\dfrac{y\left(1-x^2+y^2\right)}{\left(1+x^2+y^2\right)^2}\mathbf{i}+\dfrac{x\left(1+x^2-y^2\right)}{\left(1+x^2+y^2\right)^2}\mathbf{j}$, $\nabla T(1,1)=(\mathbf{i}+\mathbf{j})/9$, $\mathbf{u}=(2\mathbf{i}-\mathbf{j})/\sqrt{5}$, $D_{\mathbf{u}}T=1/\left(9\sqrt{5}\right)$

(b) $\mathbf{u}=-(\mathbf{i}+\mathbf{j})/\sqrt{2}$, opposite to $\nabla T(1,1)$

69. **(a)** $\nabla V(x,y)=-2e^{-2x}\cos 2y\mathbf{i}-2e^{-2x}\sin 2y\mathbf{j}$, $\mathbf{E}=-\nabla V(\pi/4,0)=2e^{-\pi/2}\mathbf{i}$

(b) $V(x,y)$ decreases most rapidly in the direction of $-\nabla V(x,y)$ which is $\mathbf{E}$.

70. $\nabla z=-0.04x\mathbf{i}-0.08y\mathbf{j}$, if $x=-20$ and $y=5$ then $\nabla z=0.8\mathbf{i}-0.4\mathbf{j}$.

(a) $\mathbf{u}=-\mathbf{i}$ points due west, $D_{\mathbf{u}}z=-0.8$, the climber will descend because z is decreasing.

(b) $\mathbf{u}=(\mathbf{i}+\mathbf{j})/\sqrt{2}$ points northeast, $D_{\mathbf{u}}z=0.2\sqrt{2}$, the climber will ascend at the rate of $0.2\sqrt{2}$ m per m of travel in the $xy-$plane.

(c) The climber will travel a level path in a direction perpendicular to $\nabla z=0.8\mathbf{i}-0.4\mathbf{j}$, by inspection $\pm(\mathbf{i}+2\mathbf{j})/\sqrt{5}$ are unit vectors in these directions; $(\mathbf{i}+2\mathbf{j})/\sqrt{5}$ makes an angle of $\tan^{-1}(1/2)\approx 27°$ with the positive y-axis so $-(\mathbf{i}+2\mathbf{j})/\sqrt{5}$ makes the same angle with the negative y-axis. The compass direction should be N 27° E or S 27° W.

71. Let $\mathbf{u}$ be the unit vector in the direction of $\mathbf{a}$, then $D_{\mathbf{u}}f(3,-2,1)=\nabla f(3,-2,1)\cdot\mathbf{u}=\|\nabla f(3,-2,1)\|\cos\theta=5\cos\theta=-5$, $\cos\theta=-1$, $\theta=\pi$ so $\nabla f(3,-2,1)$ is oppositely directed to $\mathbf{u}$; $\nabla f(3,-2,1)=-5\mathbf{u}=-10/3\mathbf{i}+5/3\mathbf{j}+10/3\mathbf{k}$.

72. **(a)** $\nabla T(1,1,1)=(\mathbf{i}+\mathbf{j}+\mathbf{k})/8$, $\mathbf{u}=-(\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{3}$, $D_{\mathbf{u}}T=-\sqrt{3}/8$

(b) $(\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{3}$ **(c)** $\sqrt{3}/8$

73. **(a)** $\nabla r=\dfrac{x}{\sqrt{x^2+y^2}}\mathbf{i}+\dfrac{y}{\sqrt{x^2+y^2}}\mathbf{j}=\mathbf{r}/r$

(b) $\nabla f(r)=\dfrac{\partial f(r)}{\partial x}\mathbf{i}+\dfrac{\partial f(r)}{\partial y}\mathbf{j}=f'(r)\dfrac{\partial r}{\partial x}\mathbf{i}+f'(r)\dfrac{\partial r}{\partial y}\mathbf{j}=f'(r)\nabla r$

74. **(a)** $\nabla\left(re^{-3r}\right)=\dfrac{(1-3r)}{r}e^{-3r}\mathbf{r}$

(b) $3r^2\mathbf{r}=\dfrac{f'(r)}{r}\mathbf{r}$ so $f'(r)=3r^3$, $f(r)=\dfrac{3}{4}r^4+C$, $f(2)=12+C=1$, $C=-11$; $f(r)=\dfrac{3}{4}r^4-11$

75. $\mathbf{u}_r=\cos\theta\mathbf{i}+\sin\theta\mathbf{j}$, $\mathbf{u}_\theta=-\sin\theta\mathbf{i}+\cos\theta\mathbf{j}$,

$$\nabla z=\frac{\partial z}{\partial x}\mathbf{i}+\frac{\partial z}{\partial y}\mathbf{j}=\left(\frac{\partial z}{\partial r}\cos\theta-\frac{1}{r}\frac{\partial z}{\partial\theta}\sin\theta\right)\mathbf{i}+\left(\frac{\partial z}{\partial r}\sin\theta+\frac{1}{r}\frac{\partial z}{\partial\theta}\cos\theta\right)\mathbf{j}$$

$$=\frac{\partial z}{\partial r}(\cos\theta\mathbf{i}+\sin\theta\mathbf{j})+\frac{1}{r}\frac{\partial z}{\partial\theta}(-\sin\theta\mathbf{i}+\cos\theta\mathbf{j})=\frac{\partial z}{\partial r}\mathbf{u}_r+\frac{1}{r}\frac{\partial z}{\partial\theta}\mathbf{u}_\theta$$

76. **(a)** $\nabla(f+g)=(f_x+g_x)\mathbf{i}+(f_y+g_y)\mathbf{j}=(f_x\mathbf{i}+f_y\mathbf{j})+(g_x\mathbf{i}+g_y\mathbf{j})=\nabla f+\nabla g$

(b) $\nabla(cf)=(cf_x)\mathbf{i}+(cf_y)\mathbf{j}=c\,(f_x\mathbf{i}+f_y\mathbf{j})=c\nabla f$

(c) $\nabla(fg)=(fg_x+gf_x)\mathbf{i}+(fg_y+gf_y)\mathbf{j}=f\,(g_x\mathbf{i}+g_y\mathbf{j})+g\,(f_x\mathbf{i}+f_y\mathbf{j})=f\nabla g+g\nabla f$

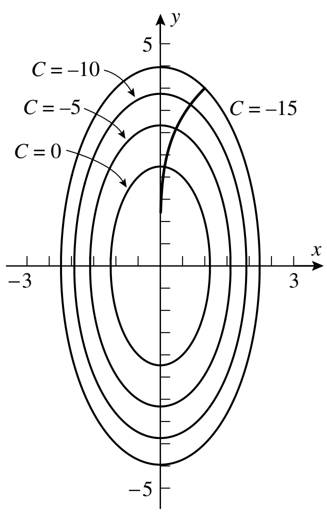
(d) $\nabla(f/g) = \dfrac{gf_x - fg_x}{g^2}\mathbf{i} + \dfrac{gf_y - fg_y}{g^2}\mathbf{j} = \dfrac{g\left(f_x\mathbf{i} + f_y\mathbf{j}\right) - f\left(g_x\mathbf{i} + g_y\mathbf{j}\right)}{g^2} = \dfrac{g\nabla f - f\nabla g}{g^2}$

(e) $\nabla\left(f^n\right) = \left(nf^{n-1}f_x\right)\mathbf{i} + \left(nf^{n-1}f_y\right)\mathbf{j} = nf^{n-1}\left(f_x\mathbf{i} + f_y\mathbf{j}\right) = nf^{n-1}\nabla f$

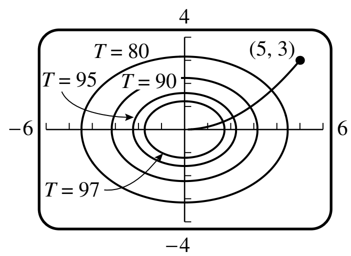
77. $\mathbf{r}'(t) = \mathbf{v}(t) = k(x,y)\nabla\mathbf{T} = -8k(x,y)x\mathbf{i} - 2k(x,y)y\mathbf{j}$; $\dfrac{dx}{dt} = -8kx$, $\dfrac{dy}{dt} = -2ky$. Divide and solve to get $y^4 = 256x$; one parametrization is $x(t) = e^{-8t},\ y(t) = 4e^{-2t}$.

78. $\mathbf{r}'(t) = \mathbf{v}(t) = k\nabla\mathbf{T} = -2k(x,y)x\mathbf{i} - 4k(x,y)y\mathbf{j}$. Divide and solve to get $y = \dfrac{3}{25}x^2$; one parametrization is $x(t) = 5e^{-2t},\ y(t) = 3e^{-4t}$.

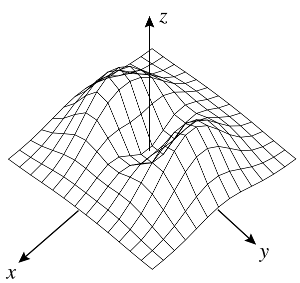
79.



80.



81. **(a)**



(c) $\nabla f = [2x - 2x(x^2 + 3y^2)]e^{-(x^2+y^2)}\mathbf{i} + [6y - 2y(x^2 + 3y^2)]e^{-(x^2+y^2)}\mathbf{j}$

(d) $\nabla f = \mathbf{0}$ if $x = y = 0$ or $x = 0, y = \pm 1$ or $x = \pm 1, y = 0$.

82.
$$\begin{aligned} dz/dt &= (\partial z/\partial x)(dx/dt) + (\partial z/\partial y)(dy/dt) \\ &= (\partial z/\partial x\mathbf{i} + \partial z/\partial y\mathbf{j}) \cdot (dx/dt\mathbf{i} + dy/dt\mathbf{j}) = \nabla z \cdot \mathbf{r}'(t) \end{aligned}$$

83. $\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}$, if $\nabla f(x,y) = 0$ throughout the region then $f_x(x,y) = f_y(x,y) = 0$ throughout the region, the result follows from Exercise 71, Section 14.5.

84. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be nonparallel unit vectors for which the directional derivative is zero. Let $\mathbf{u}$ be any other unit vector, then $\mathbf{u} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2$ for some choice of scalars c_1 and c_2,
$$\begin{aligned} D_{\mathbf{u}}f(x,y) &= \nabla f(x,y) \cdot \mathbf{u} = c_1\nabla f(x,y) \cdot \mathbf{u}_1 + c_2\nabla f(x,y) \cdot \mathbf{u}_2 \\ &= c_1D_{\mathbf{u}_1}f(x,y) + c_2D_{\mathbf{u}_2}f(x,y) = 0. \end{aligned}$$

85. $\nabla f(u,v,w) = \dfrac{\partial f}{\partial x}\mathbf{i} + \dfrac{\partial f}{\partial y}\mathbf{j} + \dfrac{\partial f}{\partial z}\mathbf{k}$

$$= \left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x}\right)\mathbf{i} + \left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y}\right)\mathbf{j}$$

$$+ \left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z}\right)\mathbf{k} = \frac{\partial f}{\partial u}\nabla u + \frac{\partial f}{\partial v}\nabla v + \frac{\partial f}{\partial w}\nabla w$$

86. (a) The distance between $(x_0 + su_1, y_0 + su_2)$ and (x_0, y_0) is $|s|\sqrt{u_1^2 + u_2^2} = |s|$, so the condition $\lim_{s\to 0} \dfrac{E(s)}{|s|} = 0$ is exactly the condition of Definition 14.4.1, with the local linear approximation of f given by $L(s) = f(x_0, y_0) + f_x(x_0, y_0)su_1 + f_y(x_0, y_0)su_2$, which in turn says that $g'(0) = f_x(x_0, y_0) + f_y(x_0, y_0)$.

(b) The function $E(s)$ of Part (a) has the same values as the function $E(x, y)$ when $x = x_0 + su_1, y = y_0 + su_2$, and the distance between (x, y) and (x_0, y_0) is $|s|$, so the limit in Part (a) is equivalent to the limit (5) of Definition 14.4.2.

(c) Let $f(x, y)$ be differentiable at (x_0, y_0) and let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ be a unit vector. Then by Parts (a) and (b) the directional derivative $D_{\mathbf{u}}\dfrac{d}{ds}[f(x_0 + su_1, y_0 + su_2)]_{s=0}$ exists and is given by $f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$.

87. (a) $\dfrac{d}{ds}f(x_0+su_1, y_0+su_2)$ at $s = 0$ is by definition equal to $\lim_{s\to 0} \dfrac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$, and from Exercise 86(a) this value is equal to $f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$.

(b) For any number $\epsilon > 0$ a number $\delta > 0$ exists such that whenever $0 < |s| < \delta$ then $\left|\dfrac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0) - f_x(x_0, y_0)su_1 - f_y(x_0, y_0)su_2}{s}\right| < \epsilon.$

(c) For any number $\epsilon > 0$ there exists a number $\delta > 0$ such that $\dfrac{|E(x, y)|}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} < \epsilon$ whenever $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$.

(d) For any number $\epsilon > 0$ there exists a number $\delta > 0$ such that $\left|\dfrac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0) - f_x(x_0, y_0)su_1 - f_y(x_0, y_0)su_2|}{s}\right| < \epsilon$ when $0 < |s| < \delta$.

(e) Since f is differentiable at (x_0, y_0), by Part (c) the Equation (5) of Definition 14.2.1 holds. By Part (d), for any $\epsilon > 0$ there exists $\delta > 0$ such that $\left|\dfrac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0) - f_x(x_0, y_0)su_1 - f_y(x_0, y_0)su_2}{s}\right| < \epsilon$ when $0 < |s| < \delta$.

By Part (a) it follows that the limit in Part (a) holds, and thus that $\dfrac{d}{ds}f(x_0 + su_1, y_0 + su_2)\Big]_{s=0} = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2,$ which proves Equation (4) of Theorem 14.6.3.

EXERCISE SET 14.7

1. At P, $\partial z/\partial x = 48$ and $\partial z/\partial y = -14$, tangent plane $48x - 14y - z = 64$, normal line $x = 1 + 48t$, $y = -2 - 14t$, $z = 12 - t$.

2. At P, $\partial z/\partial x = 14$ and $\partial z/\partial y = -2$, tangent plane $14x - 2y - z = 16$, normal line $x = 2 + 14t$, $y = 4 - 2t$, $z = 4 - t$.

3. At P, $\partial z/\partial x = 1$ and $\partial z/\partial y = -1$, tangent plane $x - y - z = 0$, normal line $x = 1 + t$, $y = -t$, $z = 1 - t$.

4. At P, $\partial z/\partial x = -1$ and $\partial z/\partial y = 0$, tangent plane $x + z = -1$, normal line $x = -1 - t$, $y = 0$, $z = -t$.

5. At P, $\partial z/\partial x = 0$ and $\partial z/\partial y = 3$, tangent plane $3y - z = -1$, normal line $x = \pi/6$, $y = 3t$, $z = 1 - t$.

6. At P, $\partial z/\partial x = 1/4$ and $\partial z/\partial y = 1/6$, tangent plane $3x + 2y - 12z = -30$, normal line $x = 4 + t/4$, $y = 9 + t/6$, $z = 5 - t$.

7. By implicit differentiation $\partial z/\partial x = -x/z$, $\partial z/\partial y = -y/z$ so at P, $\partial z/\partial x = 3/4$ and $\partial z/\partial y = 0$, tangent plane $3x - 4z = -25$, normal line $x = -3 + 3t/4$, $y = 0$, $z = 4 - t$.

8. By implicit differentiation $\partial z/\partial x = (xy)/(4z)$, $\partial z/\partial y = x^2/(8z)$ so at P, $\partial z/\partial x = 3/8$ and $\partial z/\partial y = -9/16$, tangent plane $6x - 9y - 16z = 5$, normal line $x = -3 + 3t/8$, $y = 1 - 9t/16$, $z = -2 - t$.

9. The tangent plane is horizontal if the normal $\partial z/\partial x\mathbf{i} + \partial z/\partial y\mathbf{j} - \mathbf{k}$ is parallel to $\mathbf{k}$ which occurs when $\partial z/\partial x = \partial z/\partial y = 0$.

(a) $\partial z/\partial x = 3x^2y^2$, $\partial z/\partial y = 2x^3y$; $3x^2y^2 = 0$ and $2x^3y = 0$ for all (x, y) on the x-axis or y-axis, and $z = 0$ for these points, the tangent plane is horizontal at all points on the x-axis or y-axis.

(b) $\partial z/\partial x = 2x - y - 2$, $\partial z/\partial y = -x + 2y + 4$; solve the system $2x - y - 2 = 0$, $-x + 2y + 4 = 0$, to get $x = 0$, $y = -2$. $z = -4$ at $(0, -2)$, the tangent plane is horizontal at $(0, -2, -4)$.

10. $\partial z/\partial x = 6x$, $\partial z/\partial y = -2y$, so $6x_0\mathbf{i} - 2y_0\mathbf{j} - \mathbf{k}$ is normal to the surface at a point (x_0, y_0, z_0) on the surface. $6\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ is normal to the given plane. The tangent plane and the given plane are parallel if their normals are parallel so $6x_0 = 6$, $x_0 = 1$ and $-2y_0 = 4$, $y_0 = -2$. $z = -1$ at $(1, -2)$, the point on the surface is $(1, -2, -1)$.

11. $\partial z/\partial x = -6x$, $\partial z/\partial y = -4y$ so $-6x_0\mathbf{i} - 4y_0\mathbf{j} - \mathbf{k}$ is normal to the surface at a point (x_0, y_0, z_0) on the surface. This normal must be parallel to the given line and hence to the vector $-3\mathbf{i} + 8\mathbf{j} - \mathbf{k}$ which is parallel to the line so $-6x_0 = -3$, $x_0 = 1/2$ and $-4y_0 = 8$, $y_0 = -2$. $z = -3/4$ at $(1/2, -2)$. The point on the surface is $(1/2, -2, -3/4)$.

12. (3,4,5) is a point of intersection because it satisfies both equations. Both surfaces have $(3/5)\mathbf{i} + (4/5)\mathbf{j} - \mathbf{k}$ as a normal so they have a common tangent plane at (3,4,5).

13. **(a)** $2t + 7 = (-1 + t)^2 + (2 + t)^2$, $t^2 = 1$, $t = \pm 1$ so the points of intersection are $(-2, 1, 5)$ and $(0, 3, 9)$.

(b) $\partial z/\partial x = 2x$, $\partial z/\partial y = 2y$ so at $(-2, 1, 5)$ the vector $\mathbf{n} = -4\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is normal to the surface. $\mathbf{v} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$ is parallel to the line; $\mathbf{n} \cdot \mathbf{v} = -4$ so the cosine of the acute angle is $[\mathbf{n} \cdot (-\mathbf{v})]/(\|\mathbf{n}\|\,\|-\mathbf{v}\|) = 4/\left(\sqrt{21}\sqrt{6}\right) = 4/\left(3\sqrt{14}\right)$. Similarly, at (0,3,9) the vector $\mathbf{n} = 6\mathbf{j} - \mathbf{k}$ is normal to the surface, $\mathbf{n} \cdot \mathbf{v} = 4$ so the cosine of the acute angle is $4/\left(\sqrt{37}\sqrt{6}\right) = 4/\sqrt{222}$.

14. $z = xf(u)$ where $u = x/y$, $\partial z/\partial x = xf'(u)\partial u/\partial x + f(u) = (x/y)f'(u) + f(u) = uf'(u) + f(u)$, $\partial z/\partial y = xf'(u)\partial u/\partial y = -(x^2/y^2)f'(u) = -u^2 f'(u)$. If (x_0, y_0, z_0) is on the surface then, with $u_0 = x_0/y_0$, $[u_0 f'(u_0) + f(u_0)]\,\mathbf{i} - u_0^2 f'(u_0)\,\mathbf{j} - \mathbf{k}$ is normal to the surface so the tangent plane is

$$\begin{aligned}[u_0 f'(u_0) + f(u_0)]\,x - u_0^2 f'(u_0)y - z &= [u_0 f'(u_0) + f(u_0)]x_0 - u_0^2 f'(u_0)y_0 - z_0 \\ &= \left[\frac{x_0}{y_0} f'(u_0) + f(u_0)\right] x_0 - \frac{x_0^2}{y_0^2} f'(u_0)\,y_0 - z_0 \\ &= x_0 f(u_0) - z_0 = 0\end{aligned}$$

so all tangent planes pass through the origin.

15. **(a)** $f(x, y, z) = x^2 + y^2 + 4z^2, \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 8z\mathbf{k}, \nabla f(2, 2, 1) = 4\mathbf{i} + 4\mathbf{j} + 8\mathbf{k}$, $\mathbf{n} = \mathbf{i} + \mathbf{j} + 2\mathbf{k},\ x + y + 2z = 6$

(b) $\mathbf{r}(t) = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} + t(\mathbf{i} + \mathbf{j} + 2\mathbf{k}), x(t) = 2 + t, y(t) = 2 + t, z(t) = 1 + 2t$

(c) $\cos\theta = \dfrac{\mathbf{n}\cdot\mathbf{k}}{\|\mathbf{n}\|} = \dfrac{\sqrt{2}}{\sqrt{3}}, \theta \approx 35.26^\circ$

16. **(a)** $f(x, y, z) = xz - yz^3 + yz^2$, $\mathbf{n} = \nabla f(2, -1, 1) = \mathbf{i} + 3\mathbf{k}$; tangent plane $x + 3z = 5$

(b) normal line $x = 2 + t,\ y = -1,\ z = 1 + 3t$

(c) $\cos\theta = \dfrac{\mathbf{n}\cdot\mathbf{k}}{\|\mathbf{n}\|} = \dfrac{3}{\sqrt{10}}, \theta \approx 18.43^\circ$

17. Set $f(x, y) = z + x - z^4(y - 1)$, then $f(x, y, z) = 0, \mathbf{n} = \pm\nabla f(3, 5, 1) = \pm(\mathbf{i} - \mathbf{j} - 19\mathbf{k})$, unit vectors $\pm\dfrac{1}{\sqrt{363}}(\mathbf{i} - \mathbf{j} - 19\mathbf{k})$

18. $f(x, y, z) = \sin xz - 4\cos yz$, $\nabla f(\pi, \pi, 1) = -\mathbf{i} - \pi\mathbf{k}$; unit vectors $\pm\dfrac{1}{\sqrt{1 + \pi^2}}(\mathbf{i} + \pi\mathbf{k})$

19. $f(x, y, z) = x^2 + y^2 + z^2$, if (x_0, y_0, z_0) is on the sphere then $\nabla f(x_0, y_0, z_0) = 2(x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k})$ is normal to the sphere at (x_0, y_0, z_0), the normal line is $x = x_0 + x_0 t$, $y = y_0 + y_0 t$, $z = z_0 + z_0 t$ which passes through the origin when $t = -1$.

20. $f(x, y, z) = 2x^2 + 3y^2 + 4z^2$, if (x_0, y_0, z_0) is on the ellipsoid then $\nabla f(x_0, y_0, z_0) = 2(2x_0\mathbf{i} + 3y_0\mathbf{j} + 4z_0\mathbf{k})$ is normal there and hence so is $\mathbf{n}_1 = 2x_0\mathbf{i} + 3y_0\mathbf{j} + 4z_0\mathbf{k}$; $\mathbf{n}_1$ must be parallel to $\mathbf{n}_2 = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ which is normal to the given plane so $\mathbf{n}_1 = c\mathbf{n}_2$ for some constant c. Equate corresponding components to get $x_0 = c/2$, $y_0 = -2c/3$, and $z_0 = 3c/4$; substitute into the equation of the ellipsoid yields $2\left(c^2/4\right) + 3\left(4c^2/9\right) + 4\left(9c^2/16\right) = 9$, $c^2 = 108/49$, $c = \pm 6\sqrt{3}/7$. The points on the ellipsoid are $\left(3\sqrt{3}/7, -4\sqrt{3}/7, 9\sqrt{3}/14\right)$ and $\left(-3\sqrt{3}/7, 4\sqrt{3}/7, -9\sqrt{3}/14\right)$.

21. $f(x, y, z) = x^2 + y^2 - z^2$, if (x_0, y_0, z_0) is on the surface then $\nabla f(x_0, y_0, z_0) = 2(x_0\mathbf{i} + y_0\mathbf{j} - z_0\mathbf{k})$ is normal there and hence so is $\mathbf{n}_1 = x_0\mathbf{i} + y_0\mathbf{j} - z_0\mathbf{k}$; $\mathbf{n}_1$ must be parallel to $\overrightarrow{PQ} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ so $\mathbf{n}_1 = c\,\overrightarrow{PQ}$ for some constant c. Equate components to get $x_0 = 3c$, $y_0 = 2c$ and $z_0 = 2c$ which when substituted into the equation of the surface yields $9c^2 + 4c^2 - 4c^2 = 1$, $c^2 = 1/9$, $c = \pm 1/3$ so the points are $(1, 2/3, 2/3)$ and $(-1, -2/3, -2/3)$.

22. $f_1(x, y, z) = 2x^2 + 3y^2 + z^2$, $f_2(x, y, z) = x^2 + y^2 + z^2 - 6x - 8y - 8z + 24$, $\mathbf{n}_1 = \nabla f_1(1, 1, 2) = 4\mathbf{i} + 6\mathbf{j} + 4\mathbf{k}$, $\mathbf{n}_2 = \nabla f_2(1, 1, 2) = -4\mathbf{i} - 6\mathbf{j} - 4\mathbf{k}$, $\mathbf{n}_1 = -\mathbf{n}_2$ so $\mathbf{n}_1$ and $\mathbf{n}_2$ are parallel.

23. $\mathbf{n}_1 = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}, \mathbf{n}_2 = 2\mathbf{i} - 8\mathbf{j} + 4\mathbf{k}, \mathbf{n}_1 \times \mathbf{n}_2 = -16\mathbf{i} - 10\mathbf{j} - 12\mathbf{k}$ is tangent to the line, so $x(t) = 1 + 8t, y(t) = -1 + 5t, z(t) = 2 + 6t$

24. $f(x, y, z) = \sqrt{x^2 + y^2} - z, \mathbf{n}_1 = \nabla f(4, 3, 5) = \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j} - \mathbf{k}, \mathbf{n}_2 = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}, \mathbf{n}_1 \times \mathbf{n}_2 = (16\mathbf{i} - 13\mathbf{j} + 5\mathbf{k})/5$ is tangent to the line, $x(t) = 4 + 16t, y(t) = 3 - 13t, z(t) = 5 + 5t$

25. $f(x, y, z) = x^2 + z^2 - 25,\ g(x, y, z) = y^2 + z^2 - 25, \mathbf{n}_1 = \nabla f(3, -3, 4) = 6\mathbf{i} + 8\mathbf{k}$, $\mathbf{n}_2 = \nabla g(3, -3, 4) = -6\mathbf{j} + 8\mathbf{k}, \mathbf{n}_1 \times \mathbf{n}_2 = 48\mathbf{i} - 48\mathbf{j} - 36\mathbf{k}$ is tangent to the line, $x(t) = 3 + 4t, y(t) = -3 - 4t, z(t) = 4 - 3t$

26. **(a)** $f(x, y, z) = z - 8 + x^2 + y^2, g(x, y, z) = 4x + 2y - z, \mathbf{n}_1 = 4\mathbf{j} + \mathbf{k}, \mathbf{n}_2 = 4\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{n}_1 \times \mathbf{n}_2 = -6\mathbf{i} + 4\mathbf{j} - 16\mathbf{k}$ is tangent to the line, $x(t) = 3t, y(t) = 2 - 2t, z(t) = 4 + 8t$

27. Use implicit differentiation to get $\partial z/\partial x = -c^2x/\left(a^2z\right),\ \partial z/\partial y = -c^2y/\left(b^2z\right)$. At (x_0, y_0, z_0), $z_0 \neq 0$, a normal to the surface is $-\left[c^2x_0/\left(a^2z_0\right)\right]\mathbf{i} - \left[c^2y_0/\left(b^2z_0\right)\right]\mathbf{j} - \mathbf{k}$ so the tangent plane is $-\frac{c^2x_0}{a^2z_0}x - \frac{c^2y_0}{b^2z_0}y - z = -\frac{c^2x_0^2}{a^2z_0} - \frac{c^2y_0^2}{b^2z_0} - z_0,\ \frac{x_0x}{a^2} + \frac{y_0y}{b^2} + \frac{z_0z}{c^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} = 1$

28. $\partial z/\partial x = 2x/a^2,\ \partial z/\partial y = 2y/b^2$. At (x_0, y_0, z_0) the vector $\left(2x_0/a^2\right)\mathbf{i} + \left(2y_0/b^2\right)\mathbf{j} - \mathbf{k}$ is normal to the surface so the tangent plane is $\left(2x_0/a^2\right)x + \left(2y_0/b^2\right)y - z = 2x_0^2/a^2 + 2y_0^2/b^2 - z_0$, but $z_0 = x_0^2/a^2 + y_0^2/b^2$ so $\left(2x_0/a^2\right)x + \left(2y_0/b^2\right)y - z = 2z_0 - z_0 = z_0,\ 2x_0x/a^2 + 2y_0y/b^2 = z + z_0$

29. $\mathbf{n}_1 = f_x\left(x_0, y_0\right)\mathbf{i} + f_y\left(x_0, y_0\right)\mathbf{j} - \mathbf{k}$ and $\mathbf{n}_2 = g_x\left(x_0, y_0\right)\mathbf{i} + g_y\left(x_0, y_0\right)\mathbf{j} - \mathbf{k}$ are normal, respectively, to $z = f(x, y)$ and $z = g(x, y)$ at P; $\mathbf{n}_1$ and $\mathbf{n}_2$ are perpendicular if and only if $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$,
$f_x\left(x_0, y_0\right)g_x\left(x_0, y_0\right) + f_y\left(x_0, y_0\right)g_y\left(x_0, y_0\right) + 1 = 0$,
$f_x\left(x_0, y_0\right)g_x\left(x_0, y_0\right) + f_y\left(x_0, y_0\right)g_y\left(x_0, y_0\right) = -1$.

30. $\mathbf{n}_1 = f_x\mathbf{i} + f_y\mathbf{j} - \mathbf{k} = \frac{x_0}{\sqrt{x_0^2 + y_0^2}}\mathbf{i} + \frac{y_0}{\sqrt{x_0^2 + y_0^2}}\mathbf{j} - \mathbf{k}$; similarly $\mathbf{n}_2 = -\frac{x_0}{\sqrt{x_0^2 + y_0^2}}\mathbf{i} - \frac{y_0}{\sqrt{x_0^2 + y_0^2}}\mathbf{j} - \mathbf{k}$; since a normal to the sphere is $\mathbf{N} = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$, and $\mathbf{n}_1 \cdot \mathbf{N} = \sqrt{x_0^2 + y_0^2} - z_0 = 0$, $\mathbf{n}_2 \cdot \mathbf{N} = -\sqrt{x_0^2 + y_0^2} - z_0 = 0$, the result follows.

31. $\nabla f = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$ and $\nabla g = g_x\mathbf{i} + g_y\mathbf{j} + g_z\mathbf{k}$ evaluated at (x_0, y_0, z_0) are normal, respectively, to the surfaces $f(x, y, z) = 0$ and $g(x, y, z) = 0$ at (x_0, y_0, z_0). The surfaces are orthogonal at (x_0, y_0, z_0) if and only if $\nabla f \cdot \nabla g = 0$ so $f_xg_x + f_yg_y + f_zg_z = 0$.

32. $f(x, y, z) = x^2 + y^2 + z^2 - a^2 = 0, g(x, y, z) = z^2 - x^2 - y^2 = 0$,
$f_xg_x + f_yg_y + f_zg_z = -4x^2 - 4y^2 + 4z^2 = 4g(x, y, z) = 0$

33. $z = \frac{k}{xy}$; at a point $\left(a, b, \frac{k}{ab}\right)$ on the surface, $\left\langle -\frac{k}{a^2b}, -\frac{k}{ab^2}, -1\right\rangle$ and hence $\left\langle bk, ak, a^2b^2\right\rangle$ is normal to the surface so the tangent plane is $bkx + aky + a^2b^2z = 3abk$. The plane cuts the x, y, and z-axes at the points 3a, 3b, and $\frac{3k}{ab}$, respectively, so the volume of the tetrahedron that is formed is $V = \frac{1}{3}\left(\frac{3k}{ab}\right)\left[\frac{1}{2}(3a)(3b)\right] = \frac{9}{2}k$, which does not depend on a and b.

EXERCISE SET 14.8

1. **(a)** minimum at $(2,-1)$, no maxima **(b)** maximum at $(0,0)$, no minima
(c) no maxima or minima

2. **(a)** maximum at $(-1,5)$, no minima **(b)** no maxima or minima
(c) no maxima or minima

3. $f(x,y) = (x-3)^2 + (y+2)^2$, minimum at $(3,-2)$, no maxima

4. $f(x,y) = -(x+1)^2 - 2(y-1)^2 + 4$, maximum at $(-1,1)$, no minima

5. $f_x = 6x + 2y = 0$, $f_y = 2x + 2y = 0$; critical point (0,0); $D = 8 > 0$ and $f_{xx} = 6 > 0$ at (0,0), relative minimum.

6. $f_x = 3x^2 - 3y = 0$, $f_y = -3x - 3y^2 = 0$; critical points (0,0) and $(-1,1)$; $D = -9 < 0$ at (0,0), saddle point; $D = 27 > 0$ and $f_{xx} = -6 < 0$ at $(-1,1)$, relative maximum.

7. $f_x = 2x - 2xy = 0$, $f_y = 4y - x^2 = 0$; critical points (0,0) and $(\pm 2, 1)$; $D = 8 > 0$ and $f_{xx} = 2 > 0$ at (0,0), relative minimum; $D = -16 < 0$ at $(\pm 2, 1)$, saddle points.

8. $f_x = 3x^2 - 3 = 0$, $f_y = 3y^2 - 3 = 0$; critical points $(-1, \pm 1)$ and $(1, \pm 1)$; $D = -36 < 0$ at $(-1,1)$ and $(1,-1)$, saddle points; $D = 36 > 0$ and $f_{xx} = 6 > 0$ at (1,1), relative minimum; $D = 36 > 0$ and $f_{xx} = -36 < 0$ at $(-1,-1)$, relative maximum.

9. $f_x = y + 2 = 0$, $f_y = 2y + x + 3 = 0$; critical point $(1,-2)$; $D = -1 < 0$ at $(1,-2)$, saddle point.

10. $f_x = 2x + y - 2 = 0$, $f_y = x - 2 = 0$; critical point $(2,-2)$; $D = -1 < 0$ at $(2,-2)$, saddle point.

11. $f_x = 2x + y - 3 = 0$, $f_y = x + 2y = 0$; critical point $(2,-1)$; $D = 3 > 0$ and $f_{xx} = 2 > 0$ at $(2,-1)$, relative minimum.

12. $f_x = y - 3x^2 = 0$, $f_y = x - 2y = 0$; critical points (0,0) and $(1/6, 1/12)$; $D = -1 < 0$ at (0,0), saddle point; $D = 1 > 0$ and $f_{xx} = -1 < 0$ at $(1/6, 1/12)$, relative maximum.

13. $f_x = 2x - 2/\left(x^2 y\right) = 0$, $f_y = 2y - 2/\left(xy^2\right) = 0$; critical points $(-1,-1)$ and $(1,1)$; $D = 32 > 0$ and $f_{xx} = 6 > 0$ at $(-1,-1)$ and $(1,1)$, relative minima.

14. $f_x = e^y = 0$ is impossible, no critical points.

15. $f_x = 2x = 0$, $f_y = 1 - e^y = 0$; critical point $(0,0)$; $D = -2 < 0$ at $(0,0)$, saddle point.

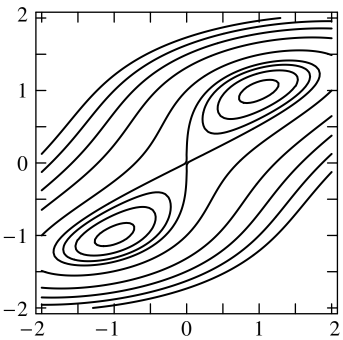
16. $f_x = y - 2/x^2 = 0$, $f_y = x - 4/y^2 = 0$; critical point (1,2); $D = 3 > 0$ and $f_{xx} = 4 > 0$ at $(1,2)$, relative minimum.

17. $f_x = e^x \sin y = 0$, $f_y = e^x \cos y = 0$, $\sin y = \cos y = 0$ is impossible, no critical points.

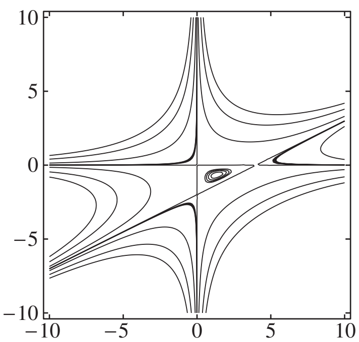
18. $f_x = y \cos x = 0$, $f_y = \sin x = 0$; $\sin x = 0$ if $x = n\pi$ for $n = 0, \pm 1, \pm 2, \ldots$ and $\cos x \neq 0$ for these values of x so $y = 0$; critical points $(n\pi, 0)$ for $n = 0, \pm 1, \pm 2, \ldots$; $D = -1 < 0$ at $(n\pi, 0)$, saddle points.

19. $f_x = -2(x+1)e^{-\left(x^2+y^2+2x\right)} = 0$, $f_y = -2ye^{-\left(x^2+y^2+2x\right)} = 0$; critical point $(-1,0)$; $D = 4e^2 > 0$ and $f_{xx} = -2e < 0$ at $(-1,0)$, relative maximum.

20. $f_x = y - a^3/x^2 = 0$, $f_y = x - b^3/y^2 = 0$; critical point $\left(a^2/b, b^2/a\right)$; if $ab > 0$ then $D = 3 > 0$ and $f_{xx} = 2b^3/a^3 > 0$ at $\left(a^2/b, b^2/a\right)$, relative minimum; if $ab < 0$ then $D = 3 > 0$ and $f_{xx} = 2b^3/a^3 < 0$ at $\left(a^2/b, b^2/a\right)$, relative maximum.

21. 

$\nabla f = (4x - 4y)\mathbf{i} - (4x - 4y^3)\mathbf{j} = \mathbf{0}$ when $x = y, x = y^3$, so $x = y = 0$ or $x = y = \pm 1$. At $(0, 0), D = -16$, a saddle point; at $(1, 1)$ and $(-1, -1), D = 32 > 0, f_{xx} = 4$, a relative minimum.

22. 

$\nabla f = (2y^2 - 2xy + 4y)\mathbf{i} + (4xy - x^2 + 4x)\mathbf{j} = \mathbf{0}$ when $2y^2 - 2xy + 4y = 0, 4xy - x^2 + 4x = 0$, with solutions $(0, 0), (0, -2), (4, 0), (4/3, -2/3)$. At $(0, 0)$, $D = -16$, a saddle point. At $(0, -2)$, $D = -16$, a saddle point. At $(4, 0), D = -16$, a saddle point. At $(4/3, -2/3), D = 16/3$, $f_{xx} = 4/3 > 0$, a relative minimum.

23. **(a)** critical point (0,0); $D = 0$

(b) $f(0, 0) = 0$, $x^4 + y^4 \geq 0$ so $f(x, y) \geq f(0, 0)$, relative minimum.

24. **(a)** critical point (0,0); $D = 0$

(b) The trace of the surface on the plane $x = 0$ has equation $z = -y^4$, which has a maximum at $(0, 0, 0)$; the trace of the surface on the plane $y = 0$ has equation $z = x^4$, which has a minimum at $(0, 0, 0)$.

25. **(a)** $f_x = 3e^y - 3x^2 = 3\left(e^y - x^2\right) = 0$, $f_y = 3xe^y - 3e^{3y} = 3e^y\left(x - e^{2y}\right) = 0$, $e^y = x^2$ and $e^{2y} = x$, $x^4 = x$, $x\left(x^3 - 1\right) = 0$ so $x = 0, 1$; critical point $(1, 0)$; $D = 27 > 0$ and $f_{xx} = -6 < 0$ at $(1, 0)$, relative maximum.

(b) $\lim_{x \to -\infty} f(x, 0) = \lim_{x \to -\infty} \left(3x - x^3 - 1\right) = +\infty$ so no absolute maximum.

26. $f_x = 8xe^y - 8x^3 = 8x(e^y - x^2) = 0$, $f_y = 4x^2e^y - 4e^{4y} = 4e^y(x^2 - e^{3y}) = 0$, $x^2 = e^y$ and $x^2 = e^{3y}$, $e^{3y} = e^y$, $e^{2y} = 1$, so $y = 0$ and $x = \pm 1$; critical points (1,0) and $(-1,0)$. $D = 128 > 0$ and $f_{xx} = -16 < 0$ at both points so a relative maximum occurs at each one.

27. $f_x = y - 1 = 0,\ f_y = x - 3 = 0$; critical point (3,1).

$$\begin{aligned} \text{Along } y = 0: &\quad u(x) = -x;\ \text{no critical points,} \\ \text{along } x = 0: &\quad v(y) = -3y;\ \text{no critical points,} \\ \text{along } y = -\frac{4}{5}x + 4: &\quad w(x) = -\frac{4}{5}x^2 + \frac{27}{5}x - 12;\ \text{critical point } (27/8, 13/10). \end{aligned}$$

(x,y)	$(3,1)$	$(0,0)$	$(5,0)$	$(0,4)$	$(27/8, 13/10)$
$f(x,y)$	-3	0	-5	-12	$-231/80$

Absolute maximum value is 0, absolute minimum value is -12.

28. $f_x = y - 2 = 0,\ f_y = x = 0$; critical point (0,2), but (0,2) is not in the interior of R.

$$\begin{aligned} \text{Along } y = 0: &\quad u(x) = -2x;\ \text{no critical points,} \\ \text{along } x = 0: &\quad v(y) = 0;\ \text{no critical points,} \\ \text{along } y = 4 - x: &\quad w(x) = 2x - x^2;\ \text{critical point } (1, 3). \end{aligned}$$

(x,y)	$(0,0)$	$(0,4)$	$(4,0)$	$(1,3)$
$f(x,y)$	0	0	-8	1

Absolute maximum value is 1, absolute minimum value is -8.

29. $f_x = 2x - 2 = 0,\ f_y = -6y + 6 = 0$; critical point (1,1).

$$\begin{aligned} \text{Along } y = 0: &\quad u_1(x) = x^2 - 2x;\ \text{critical point } (1, 0), \\ \text{along } y = 2: &\quad u_2(x) = x^2 - 2x;\ \text{critical point } (1, 2) \\ \text{along } x = 0: &\quad v_1(y) = -3y^2 + 6y;\ \text{critical point } (0, 1), \\ \text{along } x = 2: &\quad v_2(y) = -3y^2 + 6y;\ \text{critical point } (2, 1) \end{aligned}$$

(x,y)	$(1,1)$	$(1,0)$	$(1,2)$	$(0,1)$	$(2,1)$	$(0,0)$	$(0,2)$	$(2,0)$	$(2,2)$
$f(x,y)$	2	-1	-1	3	3	0	0	0	0

Absolute maximum value is 3, absolute minimum value is -1.

30. $f_x = e^y - 2x = 0,\ f_y = xe^y - e^y = e^y(x - 1) = 0$; critical point $(1, \ln 2)$.

$$\begin{aligned} \text{Along } y = 0: &\quad u_1(x) = x - x^2 - 1;\ \text{critical point } (1/2, 0), \\ \text{along } y = 1: &\quad u_2(x) = ex - x^2 - e;\ \text{critical point } (e/2, 1), \\ \text{along } x = 0: &\quad v_1(y) = -e^y;\ \text{no critical points,} \\ \text{along } x = 2: &\quad v_2(y) = e^y - 4;\ \text{no critical points.} \end{aligned}$$

(x,y)	$(0,0)$	$(0,1)$	$(2,1)$	$(2,0)$	$(1,\ln 2)$	$(1/2,0)$	$(e/2,1)$
$f(x,y)$	-1	$-e$	$e-4$	-3	-1	$-3/4$	$e(e-4)/4 \approx -0.87$

Absolute maximum value is $-3/4$, absolute minimum value is -3.

31. $f_x = 2x - 1 = 0,\ f_y = 4y = 0$; critical point $(1/2, 0)$.

Along $x^2 + y^2 = 4 : y^2 = 4 - x^2$, $u(x) = 8 - x - x^2$ for $-2 \le x \le 2$; critical points $(-1/2, \pm\sqrt{15}/2)$.

(x,y)	$(1/2,0)$	$\left(-1/2, \sqrt{15}/2\right)$	$\left(-1/2, -\sqrt{15}/2\right)$	$(-2,0)$	$(2,0)$
$f(x,y)$	$-1/4$	$33/4$	$33/4$	6	2

Absolute maximum value is $33/4$, absolute minimum value is $-1/4$.

32. $f_x = y^2 = 0,\ f_y = 2xy = 0$; no critical points in the interior of R.

$$\begin{aligned} \text{Along } y = 0 &: \quad u(x) = 0;\ \text{no critical points,} \\ \text{along } x = 0 &: \quad v(y) = 0;\ \text{no critical points} \\ \text{along } x^2 + y^2 = 1 &: \quad w(x) = x - x^3 \text{ for } 0 \le x \le 1;\ \text{critical point } \left(1/\sqrt{3}, \sqrt{2/3}\right). \end{aligned}$$

(x,y)	$(0,0)$	$(0,1)$	$(1,0)$	$\left(1/\sqrt{3}, \sqrt{2/3}\right)$
$f(x,y)$	0	0	0	$2\sqrt{3}/9$

Absolute maximum value is $\dfrac{2}{9}\sqrt{3}$, absolute minimum value is 0.

33. Maximize $P = xyz$ subject to $x + y + z = 48$, $x > 0$, $y > 0$, $z > 0$. $z = 48 - x - y$ so $P = xy(48 - x - y) = 48xy - x^2y - xy^2$, $P_x = 48y - 2xy - y^2 = 0$, $P_y = 48x - x^2 - 2xy = 0$. But $x \ne 0$ and $y \ne 0$ so $48 - 2x - y = 0$ and $48 - x - 2y = 0$; critical point (16,16). $P_{xx}P_{yy} - P_{xy}^2 > 0$ and $P_{xx} < 0$ at (16,16), relative maximum. $z = 16$ when $x = y = 16$, the product is maximum for the numbers 16,16,16.

34. Minimize $S = x^2 + y^2 + z^2$ subject to $x + y + z = 27$, $x > 0$, $y > 0$, $z > 0$. $z = 27 - x - y$ so $S = x^2 + y^2 + (27 - x - y)^2$, $S_x = 4x + 2y - 54 = 0$, $S_y = 2x + 4y - 54 = 0$; critical point (9,9); $S_{xx}S_{yy} - S_{xy}^2 = 12 > 0$ and $S_{xx} = 4 > 0$ at (9,9), relative minimum. $z = 9$ when $x = y = 9$, the sum of the squares is minimum for the numbers 9,9,9.

35. Maximize $w = xy^2z^2$ subject to $x + y + z = 5$, $x > 0$, $y > 0$, $z > 0$. $x = 5 - y - z$ so $w = (5 - y - z)y^2z^2 = 5y^2z^2 - y^3z^2 - y^2z^3$, $w_y = 10yz^2 - 3y^2z^2 - 2yz^3 = yz^2(10 - 3y - 2z) = 0$, $w_z = 10y^2z - 2y^3z - 3y^2z^2 = y^2z(10 - 2y - 3z) = 0$, $10 - 3y - 2z = 0$ and $10 - 2y - 3z = 0$; critical point when $y = z = 2$; $w_{yy}w_{zz} - w_{yz}^2 = 320 > 0$ and $w_{yy} = -24 < 0$ when $y = z = 2$, relative maximum. $x = 1$ when $y = z = 2$, xy^2z^2 is maximum at (1,2,2).

36. Minimize $w = D^2 = x^2 + y^2 + z^2$ subject to $x^2 - yz = 5$. $x^2 = 5 + yz$ so $w = 5 + yz + y^2 + z^2$, $w_y = z + 2y = 0$, $w_z = y + 2z = 0$; critical point when $y = z = 0$; $w_{yy}w_{zz} - w_{yz}^2 = 3 > 0$ and $w_{yy} = 2 > 0$ when $y = z = 0$, relative minimum. $x^2 = 5$, $x = \pm\sqrt{5}$ when $y = z = 0$. The points $\left(\pm\sqrt{5}, 0, 0\right)$ are closest to the origin.

37. The diagonal of the box must equal the diameter of the sphere, thus we maximize $V = xyz$ or, for convenience, $w = V^2 = x^2y^2z^2$ subject to $x^2+y^2+z^2 = 4a^2$, $x > 0$, $y > 0$, $z > 0$; $z^2 = 4a^2 - x^2 - y^2$ hence $w = 4a^2x^2y^2 - x^4y^2 - x^2y^4$, $w_x = 2xy^2(4a^2 - 2x^2 - y^2) = 0$, $w_y = 2x^2y\left(4a^2 - x^2 - 2y^2\right) = 0$, $4a^2 - 2x^2 - y^2 = 0$ and $4a^2 - x^2 - 2y^2 = 0$; critical point $\left(2a/\sqrt{3}, 2a/\sqrt{3}\right)$;
$w_{xx}w_{yy} - w_{xy}^2 = \dfrac{4096}{27}a^8 > 0$ and $w_{xx} = -\dfrac{128}{9}a^4 < 0$ at $\left(2a/\sqrt{3}, 2a/\sqrt{3}\right)$, relative maximum. $z = 2a/\sqrt{3}$ when $x = y = 2a/\sqrt{3}$, the dimensions of the box of maximum volume are $2a/\sqrt{3}, 2a/\sqrt{3}, 2a/\sqrt{3}$.

38. Maximize $V = xyz$ subject to $x+y+z = 1$, $x > 0$, $y > 0$, $z > 0$. $z = 1-x-y$ so $V = xy - x^2y - xy^2$, $V_x = y(1 - 2x - y) = 0$, $V_y = x(1 - x - 2y) = 0$, $1 - 2x - y = 0$ and $1 - x - 2y = 0$; critical point $(1/3, 1/3)$; $V_{xx}V_{yy} - V_{xy}^2 = 1/3 > 0$ and $V_{xx} = -2/3 < 0$ at $(1/3, 1/3)$, relative maximum. The maximum volume is $V = (1/3)(1/3)(1/3) = 1/27$.

39. Let x, y, and z be, respectively, the length, width, and height of the box. Minimize
$C = 10(2xy) + 5(2xz + 2yz) = 10(2xy + xz + yz)$ subject to $xyz = 16$. $z = 16/(xy)$
so $C = 20(xy + 8/y + 8/x)$, $C_x = 20(y - 8/x^2) = 0$, $C_y = 20(x - 8/y^2) = 0$;
critical point (2,2); $C_{xx}C_{yy} - C_{xy}^2 = 1200 > 0$
and $C_{xx} = 40 > 0$ at (2,2), relative minimum. $z = 4$ when $x = y = 2$. The cost of materials is minimum if the length and width are 2 ft and the height is 4 ft.

40. Maximize the profit $P = 500(y-x)(x-40) + [45,000 + 500(x-2y)](y-60)$
$= 500(-x^2 - 2y^2 + 2xy - 20x + 170y - 5400)$.
$P_x = 1000(-x+y-10) = 0$, $P_y = 1000(-2y+x+85) = 0$; critical point (65,75);
$P_{xx}P_{yy} - P_{xy}^2 = 1{,}000{,}000 > 0$ and $P_{xx} = -1000 < 0$ at (65,75), relative maximum. The profit will be maximum when $x = 65$ and $y = 75$.

41. **(a)** $x = 0 : f(0,y) = -3y^2$, minimum -3, maximum 0;
$x = 1, f(1,y) = 4 - 3y^2 + 2y, \dfrac{\partial f}{\partial y}(1,y) = -6y + 2 = 0$ at $y = 1/3$, minimum 3, maximum 13/3;
$y = 0, f(x,0) = 4x^2$, minimum 0, maximum 4;
$y = 1, f(x,1) = 4x^2 + 2x - 3, \dfrac{\partial f}{\partial x}(x,1) = 8x + 2 \neq 0$ for $0 < x < 1$, minimum -3, maximum 3

(b) $f(x,x) = 3x^2$, minimum 0, maximum 3; $f(x,1-x) = -x^2+8x-3$, $\dfrac{d}{dx}f(x,1-x) = -2x+8 \neq 0$ for $0 < x < 1$, maximum 4, minimum -3

(c) $f_x(x,y) = 8x + 2y = 0$, $f_y(x,y) = -6y + 2x = 0$, solution is $(0,0)$, which is not an interior point of the square, so check the sides: minimum -3, maximum 13/3.

42. Maximize $A = ab\sin\alpha$ subject to $2a + 2b = \ell$, $a > 0$, $b > 0$, $0 < \alpha < \pi$. $b = (\ell - 2a)/2$ so $A = (1/2)(\ell a - 2a^2)\sin\alpha$, $A_a = (1/2)(\ell - 4a)\sin\alpha$, $A_\alpha = (a/2)(\ell - 2a)\cos\alpha$; $\sin\alpha \neq 0$ so from $A_a = 0$ we get $a = \ell/4$ and then from $A_\alpha = 0$ we get $\cos\alpha = 0$, $\alpha = \pi/2$. $A_{aa}A_{\alpha\alpha} - A_{a\alpha}^2 = \ell^2/8 > 0$ and $A_{aa} = -2 < 0$ when $a = \ell/4$ and $\alpha = \pi/2$, the area is maximum.

43. Minimize $S = xy + 2xz + 2yz$ subject to $xyz = V$, $x > 0$, $y > 0$, $z > 0$ where x, y, and z are, respectively, the length, width, and height of the box. $z = V/(xy)$ so $S = xy + 2V/y + 2V/x$, $S_x = y - 2V/x^2 = 0$, $S_y = x - 2V/y^2 = 0$; critical point $(\sqrt[3]{2V}, \sqrt[3]{2V})$; $S_{xx}S_{yy} - S_{xy}^2 = 3 > 0$ and $S_{xx} = 2 > 0$ at this point so there is a relative minimum there. The length and width are each $\sqrt[3]{2V}$, the height is $z = \sqrt[3]{2V}/2$.

44. The altitude of the trapezoid is $x\sin\phi$ and the lengths of the lower and upper bases are, respectively, $27 - 2x$ and $27 - 2x + 2x\cos\phi$ so we want to maximize
$A = (1/2)(x\sin\phi)[(27-2x) + (27 - 2x + 2x\cos\phi)] = 27x\sin\phi - 2x^2\sin\phi + x^2\sin\phi\cos\phi$.
$A_x = \sin\phi(27 - 4x + 2x\cos\phi)$,
$A_\phi = x(27\cos\phi - 2x\cos\phi - x\sin^2\phi + x\cos^2\phi) = x(27\cos\phi - 2x\cos\phi + 2x\cos^2\phi - x)$.
$\sin\phi \neq 0$ so from $A_x = 0$ we get $\cos\phi = (4x-27)/(2x)$, $x \neq 0$ so from $A_\phi = 0$ we get $(27 - 2x + 2x\cos\phi)\cos\phi - x = 0$ which, for $\cos\phi = (4x-27)/(2x)$, yields $4x - 27 - x = 0$, $x = 9$. If $x = 9$ then $\cos\phi = 1/2$, $\phi = \pi/3$. The critical point occurs when $x = 9$ and $\phi = \pi/3$; $A_{xx}A_{\phi\phi} - A_{x\phi}^2 = 729/2 > 0$ and $A_{xx} = -3\sqrt{3}/2 < 0$ there, the area is maximum when $x = 9$ and $\phi = \pi/3$.

45. **(a)** $\dfrac{\partial g}{\partial m} = \displaystyle\sum_{i=1}^{n} 2(mx_i + b - y_i)x_i = 2\left(m\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_iy_i\right) = 0$ if

$$\left(\sum_{i=1}^{n} x_i^2\right)m + \left(\sum_{i=1}^{n} x_i\right)b = \sum_{i=1}^{n} x_iy_i,$$

$$\frac{\partial g}{\partial b} = \sum_{i=1}^{n} 2(mx_i + b - y_i) = 2\left(m\sum_{i=1}^{n} x_i + bn - \sum_{i=1}^{n} y_i\right) = 0 \text{ if } \left(\sum_{i=1}^{n} x_i\right)m + nb = \sum_{i=1}^{n} y_i$$

(b) $$\sum_{i=1}^{n}(x_i-\bar{x})^2 = \sum_{i=1}^{n}\left(x_i^2-2\bar{x}x_i+\bar{x}^2\right) = \sum_{i=1}^{n}x_i^2 - 2\bar{x}\sum_{i=1}^{n}x_i + n\bar{x}^2$$
$$= \sum_{i=1}^{n}x_i^2 - \frac{2}{n}\left(\sum_{i=1}^{n}x_i\right)^2 + \frac{1}{n}\left(\sum_{i=1}^{n}x_i\right)^2$$
$$= \sum_{i=1}^{n}x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n}x_i\right)^2 \geq 0 \text{ so } n\sum_{i=1}^{n}x_i^2 - \left(\sum_{i=1}^{n}x_i\right)^2 \geq 0$$

This is an equality if and only if $\sum_{i=1}^{n}(x_i-\bar{x})^2 = 0$, which means $x_i = \bar{x}$ for each i.

(c) The system of equations $Am+Bb=C, Dm+Eb=F$ in the unknowns m and b has a unique solution provided $AE \neq BD$, and if so the solution is $m = \dfrac{CE-BF}{AE-BD}, b = \dfrac{F-Dm}{E}$, which after the appropriate substitution yields the desired result.

46. (a) $g_{mm} = 2\sum_{i=1}^{n}x_i^2,\ g_{bb} = 2n,\ g_{mb} = 2\sum_{i=1}^{n}x_i,$

$$D = g_{mm}g_{bb} - g_{mb}^2 = 4\left[n\sum_{i=1}^{n}x_i^2 - \left(\sum_{i=1}^{n}x_i\right)^2\right] > 0 \text{ and } g_{mm} > 0$$

(b) $g(m,b)$ is of the second-degree in m and b so the graph of $z = g(m,b)$ is a quadric surface.

(c) The function $z = g(m,b)$, as a function of m and b, has only one critical point, found in Exercise 47, and tends to $+\infty$ as either $|m|$ or $|b|$ tends to infinity, since g_{mm} and g_{bb} are both positive. Thus the only critical point must be a minimum.

47. $n=3, \sum_{i=1}^{3}x_i = 3, \sum_{i=1}^{3}y_i = 7, \sum_{i=1}^{3}x_iy_i = 13, \sum_{i=1}^{3}x_i^2 = 11, y = \dfrac{3}{4}x + \dfrac{19}{12}$

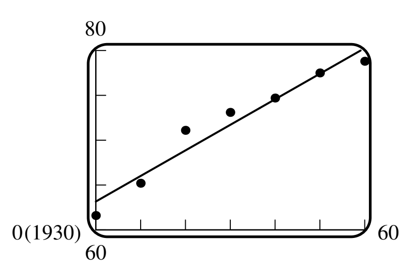
48. $n=4, \sum_{i=1}^{4}x_i = 7, \sum_{i=1}^{4}y_i = 4, \sum_{i=1}^{4}x_i^2 = 21, \sum_{i=1}^{4}x_iy_i = -2, y = -\dfrac{36}{35}x + \dfrac{14}{5}$

49. $\sum_{i=1}^{4}x_i = 10, \sum_{i=1}^{4}y_i = 8.2, \sum_{i=1}^{4}x_i^2 = 30, \sum_{i=1}^{4}x_iy_i = 23, n=4;\ m=0.5,\ b=0.8,\ y=0.5x+0.8.$

50. $\sum_{i=1}^{5}x_i = 15, \sum_{i=1}^{5}y_i = 15.1, \sum_{i=1}^{5}x_i^2 = 55, \sum_{i=1}^{5}x_iy_i = 39.8, n=5; m=-0.55, b=4.67, y=4.67-0.55x$

51. (a) $y = \dfrac{8843}{140} + \dfrac{57}{200}t \approx 63.1643 + 0.285t$

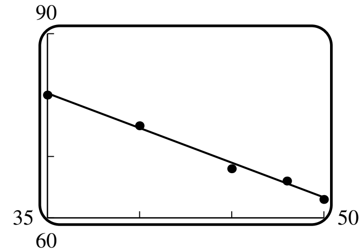
(b)



(c) $y = \dfrac{2909}{35} \approx 83.1143$

52. **(a)** $y \approx 119.84 - 1.13x$

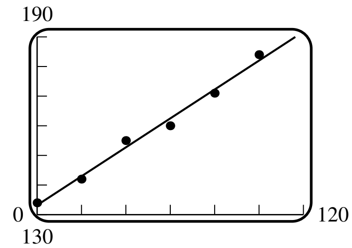
(b)



(c) about 52 units

53. **(a)** $P = \frac{2798}{21} + \frac{171}{350}T \approx 133.2381 + 0.4886T$

(b)



(c) $T \approx -\frac{139{,}900}{513} \approx -272.7096°$ C

54. **(a)** for example, $z = y$

(b) For example, on $0 \le x \le 1, 0 \le y \le 1$ let $z = \begin{cases} y & \text{if } 0 < x < 1, 0 < y < 1 \\ 1/2 & \text{if } x = 0, 1 \text{ or } y = 0, 1 \end{cases}$

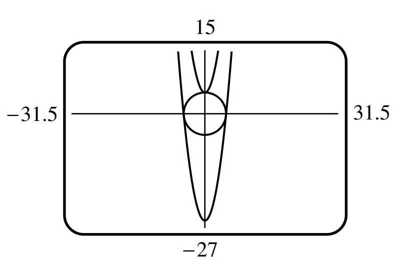
55. $f(x_0, y_0) \ge f(x, y)$ for all (x, y) inside a circle centered at (x_0, y_0) by virtue of Definition 14.8.1. If r is the radius of the circle, then in particular $f(x_0, y_0) \ge f(x, y_0)$ for all x satisfying $|x - x_0| < r$ so $f(x, y_0)$ has a relative maximum at x_0. The proof is similar for the function $f(x_0, y)$.

EXERCISE SET 14.9

1. **(a)** $xy = 4$ is tangent to the line, so the maximum value of f is 4.
(b) $xy = 2$ intersects the curve and so gives a smaller value of f.
(c) Maximize $f(x, y) = xy$ subject to the constraint $g(x, y) = x + y - 4 = 0, \nabla f = \lambda \nabla g$, $y\mathbf{i} + x\mathbf{j} = \lambda(\mathbf{i} + \mathbf{j})$, so solve the equations $y = \lambda, x = \lambda$ with solution $x = y = \lambda$, but $x + y = 4$, so $x = y = 2$, and the maximum value of f is $f = xy = 4$.

2. **(a)** $x^2 + y^2 = 25$ is tangent to the line at $(3, 4)$, so the minimum value of f is 25.
(b) A larger value of f yields a circle of a larger radius, and hence intersects the line.
(c) Minimize $f(x, y) = x^2 + y^2$ subject to the constraint $g(x, y) = 3x + 4y - 25 = 0, \nabla f = \lambda \nabla g$, $2x\mathbf{i} + 2y\mathbf{j} = 3\lambda\mathbf{i} + 4\lambda\mathbf{j}$, so solve $2x = 3\lambda, 2y = 4\lambda$ and $3x + 4y - 25 = 0$; solution is $x = 3, y = 4$, minimum $= 25$.

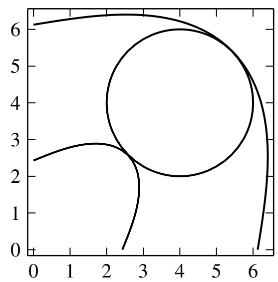
3. **(a)**



(b) one extremum at $(0, 5)$ and one at approximately $(\pm 5, 0)$, so minimum value -5, maximum value ≈ 25

(c) Find the minimum and maximum values of $f(x,y) = x^2 - y$ subject to the constraint $g(x,y) = x^2 + y^2 - 25 = 0, \nabla f = \lambda \nabla g, 2x\mathbf{i} - \mathbf{j} = 2\lambda x\mathbf{i} + 2\lambda y\mathbf{j}$, so solve $2x = 2\lambda x, -1 = 2\lambda y, x^2 + y^2 - 25 = 0$. If $x = 0$ then $y = \pm 5, f = \mp 5$, and if $x \neq 0$ then $\lambda = 1, y = -1/2, x^2 = 25 - 1/4 = 99/4, f = 99/4 + 1/2 = 101/4$, so the maximum value of f is $101/4$ at $(\pm 3\sqrt{11}/2, -1/2)$ and the minimum value of f is -5 at $(0,5)$.

4. **(a)**



(b) $f \approx 15$

(d) Set $f(x,y) = x^3 + y^3 - 3xy,\ g(x,y) = (x-4)^2 + (y-4)^2 - 4$; minimize f subject to the constraint $g = 0:\ \nabla f = \lambda g, (3x^2 - 3y)\mathbf{i} + (3y^2 - 3x)\mathbf{j} = 2\lambda(x-4)\mathbf{i} + 2\lambda(y-4)\mathbf{j}$, so solve (use a CAS) $3x^2 - 3y = 2\lambda(x-4), 3y^2 - 3x = 2\lambda(y-4)$ and $(x-4)^2 + (y-4)^2 - 4 = 0$; minimum value $f = 14.52$ at $(2.5858, 2.5858)$

5. $y = 8x\lambda,\ x = 16y\lambda;\ y/(8x) = x/(16y),\ x^2 = 2y^2$ so $4\left(2y^2\right) + 8y^2 = 16,\ y^2 = 1,\ y = \pm 1$. Test $\left(\pm\sqrt{2}, -1\right)$ and $(\pm\sqrt{2}, 1)$. $f\left(-\sqrt{2}, -1\right) = f\left(\sqrt{2}, 1\right) = \sqrt{2},\ f\left(-\sqrt{2}, 1\right) = f\left(\sqrt{2}, -1\right) = -\sqrt{2}$. Maximum $\sqrt{2}$ at $\left(-\sqrt{2}, -1\right)$ and $\left(\sqrt{2}, 1\right)$, minimum $-\sqrt{2}$ at $\left(-\sqrt{2}, 1\right)$ and $\left(\sqrt{2}, -1\right)$.

6. $2x = 2x\lambda, -2y = 2y\lambda, x^2 + y^2 = 25$. If $x \neq 0$ then $\lambda = 1$ and $y = 0$ so $x^2 + 0^2 = 25,\ x = \pm 5$. If $x = 0$ then $0^2 + y^2 = 25,\ y = \pm 5$. Test $(\pm 5, 0)$ and $(0, \pm 5)$: $f(\pm 5, 0) = 25,\ f(0, \pm 5) = -25$, maximum 25 at $(\pm 5, 0)$, minimum -25 at $(0, \pm 5)$.

7. $12x^2 = 4x\lambda,\ 2y = 2y\lambda$. If $y \neq 0$ then $\lambda = 1$ and $12x^2 = 4x,\ 12x(x - 1/3) = 0,\ x = 0$ or $x = 1/3$ so from $2x^2 + y^2 = 1$ we find that $y = \pm 1$ when $x = 0,\ y = \pm\sqrt{7}/3$ when $x = 1/3$. If $y = 0$ then $2x^2 + (0)^2 = 1,\ x = \pm 1/\sqrt{2}$. Test $(0, \pm 1)$, $\left(1/3, \pm\sqrt{7}/3\right)$, and $\left(\pm 1/\sqrt{2}, 0\right)$. $f(0, \pm 1) = 1$, $f\left(1/3, \pm\sqrt{7}/3\right) = 25/27,\ f\left(1/\sqrt{2}, 0\right) = \sqrt{2},\ f\left(-1/\sqrt{2}, 0\right) = -\sqrt{2}$. Maximum $\sqrt{2}$ at $\left(1/\sqrt{2}, 0\right)$, minimum $-\sqrt{2}$ at $\left(-1/\sqrt{2}, 0\right)$.

8. $1 = 2x\lambda,\ -3 = 6y\lambda;\ 1/(2x) = -1/(2y),\ y = -x$ so $x^2 + 3(-x)^2 = 16,\ x = \pm 2$. Test $(-2, 2)$ and $(2, -2)$. $f(-2, 2) = -9,\ f(2, -2) = 7$. Maximum 7 at $(2, -2)$, minimum -9 at $(-2, 2)$.

9. $2 = 2x\lambda, 1 = 2y\lambda, -2 = 2z\lambda;\ 1/x = 1/(2y) = -1/z$ thus $x = 2y,\ z = -2y$ so $(2y)^2 + y^2 + (-2y)^2 = 4,\ y^2 = 4/9,\ y = \pm 2/3$. Test $(-4/3, -2/3, 4/3)$ and $(4/3, 2/3, -4/3)$. $f(-4/3, -2/3, 4/3) = -6,\ f(4/3, 2/3, -4/3) = 6$. Maximum 6 at $(4/3, 2/3, -4/3)$, minimum -6 at $(-4/3, -2/3, 4/3)$.

10. $3 = 4x\lambda,\ 6 = 8y\lambda,\ 2 = 2z\lambda;\ 3/(4x) = 3/(4y) = 1/z$ thus $y = x,\ z = 4x/3$, so $2x^2 + 4x^2 + (4x/3)^2 = 70,\ x^2 = 9,\ x = \pm 3$. Test $(-3, -3, -4)$ and $(3,3,4)$. $f(-3, -3, -4) = -35,\ f(3, 3, 4) = 35$. Maximum 35 at $(3, 3, 4)$, minimum -35 at $(-3, -3, -4)$.

11. $yz = 2x\lambda,\ xz = 2y\lambda,\ xy = 2z\lambda;\ yz/(2x) = xz/(2y) = xy/(2z)$ thus $y^2 = x^2,\ z^2 = x^2$ so $x^2 + x^2 + x^2 = 1,\ x = \pm 1/\sqrt{3}$. Test the eight possibilities with $x = \pm 1/\sqrt{3},\ y = \pm 1/\sqrt{3}$, and $z = \pm 1/\sqrt{3}$ to find the maximum is $1/\left(3\sqrt{3}\right)$ at $\left(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\right)$, $\left(1/\sqrt{3}, -1/\sqrt{3}, -1/\sqrt{3}\right)$, $\left(-1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3}\right)$, and $\left(-1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3}\right)$; the minimum is $-1/\left(3\sqrt{3}\right)$ at $\left(1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3}\right)$, $\left(1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3}\right)$, $\left(-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\right)$, and $\left(-1/\sqrt{3}, -1/\sqrt{3}, -1/\sqrt{3}\right)$.

12. $4x^3 = 2\lambda x, 4y^3 = 2\lambda y, 4z^3 = 2\lambda z$; if x (or y or z) $\neq 0$ then $\lambda = 2x^2$ (or $2y^2$ or $2z^2$).
Assume for the moment that $|x| \leq |y| \leq |z|$. Then:
Case I: $x, y, z \neq 0$ so $\lambda = 2x^2 = 2y^2 = 2z^2, x = \pm y = \pm z$, $3x^2 = 1, x = \pm 1/\sqrt{3}$,
$f(x, y, z) = 3/9 = 1/3$
Case II: $x = 0, y, z \neq 0$; then $y = \pm z, 2y^2 = 1, y = \pm z = \pm 1/\sqrt{2}, f(x, y, z) = 2/4 = 1/2$
Case III: $x = y = 0, z \neq 0$; then $z^2 = 1, z = \pm 1, f(x, y, z) = 1$
Thus f has a maximum value of 1 at $(0, 0, \pm 1), (0, \pm 1, 0)$, and $(\pm 1, 0, 0)$ and a minimum value of $1/3$ at $(\pm 1/\sqrt{3}, \pm 1/\sqrt{3}, \pm 1/\sqrt{3})$.

13. $f(x, y) = x^2 + y^2$; $2x = 2\lambda$, $2y = -4\lambda$; $y = -2x$ so $2x - 4(-2x) = 3$, $x = 3/10$. The point is $(3/10, -3/5)$.

14. $f(x, y) = (x-4)^2 + (y-2)^2$, $g(x, y) = y - 2x - 3$; $2(x-4) = -2\lambda$, $2(y-2) = \lambda$; $x - 4 = -2(y-2)$, $x = -2y + 8$ so $y = 2(-2y + 8) + 3$, $y = 19/5$. The point is $(2/5, 19/5)$.

15. $f(x, y, z) = x^2 + y^2 + z^2$; $2x = \lambda$, $2y = 2\lambda$, $2z = \lambda$; $y = 2x$, $z = x$ so $x + 2(2x) + x = 1$, $x = 1/6$. The point is $(1/6, 1/3, 1/6)$.

16. $f(x, y, z) = (x-1)^2 + (y+1)^2 + (z-1)^2$; $2(x-1) = 4\lambda$, $2(y+1) = 3\lambda$, $2(z-1) = \lambda$; $x = 4z - 3$, $y = 3z - 4$ so $4(4z-3) + 3(3z-4) + z = 2$, $z = 1$. The point is $(1, -1, 1)$.

17. $f(x, y) = (x-1)^2 + (y-2)^2$; $2(x-1) = 2x\lambda$, $2(y-2) = 2y\lambda$; $(x-1)/x = (y-2)/y$, $y = 2x$ so $x^2 + (2x)^2 = 45$, $x = \pm 3$. $f(-3, -6) = 80$ and $f(3, 6) = 20$ so (3,6) is closest and $(-3, -6)$ is farthest.

18. $f(x, y, z) = x^2 + y^2 + z^2$; $2x = y\lambda$, $2y = x\lambda$, $2z = -2z\lambda$. If $z \neq 0$ then $\lambda = -1$ so $2x = -y$ and $2y = -x$, $x = y = 0$; substitute into $xy - z^2 = 1$ to get $z^2 = -1$ which has no real solution. If $z = 0$ then $xy - (0)^2 = 1$, $y = 1/x$, and also (from $2x = y\lambda$ and $2y = x\lambda$), $2x/y = 2y/x$, $y^2 = x^2$ so $(1/x)^2 = x^2$, $x^4 = 1$, $x = \pm 1$. Test (1,1,0) and $(-1, -1, 0)$ to see that they are both closest to the origin.

19. $f(x, y, z) = x + y + z$, $x^2 + y^2 + z^2 = 25$ where x, y, and z are the components of the vector; $1 = 2x\lambda$, $1 = 2y\lambda$, $1 = 2z\lambda$; $1/(2x) = 1/(2y) = 1/(2z)$; $y = x$, $z = x$ so $x^2 + x^2 + x^2 = 25$, $x = \pm 5/\sqrt{3}$. $f\left(-5/\sqrt{3}, -5/\sqrt{3}, -5/\sqrt{3}\right) = -5\sqrt{3}$ and $f\left(5/\sqrt{3}, 5/\sqrt{3}, 5/\sqrt{3}\right) = 5\sqrt{3}$ so the vector is $5(\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3}$.

20. $x^2 + y^2 = 25$ is the constraint; solve $8x - 4y = 2x\lambda$, $-4x + 2y = 2y\lambda$. If $x = 0$ then $y = 0$ and conversely; but $x^2 + y^2 = 25$, so x and y are nonzero. Thus $\lambda = (4x - 2y)/x = (-2x + y)/y$, so $0 = 2x^2 + 3xy - 2y^2 = (2x - y)(x + 2y)$, hence $y = 2x$ or $x = -2y$. If $y = 2x$ then $x^2 + (2x)^2 = 25$, $x = \pm\sqrt{5}$. If $x = -2y$ then $\left(-2y^2\right) + y^2 = 25$, $y = \pm\sqrt{5}$. $T\left(-\sqrt{5}, -2\sqrt{5}\right) = T\left(\sqrt{5}, 2\sqrt{5}\right) = 0$ and $T\left(2\sqrt{5}, -\sqrt{5}\right) = T\left(-2\sqrt{5}, \sqrt{5}\right) = 125$. The highest temperature is 125 and the lowest is 0.

21. Minimize $f = x^2 + y^2 + z^2$ subject to $g(x, y, z) = x + y + z - 27 = 0$. $\nabla f = \lambda \nabla g$,
$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda\mathbf{i} + \lambda\mathbf{j} + \lambda\mathbf{k}$, solution $x = y = z = 9$, minimum value 243

22. Maximize $f(x, y, z) = xy^2z^2$ subject to $g(x, y, z) = x + y + z - 5 = 0, \nabla f = \lambda \nabla g = \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k})$,
$\lambda = y^2z^2 = 2xyz^2 = 2xy^2z$, $\lambda = 0$ is impossible, hence $x, y, z \neq 0$, and $z = y = 2x, 5x - 5 = 0$,
$x = 1, y = z = 2$, maximum value 16 at $(1, 2, 2)$

23. Minimize $f = x^2 + y^2 + z^2$ subject to $x^2 - yz = 5, \nabla f = \lambda \nabla g, 2x = 2x\lambda, 2y = -z\lambda, 2z = -y\lambda$. If $\lambda \neq \pm 2$, then $y = z = 0, x = \pm\sqrt{5}, f = 5$; if $\lambda = \pm 2$ then $x = 0$, and since $-yz = 5$, $y = -z = \pm\sqrt{5}, f = 10$, thus the minimum value is 5 at $(\pm\sqrt{5}, 0, 0)$.

24. The diagonal of the box must equal the diameter of the sphere so maximize $V = xyz$ or, for convenience, maximize $f = V^2 = x^2y^2z^2$ subject to $g(x, y, z) = x^2 + y^2 + z^2 - 4a^2 = 0, \nabla f = \lambda \nabla g$, $2xy^2z^2 = 2\lambda x, 2x^2yz^2 = 2\lambda y, 2x^2y^2z = 2\lambda z$. Since $V \neq 0$ it follows that $x, y, z \neq 0$, hence $x = \pm y = \pm z, 3x^2 = 4a^2, x = \pm 2a/\sqrt{3}$, maximum volume $8a^3/(3\sqrt{3})$.

25. Let x, y, and z be, respectively, the length, width, and height of the box. Minimize $f(x, y, z) = 10(2xy) + 5(2xz + 2yz) = 10(2xy + xz + yz)$ subject to $g(x, y, z) = xyz - 16 = 0$, $\nabla f = \lambda \nabla g, 20y + 10z = \lambda yz, 20x + 10z = \lambda xz, 10x + 10y = \lambda xy$. Since $V = xyz = 16, x, y, z \neq 0$, thus $\lambda z = 20 + 10(z/y) = 20 + 10(z/x)$, so $x = y$. From this and $10x + 10y = \lambda xy$ it follows that $20 = \lambda x$, so $10z = 20x, z = 2x = 2y, V = 2x^3 = 16$ and thus $x = y = 2$ ft, $z = 4$ ft, $f(2, 2, 4) = 240$ cents.

26. **(a)** If $g(x, y) = x = 0$ then $8x + 2y = \lambda, -6y + 2x = 0$; but $x = 0$, so $y = \lambda = 0$,

$f(0, 0) = 0$ maximum, $f(0, 1) = -3$, minimum.

If $g(x, y) = x - 1 = 0$ then $8x + 2y = \lambda, -6y + 2x = 0$; but $x = 1$, so $y = 1/3$,

$f(1, 1/3) = 13/3$ maximum, $f(1, 0) = 4$, $f(1, 1) = 3$ minimum.

If $g(x, y) = y = 0$ then $8x + 2y = 0, -6y + 2x = \lambda$; but $y = 0$ so $x = \lambda = 0$,

$f(0, 0) = 0$ minimum, $f(1, 0) = 4$, maximum.

If $g(x, y) = y - 1 = 0$ then $8x + 2y = 0, -6y + 2x = \lambda$; but $y = 1$ so $x = -1/4$, no solution,

$f(0, 1) = -3$ minimum, $f(1, 1) = 3$ maximum.

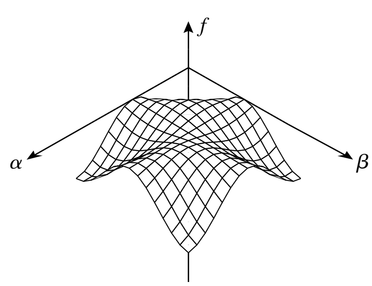
(b) If $g(x, y) = x - y = 0$ then $8x + 2y = \lambda, -6y + 2x = -\lambda$; but $x = y$ so solution $x = y = \lambda = 0, f(0, 0) = 0$ minimum, $f(1, 1) = 3$ maximum. If $g(x, y) = 1 - x - y = 0$ then $8x + 2y = -1, -6y + 2x = -1$; but $x + y = 1$ so solution is $x = -2/13, y = 3/2$ which is not on diagonal, $f(0, 1) = -3$ minimum, $f(1, 0) = 4$ maximum.

27. Maximize $A(a, b, \alpha) = ab \sin \alpha$ subject to $g(a, b, \alpha) = 2a + 2b - \ell = 0, \nabla_{(a,b,\alpha)} f = \lambda \nabla_{(a,b,\alpha)} g$, $b \sin \alpha = 2\lambda, a \sin \alpha = 2\lambda, ab \cos \alpha = 0$ with solution $a = b\ (= \ell/4), \alpha = \pi/2$ maximum value if parallelogram is a square.

28. Minimize $f(x, y, z) = xy + 2xz + 2yz$ subject to $g(x, y, z) = xyz - V = 0, \nabla f = \lambda \nabla g$, $y + 2z = \lambda yz, x + 2z = \lambda xz, 2x + 2y = \lambda xy; \lambda = 0$ leads to $x = y = z = 0$, impossible, so solve for $\lambda = 1/z + 2/x = 1/z + 2/y = 2/y + 2/x$, so $x = y = 2z, x^3 = 2V$, minimum value $3(2V)^{2/3}$

29. **(a)** Maximize $f(\alpha, \beta, \gamma) = \cos \alpha \cos \beta \cos \gamma$ subject to $g(\alpha, \beta, \gamma) = \alpha + \beta + \gamma - \pi = 0$, $\nabla f = \lambda \nabla g, -\sin \alpha \cos \beta \cos \gamma = \lambda, -\cos \alpha \sin \beta \cos \gamma = \lambda, -\cos \alpha \cos \beta \sin \gamma = \lambda$ with solution $\alpha = \beta = \gamma = \pi/3$, maximum value 1/8

(b) for example, $f(\alpha, \beta) = \cos \alpha \cos \beta \cos(\pi - \alpha - \beta)$



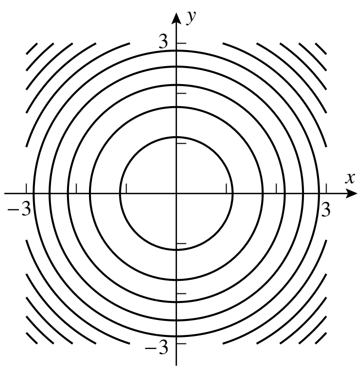
30. Find maxima and minima $z = x^2 + 4y^2$ subject to the constraint $g(x, y) = x^2 + y^2 - 1 = 0$, $\nabla z = \lambda \nabla g, 2x\mathbf{i} + 8y\mathbf{j} = 2\lambda x\mathbf{i} + 2\lambda y\mathbf{j}$, solve $2x = 2\lambda x, 8y = 2\lambda y$. If $y \neq 0$ then $\lambda = 4, x = 0, y^2 = 1$ and $z = x^2 + 4y^2 = 4$. If $y = 0$ then $x^2 = 1$ and $z = 1$, so the maximum height is obtained for $(x, y) = (0, \pm 1), z = 4$ and the minimum height is $z = 1$ at $(\pm 1, 0)$.

CHAPTER 14 SUPPLEMENTARY EXERCISES

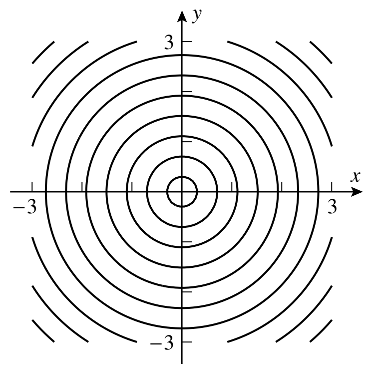
1. **(a)** They approximate the profit per unit of any additional sales of the standard or high-resolution monitors, respectively.

(b) The rates of change with respect to the two directions x and y, and with respect to time.

3. $z = \sqrt{x^2 + y^2} = c$ implies $x^2 + y^2 = c^2$, which is the equation of a circle; $x^2 + y^2 = c$ is also the equation of a circle (for $c > 0$).



$z = x^2 + y^2$

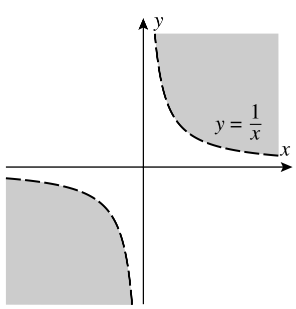


$z = \sqrt{x^2 + y^2}$

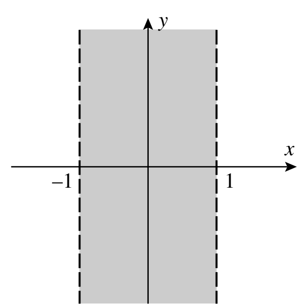
5. **(b)** $f(x, y, z) = z - x^2 - y^2$

7. **(a)** $f(\ln y, e^x) = e^{\ln y} \ln e^x = xy$ **(b)** $e^{r+s} \ln(rs)$

8. **(a)**



(b)



9. $w_x = 2x \sec^2(x^2 + y^2) + \sqrt{y},\ w_{xy} = 8xy \sec^2(x^2 + y^2) \tan(x^2 + y^2) + \frac{1}{2} y^{-1/2}$,

$w_y = 2y \sec^2(x^2 + y^2) + \frac{1}{2} xy^{-1/2},\ w_{yx} = 8xy \sec^2(x^2 + y^2) \tan(x^2 + y^2) + \frac{1}{2} y^{-1/2}$

10. $\partial w/\partial x = \dfrac{1}{x - y} - \sin(x + y), \partial^2 w/\partial x^2 = -\dfrac{1}{(x - y)^2} - \cos(x + y)$,

$\partial w/\partial y = -\dfrac{1}{x - y} - \sin(x + y),\ \partial^2 w/\partial y^2 = -\dfrac{1}{(x - y)^2} - \cos(x + y) = \partial^2 w/\partial x^2$

11. $F_x = -6xz, F_{xx} = -6z, F_y = -6yz, F_{yy} = -6z, F_z = 6z^2 - 3x^2 - 3y^2,$
$F_{zz} = 12z, F_{xx} + F_{yy} + F_{zz} = -6z - 6z + 12z = 0$

12. $f_x = yz + 2x, f_{xy} = z, f_{xyz} = 1, f_{xyzx} = 0;\ f_z = xy - (1/z), f_{zx} = y, f_{zxx} = 0, f_{zxxy} = 0$

13. **(a)** $P = \dfrac{10T}{V}$,
$$\frac{dP}{dt} = \frac{\partial P}{\partial T}\frac{dT}{dt} + \frac{\partial P}{\partial V}\frac{dV}{dt} = \frac{10}{V}\cdot 3 - \frac{10T}{V^2}\cdot 0 = \frac{30}{V} = \frac{30}{2.5} = 12\ \text{N/(m}^2\text{min)} = 12\ \text{Pa/min}$$
(b) $$\frac{dP}{dt} = \frac{\partial P}{\partial T}\frac{dT}{dt} + \frac{\partial P}{\partial V}\frac{dV}{dt} = \frac{10}{V}\cdot 0 - \frac{10T}{V^2}\cdot(-3) = \frac{30T}{V^2} = \frac{30\cdot 50}{(2.5)^2} = 240\ \text{Pa/min}$$

14. **(a)** $z = 1 - y^2$, slope $= \dfrac{\partial z}{\partial y} = -2y = 4$ **(b)** $z = 1 - 4x^2, \dfrac{\partial z}{\partial x} = -8x = -8$

15. $x^4 - x + y - x^3y = (x^3 - 1)(x - y)$, limit $= -1$, not defined on the line $y = x$ so not continuous at $(0, 0)$

16. $\dfrac{x^4 - y^4}{x^2 + y^2} = x^2 - y^2$, limit $= \displaystyle\lim_{(x,y)\to(0,0)}(x^2 - y^2) = 0$, continuous

17. Use the unit vectors $\mathbf{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle, \mathbf{v} = \langle 0, -1\rangle, \mathbf{w} = \langle -\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\rangle = -\frac{\sqrt{2}}{\sqrt{5}}\mathbf{u} + \frac{1}{\sqrt{5}}\mathbf{v}$, so that
$$D_{\mathbf{w}}f = -\frac{\sqrt{2}}{\sqrt{5}}D\mathbf{u}f + \frac{1}{\sqrt{5}}D\mathbf{v}f = -\frac{\sqrt{2}}{\sqrt{5}}2\sqrt{2} + \frac{1}{\sqrt{5}}(-3) = -\frac{7}{\sqrt{5}}$$

18. **(a)** $\mathbf{n} = z_x\mathbf{i} + z_y\mathbf{j} - \mathbf{k} = 8\mathbf{i} + 8\mathbf{j} - \mathbf{k}$, tangent plane $8x + 8y - z = 4 + 8\ln 2$, normal line $x(t) = 1 + 8t, y(t) = \ln 2 + 8t, z(t) = 4 - t$

(b) $\mathbf{n} = 3\mathbf{i} + 10\mathbf{j} - 14\mathbf{k}$, tangent plane $3x + 10y - 14z = 30$, normal line $x(t) = 2 + 3t, y(t) = 1 + 10t, z(t) = -1 - 14t$

19. The origin is not such a point, so assume that the normal line at $(x_0, y_0, z_0) \neq (0, 0, 0)$ passes through the origin, then $\mathbf{n} = z_x\mathbf{i} + z_y\mathbf{j} - \mathbf{k} = -y_0\mathbf{i} - x_0\mathbf{j} - \mathbf{k}$; the line passes through the origin and is normal to the surface if it has the form $\mathbf{r}(t) = -y_0t\mathbf{i} - x_0t\mathbf{j} - t\mathbf{k}$ and $(x_0, y_0, z_0) = (x_0, y_0, 2 - x_0y_0)$ lies on the line if $-y_0t = x_0, -x_0t = y_0, -t = 2 - x_0y_0$, with solutions $x_0 = y_0 = -1$, $x_0 = y_0 = 1, x_0 = y_0 = 0$; thus the points are $(0, 0, 2), (1, 1, 1), (-1, -1, 1)$.

20. $\mathbf{n} = \frac{2}{3}x_0^{-1/3}\mathbf{i} + \frac{2}{3}y_0^{-1/3}\mathbf{j} + \frac{2}{3}z_0^{-1/3}\mathbf{k}$, tangent plane $x_0^{-1/3}x + y_0^{-1/3}y + z_0^{-1/3}z = x_0^{2/3} + y_0^{2/3} + z_0^{2/3} = 1$; intercepts are $x = x_0^{1/3}, y = y_0^{1/3}, z = z_0^{1/3}$, sum of squares of intercepts is $x_0^{2/3} + y_0^{2/3} + z_0^{2/3} = 1$.

21. A tangent to the line is $6\mathbf{i} + 4\mathbf{j} + \mathbf{k}$, a normal to the surface is $\mathbf{n} = 18x\mathbf{i} + 8y\mathbf{j} - \mathbf{k}$, so solve
$18x = 6k, 8y = 4k, -1 = k;\ k = -1, x = -1/3, y = -1/2, z = 2$

22. $\Delta w = (1.1)^2(-0.1) - 2(1.1)(-0.1) + (-0.1)^2(1.1) - 0 = 0.11,$
$dw = (2xy - 2y + y^2)dx + (x^2 - 2x + 2yx)dy = -(-0.1) = 0.1$

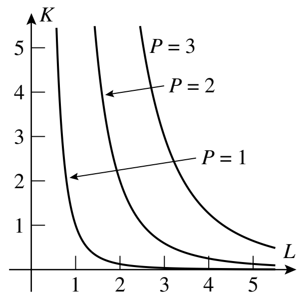
23. $dV = \frac{2}{3}xhdx + \frac{1}{3}x^2dh = \frac{2}{3}2(-0.1) + \frac{1}{3}(0.2) = -0.06667\ \text{m}^3; \Delta V = -0.07267\ \text{m}^3$

24. $\nabla f = (2x + 3y - 6)\mathbf{i} + (3x + 6y + 3)\mathbf{j} = \mathbf{0}$ if $2x + 3y = 6, x + 2y = -1, x = 15, y = -8, D = 3 > 0$, $f_{xx} = 2 > 0$, so f has a relative minimum at $(15, -8)$.

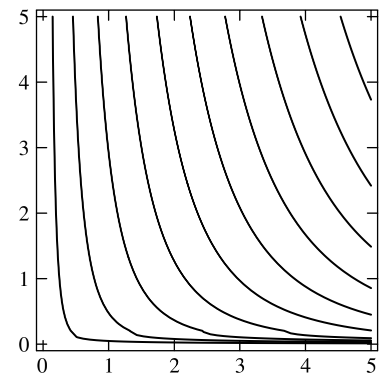
25. $\nabla f = (2xy - 6x)\mathbf{i} + (x^2 - 12y)\mathbf{j} = \mathbf{0}$ if $2xy - 6x = 0, x^2 - 12y = 0$; if $x = 0$ then $y = 0$, and if $x \neq 0$ then $y = 3, x = \pm 6$, thus the gradient vanishes at $(0, 0), (-6, 3), (6, 3)$; $f_{xx} = 0$ at all three points, $f_{yy} = -12 < 0, D = -4x^2$, so $(\pm 6, 3)$ are saddle points, and near the origin we write $f(x, y) = (y - 3)x^2 - 6y^2$; since $y - 3 < 0$ when $|y| < 3$, f has a local maximum by inspection.

26. $\nabla f = (3x^2 - 3y)\mathbf{i} - (3x - y)\mathbf{j} = \mathbf{0}$ if $y = x^2, 3x = y$, so $x = y = 0$ or $x = 3, y = 9$; at $x = y = 0, D = -9$, saddle point; at $x = 3, y = 9, D = 9, f_{xx} = 18 > 0$, relative minimum

27. **(a)**



(b)



28. **(a)** $\partial P/\partial L = c\alpha L^{\alpha-1}K^{\beta}, \partial P/\partial K = c\beta L^{\alpha}K^{\beta-1}$

(b) the rates of change of output with respect to labor and capital equipment, respectively

(c) $K(\partial P/\partial K) + L(\partial P/\partial L) = c\beta L^{\alpha}K^{\beta} + c\alpha L^{\alpha}K^{\beta} = (\alpha + \beta)P = P$

29. **(a)** Maximize $P = 1000L^{0.6}(200{,}000 - L)^{0.4}$ subject to $50L + 100K = 200{,}000$ or $L = 2K = 4000$. $600\left(\dfrac{K}{L}\right)^{0.4} = \lambda$, $400\left(\dfrac{L}{K}\right)^{0.6} = 2\lambda$, $L + 2K = 4000$; so $\dfrac{2}{3}\left(\dfrac{L}{K}\right) = 2$, thus $L = 3K$, $L = 2400, K = 800, P(2400, 800) = 1000 \cdot 2400^{0.6} \cdot 800^{0.4} = 1000 \cdot 3^{0.6} \cdot 800 = 800{,}000 \cdot 3^{0.6} \approx$ \$1,546,545.64

(b) The value of labor is $50L = 120{,}000$ and the value of capital is $100K = 80{,}000$.

30. **(a)** $y^2 = 8 - 4x^2$, find extrema of $f(x) = x^2(8 - 4x^2) = -4x^4 + 8x^2$ defined for $-\sqrt{2} \leq x \leq \sqrt{2}$. Then $f'(x) = -16x^3 + 16x = 0$ when $x = 0, \pm 1, f''(x) = -48x^2 + 16$, so f has a relative maximum at $x = \pm 1, y = \pm 2$ and a relative minimum at $x = 0, y = \pm 2\sqrt{2}$. At the endpoints $x = \pm\sqrt{2}, y = 0$ we obtain the minimum $f = 0$ again.

(b) $f(x, y) = x^2y^2, g(x, y) = 4x^2 + y^2 - 8 = 0, \nabla f = 2xy^2\mathbf{i} + 2x^2y\mathbf{j} = \lambda\nabla g = 8\lambda x\mathbf{i} + 2\lambda y\mathbf{j}$, so solve $2xy^2 = \lambda 8x, 2x^2y = \lambda 2y$. If $x = 0$ then $y = \pm 2\sqrt{2}$, and if $y = 0$ then $x = \pm\sqrt{2}$. In either case f has a relative and absolute minimum. Assume $x, y \neq 0$, then $y^2 = 4\lambda, x^2 = \lambda$, use $g = 0$ to obtain $x^2 = 1, x = \pm 1, y = \pm 2$, and $f = 4$ is a relative and absolute maximum at $(\pm 1, \pm 2)$.

31. Let the first octant corner of the box be (x, y, z), so that $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$. Maximize $V = 8xyz$ subject to $g(x, y, z) = (x/a)^2 + (y/b)^2 + (z/c)^2 = 1$, solve $\nabla V = \lambda\nabla g$, or $8(yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}) = (2\lambda x/a^2)\mathbf{i} + (2\lambda y/b^2)\mathbf{j} + (2\lambda z/c^2)\mathbf{k}, 8a^2yz = 2\lambda x, 8b^2xz = 2\lambda y, 8c^2xy = 2\lambda z$. For the maximum volume, $x, y, z \neq 0$; divide the first equation by the second to obtain $a^2y^2 = b^2x^2$; the first by the third to obtain $a^2z^2 = c^2x^2$, and finally $b^2z^2 = c^2y^2$. From $g = 1$ get $3(x/a)^2 = 1, x = \pm a/\sqrt{3}$, and then $y = \pm b/\sqrt{3}, z = \pm c/\sqrt{3}$. The dimensions of the box are $\dfrac{2a}{\sqrt{3}} \times \dfrac{2b}{\sqrt{3}} \times \dfrac{2c}{\sqrt{3}}$, and the maximum volume is $8abc/(3\sqrt{3})$.

32. **(a)** $\dfrac{dy}{dx} = -\dfrac{6x - 5y + y\sec^2 xy}{-5x + x\sec^2 xy}$. **(b)** $\dfrac{dy}{dx} = -\dfrac{\ln y + \cos(x-y)}{x/y - \cos(x-y)}$

33. $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y}, \ \dfrac{d^2y}{dx^2} = -\dfrac{f_y(d/dx)f_x - f_x(d/dx)f_y}{f_y^2} = -\dfrac{f_y(f_{xx} + f_{xy}(dy/dx)) - f_x(f_{xy} + f_{yy}(dy/dx))}{f_y^2}$

$$= -\frac{f_y(f_{xx} + f_{xy}(-f_x/f_y)) - f_x(f_{xy} + f_{yy}(-f_x/f_y))}{f_y^2} = \frac{-f_y^2 f_{xx} + 2f_x f_y f_{xy} - f_x^2 f_{yy}}{f_y^3}$$

34. Denote the currents I_1, I_2, I_3 by x, y, z respectively. Then minimize $F(x,y,z) = x^2R_1 + y^2R_2 + z^2R_3$ subject to $g(x,y,z) = x + y + z - I = 0$, so solve $\nabla F = \lambda \nabla g, 2xR_1\mathbf{i} + 2yR_2\mathbf{j} + 2zR_3\mathbf{k} = \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k})$, $\lambda = 2xR_1 = 2yR_2 = 2zR_3$, so the minimum value of F occurs when $I_1 : I_2 : I_3 = \dfrac{1}{R_1} : \dfrac{1}{R_2} : \dfrac{1}{R_3}$.

35. Solve $(t-1)^2/4 + 16e^{-2t} + (2 - \sqrt{t})^2 = 1$ for t to get $t = 1.833223, 2.839844$; the particle strikes the surface at the points $P_1(0.83322, 0.639589, 0.646034)$, $P_2(1.83984, 0.233739, 0.314816)$. The velocity vectors are given by $\mathbf{v} = \dfrac{dx}{dt}\mathbf{i} + \dfrac{dy}{dt}\mathbf{j} + \dfrac{dz}{dt}\mathbf{k} = \mathbf{i} - 4e^{-t}\mathbf{j} - 1/(2\sqrt{t})\mathbf{k}$, and a normal to the surface is $\mathbf{n} = \nabla(x^2/4 + y^2 + z^2) = x/2\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$. At the points P_i these are
$\mathbf{v}_1 = \mathbf{i} - 0.639589\mathbf{j} - 0.369286\mathbf{k}, \mathbf{v}_2 = \mathbf{i} - 0.233739\mathbf{j} + 0.296704\mathbf{k}$;
$\mathbf{n}_1 = 0.41661\mathbf{i} + 1.27918\mathbf{j} + 1.29207\mathbf{k}$ and $\mathbf{n}_2 = 0.91992\mathbf{i} + 0.46748\mathbf{j} + 0.62963\mathbf{k}$ so
$\cos^{-1}[(\mathbf{v}_i \cdot \mathbf{n}_i)/(\|\mathbf{v}_i\|\,\|\mathbf{n}_i\|)] = 112.3°, 61.1°$; the acute angles are $67.7°, 61.1°$.

36. **(a)** $F'(x) = \displaystyle\int_0^1 e^y \cos(xe^y)dy = \frac{\sin(ex) - \sin x}{x}$

(b) Use a CAS to get $x = \dfrac{\pi}{e+1}$ so the maximum value of $F(x)$ is

$$F\left(\frac{\pi}{e+1}\right) = \int_0^1 \sin\left(\frac{\pi}{e+1}e^y\right) dy \approx 0.909026.$$

37. Let x, y, z be the lengths of the sides opposite angles α, β, γ, located at A, B, C respectively. Then $x^2 = y^2 + z^2 - 2yz\cos\alpha$ and $x^2 = 100 + 400 - 2(10)(20)/2 = 300, x = 10\sqrt{3}$ and

$$2x\frac{dx}{dt} = 2y\frac{dy}{dt} + 2z\frac{dz}{dt} - 2\left(y\frac{dz}{dt}\cos\alpha + z\frac{dy}{dt}\cos\alpha - yz(\sin\alpha)\frac{d\alpha}{dt}\right)$$

$$= 2(10)(4) + 2(20)(2) - 2\left(10(2)\frac{1}{2} + 20(4)\frac{1}{2} - 10(20)\frac{\sqrt{3}}{2}\frac{\pi}{60}\right) = 60 + \frac{10\pi}{\sqrt{3}}$$

so $\dfrac{dx}{dt} = \sqrt{3} + \dfrac{\pi}{6}$, the length of BC is increasing.

38. **(a)** $\dfrac{d}{dt}\left(\dfrac{\partial z}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right)\dfrac{dx}{dt} + \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right)\dfrac{dy}{dt} = \dfrac{\partial^2 z}{\partial x^2}\dfrac{dx}{dt} + \dfrac{\partial^2 z}{\partial y \partial x}\dfrac{dy}{dt}$ by the Chain Rule, and

$$\frac{d}{dt}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{dx}{dt} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\frac{dy}{dt} = \frac{\partial^2 z}{\partial x \partial y}\frac{dx}{dt} + \frac{\partial^2 z}{\partial y^2}\frac{dy}{dt}$$

(b) $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$,

$$\frac{d^2z}{dt^2} = \frac{dx}{dt}\left(\frac{\partial^2 z}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2 z}{\partial y \partial x}\frac{dy}{dt}\right) + \frac{\partial z}{\partial x}\frac{d^2x}{dt^2} + \frac{dy}{dt}\left(\frac{\partial^2 z}{\partial x \partial y}\frac{dx}{dt} + \frac{\partial^2 z}{\partial y^2}\frac{dy}{dt}\right) + \frac{\partial z}{\partial y}\frac{d^2y}{dt^2}$$