CHAPTER 13
Vector-Valued Functions

EXERCISE SET 13.1

1. $(-\infty, +\infty)$; $\mathbf{r}(\pi) = -\mathbf{i} - 3\pi\mathbf{j}$

2. $[-1/3, +\infty)$; $\mathbf{r}(1) = \langle 2, 1\rangle$

3. $[2, +\infty)$; $\mathbf{r}(3) = -\mathbf{i} - \ln 3\mathbf{j} + \mathbf{k}$

4. $[-1, 1)$; $\mathbf{r}(0) = \langle 2, 0, 0\rangle$

5. $\mathbf{r} = 3\cos t\mathbf{i} + (t + \sin t)\mathbf{j}$

6. $\mathbf{r} = (t^2 + 1)\mathbf{i} + e^{-2t}\mathbf{j}$

7. $\mathbf{r} = 2t\mathbf{i} + 2\sin 3t\mathbf{j} + 5\cos 3t\mathbf{k}$

8. $\mathbf{r} = t\sin t\mathbf{i} + \ln t\mathbf{j} + \cos^2 t\mathbf{k}$

9. $x = 3t^2$, $y = -2$

10. $x = \sin^2 t$, $y = 1 - \cos 2t$

11. $x = 2t - 1$, $y = -3\sqrt{t}$, $z = \sin 3t$

12. $x = te^{-t}$, $y = 0$, $z = -5t^2$

13. the line in 2-space through the point $(2, 0)$ and parallel to the vector $-3\mathbf{i} - 4\mathbf{j}$

14. the circle of radius 3 in the xy-plane, with center at the origin

15. the line in 3-space through the point $(0, -3, 1)$ and parallel to the vector $2\mathbf{i} + 3\mathbf{k}$

16. the circle of radius 2 in the plane $x = 3$, with center at $(3, 0, 0)$

17. an ellipse in the plane $z = -1$, center at $(0, 0, -1)$, major axis of length 6 parallel to x-axis, minor axis of length 4 parallel to y-axis

18. a parabola in the plane $x = -2$, vertex at $(-2, 0, -1)$, opening upward

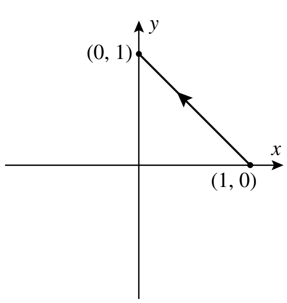
19. **(a)** The line is parallel to the vector $-2\mathbf{i} + 3\mathbf{j}$; the slope is $-3/2$.

(b) $y = 0$ in the xz-plane so $1 - 2t = 0$, $t = 1/2$ thus $x = 2 + 1/2 = 5/2$ and $z = 3(1/2) = 3/2$; the coordinates are $(5/2, 0, 3/2)$.

20. **(a)** $x = 3 + 2t = 0$, $t = -3/2$ so $y = 5(-3/2) = -15/2$

(b) $x = t$, $y = 1 + 2t$, $z = -3t$ so $3(t) - (1 + 2t) - (-3t) = 2$, $t = 3/4$; the point of intersection is $(3/4, 5/2, -9/4)$.

21. **(a)**



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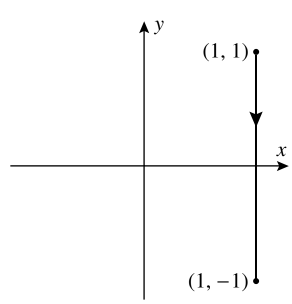
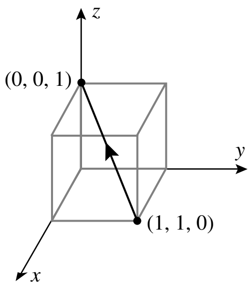
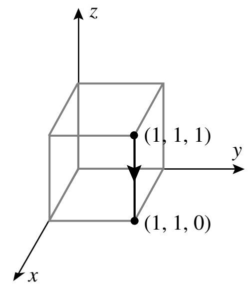


22. (a)



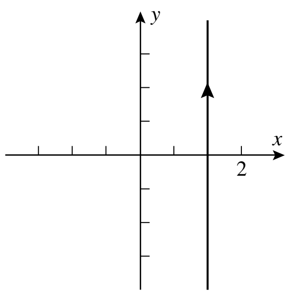
(b)



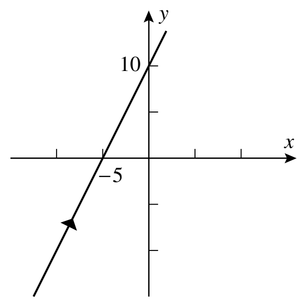
23. $\mathbf{r} = (1 - t)(3\mathbf{i} + 4\mathbf{j}), 0 \le t \le 1$

24. $\mathbf{r} = (1 - t)4\mathbf{k} + t(2\mathbf{i} + 3\mathbf{j}), 0 \le t \le 1$

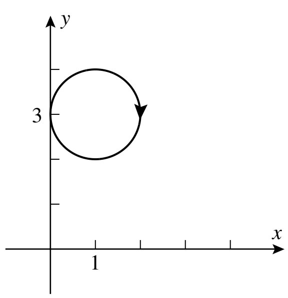
25. $x = 2$



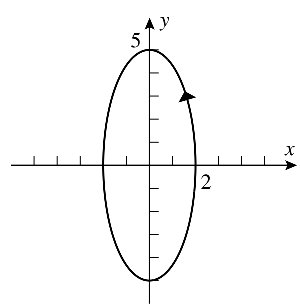
26. $y = 2x + 10$



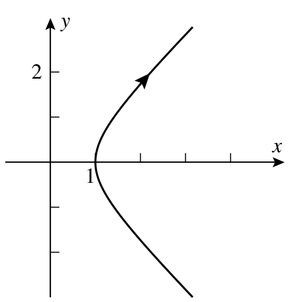
27. $(x - 1)^2 + (y - 3)^2 = 1$



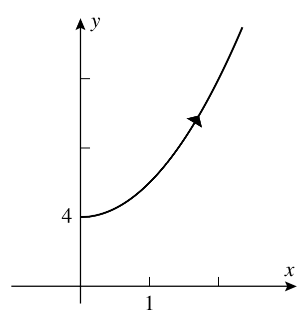
28. $x^2/4 + y^2/25 = 1$



29. $x^2 - y^2 = 1,\ x \ge 1$



30. $y = 2x^2 + 4,\ x \ge 0$



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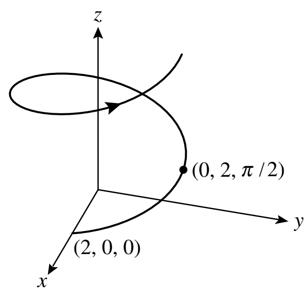

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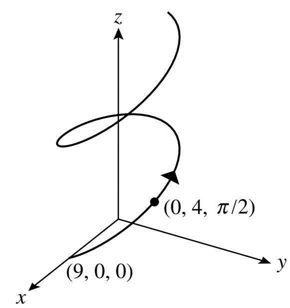

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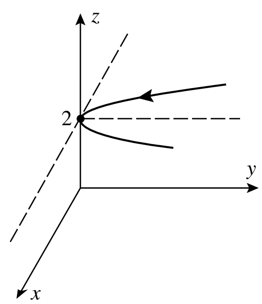

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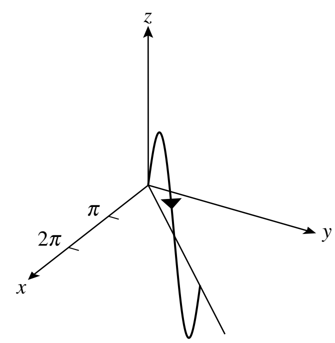
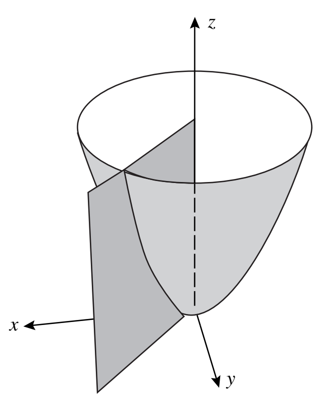
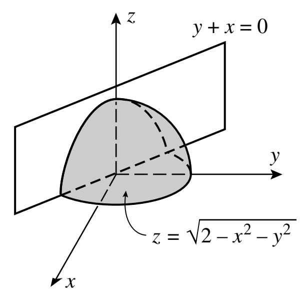


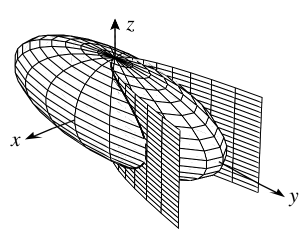
35. $x = t, y = t, z = 2t^2$



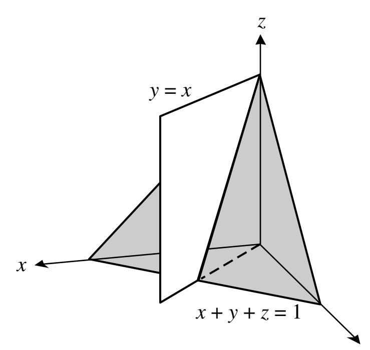
36. $x = t, y = -t, z = \sqrt{2}\sqrt{1 - t^2}$



37. $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} \pm \dfrac{1}{3}\sqrt{81 - 9t^2 - t^4}\,\mathbf{k}$



38. $\mathbf{r} = t\mathbf{i} + t\mathbf{j} + (1 - 2t)\mathbf{k}$



39. $x^2 + y^2 = (t\sin t)^2 + (t\cos t)^2 = t^2(\sin^2 t + \cos^2 t) = t^2 = z$

40. $x - y + z + 1 = t - (1+t)/t + (1-t^2)/t + 1 = [t^2 - (1+t) + (1-t^2) + t]/t = 0$

41. $x = \sin t$, $y = 2\cos t$, $z = \sqrt{3}\sin t$ so $x^2 + y^2 + z^2 = \sin^2 t + 4\cos^2 t + 3\sin^2 t = 4$ and $z = \sqrt{3}x$; it is the curve of intersection of the sphere $x^2 + y^2 + z^2 = 4$ and the plane $z = \sqrt{3}x$, which is a circle with center at $(0,0,0)$ and radius 2.

42. $x = 3\cos t$, $y = 3\sin t$, $z = 3\sin t$ so $x^2 + y^2 = 9\cos^2 t + 9\sin^2 t = 9$ and $z = y$; it is the curve of intersection of the circular cylinder $x^2 + y^2 = 9$ and the plane $z = y$, which is an ellipse with major axis of length $6\sqrt{2}$ and minor axis of length 6.

43. The helix makes one turn as t varies from 0 to 2π so $z = c(2\pi) = 3$, $c = 3/(2\pi)$.

44. $0.2t = 10$, $t = 50$; the helix has made one revolution when $t = 2\pi$ so when $t = 50$ it has made $50/(2\pi) = 25/\pi \approx 7.96$ revolutions.

45. $x^2 + y^2 = t^2\cos^2 t + t^2\sin^2 t = t^2$, $\sqrt{x^2 + y^2} = t = z$; a conical helix.

46. The curve wraps around an elliptic cylinder with axis along the z-axis; an elliptical helix.

47. **(a)** III, since the curve is a subset of the plane $y = -x$

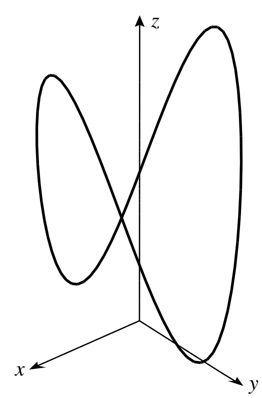
(b) IV, since only x is periodic in t, and y, z increase without bound

(c) II, since all three components are periodic in t

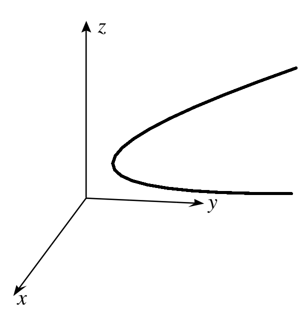
(d) I, since the projection onto the yz-plane is a circle and the curve increases without bound in the x-direction

49. **(a)** Let $x = 3\cos t$ and $y = 3\sin t$, then $z = 9\cos^2 t$.

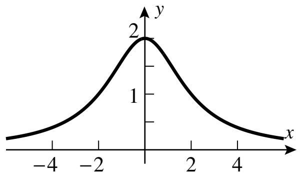
(b)



50. The plane is parallel to a line on the surface of the cone and does not go through the vertex so the curve of intersection is a parabola. Eliminate z to get $y + 2 = \sqrt{x^2 + y^2}$, $(y+2)^2 = x^2 + y^2$, $y = x^2/4 - 1$; let $x = t$, then $y = t^2/4 - 1$ and $z = t^2/4 + 1$.



51. **(a)**



(b) In Part (a) set $x = 2t$;
then $y = 2/(1 + (x/2)^2) = 8/(4 + x^2)$

EXERCISE SET 13.2

1. $9\mathbf{i} + 6\mathbf{j}$

2. $\langle \sqrt{2}/2, \sqrt{2}/2 \rangle$

3. $\langle 1/3, 0 \rangle$

4. $\mathbf{j}$

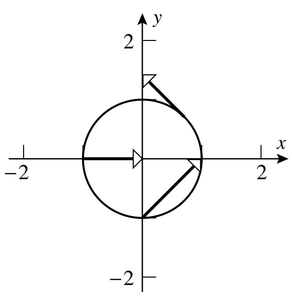
5. $2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$

6. $\langle 3, 1/2, \sin 2 \rangle$

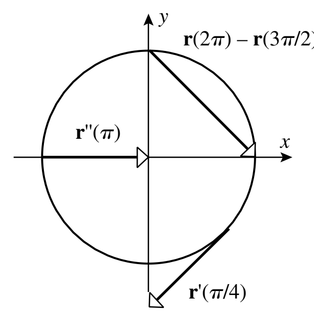
7. **(a)** continuous, $\lim_{t \to 0} \mathbf{r}(t) = \mathbf{0} = \mathbf{r}(0)$ **(b)** not continuous, $\lim_{t \to 0} \mathbf{r}(t)$ does not exist

8. **(a)** not continuous, $\lim_{t \to 0} \mathbf{r}(t)$ does not exist. **(b)** continuous, $\lim_{t \to 0} \mathbf{r}(t) = 5\mathbf{i} - \mathbf{j} + \mathbf{k} = \mathbf{r}(0)$

9.



10.



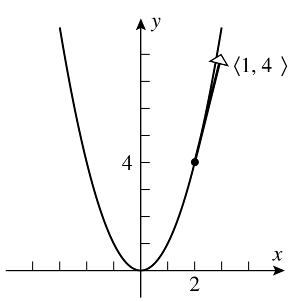
11. $\mathbf{r}'(t) = 5\mathbf{i} + (1 - 2t)\mathbf{j}$

12. $\mathbf{r}'(t) = \sin t\mathbf{j}$

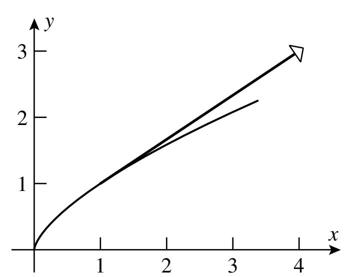
13. $\mathbf{r}'(t) = -\dfrac{1}{t^2}\mathbf{i} + \sec^2 t\mathbf{j} + 2e^{2t}\mathbf{k}$

14. $\mathbf{r}'(t) = \dfrac{1}{1 + t^2}\mathbf{i} + (\cos t - t \sin t)\mathbf{j} - \dfrac{1}{2\sqrt{t}}\mathbf{k}$

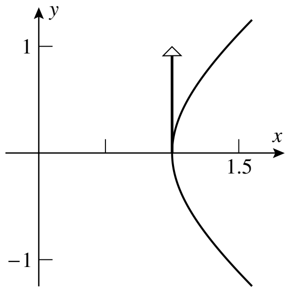
15. $\mathbf{r}'(t) = \langle 1, 2t \rangle$,
$\mathbf{r}'(2) = \langle 1, 4 \rangle$,
$\mathbf{r}(2) = \langle 2, 4 \rangle$



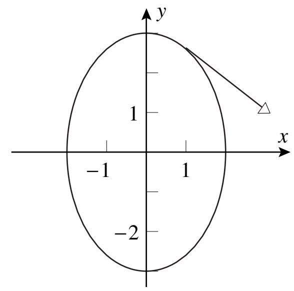
16. $\mathbf{r}'(t) = 3t^2\mathbf{i} + 2t\mathbf{j}$,
$\mathbf{r}'(1) = 3\mathbf{i} + 2\mathbf{j}$
$\mathbf{r}(1) = \mathbf{i} + \mathbf{j}$



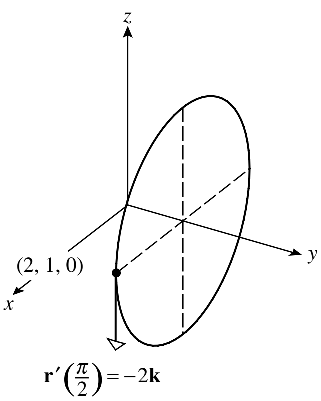
17. $\mathbf{r}'(t) = \sec t \tan t\mathbf{i} + \sec^2 t\mathbf{j}$,
$\mathbf{r}'(0) = \mathbf{j}$
$\mathbf{r}(0) = \mathbf{i}$



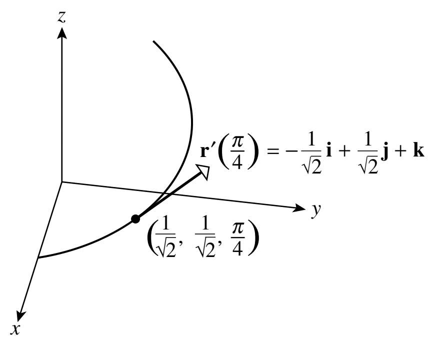
18. $\mathbf{r}'(t) = 2\cos t\mathbf{i} - 3\sin t\mathbf{j}$,
$\mathbf{r}'\left(\frac{\pi}{6}\right) = \sqrt{3}\mathbf{i} - \frac{3}{2}\mathbf{j}$
$\mathbf{r}\left(\frac{\pi}{6}\right) = \mathbf{i} + \frac{3\sqrt{3}}{2}\mathbf{j}$



19. $\mathbf{r}'(t) = 2\cos t\mathbf{i} - 2\sin t\mathbf{k}$,
$\mathbf{r}'(\pi/2) = -2\mathbf{k}$,
$\mathbf{r}(\pi/2) = 2\mathbf{i} + \mathbf{j}$



20. $\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}$,
$\mathbf{r}'(\pi/4) = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \mathbf{k}$,
$\mathbf{r}(\pi/4) = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \frac{\pi}{4}\mathbf{k}$



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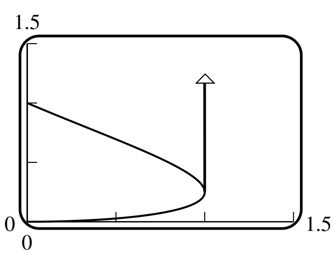

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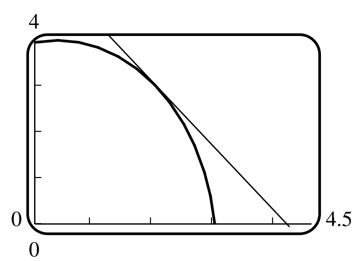


23. $\mathbf{r}'(t) = 2t\mathbf{i} - \frac{1}{t}\mathbf{j}$, $\mathbf{r}'(1) = 2\mathbf{i} - \mathbf{j}$, $\mathbf{r}(1) = \mathbf{i} + 2\mathbf{j}$; $x = 1 + 2t$, $y = 2 - t$

24. $\mathbf{r}'(t) = 2e^{2t}\mathbf{i} + 6\sin 3t\mathbf{j}$, $\mathbf{r}'(0) = 2\mathbf{i}$, $\mathbf{r}(0) = \mathbf{i} - 2\mathbf{j}$; $x = 1 + 2t$, $y = -2$

25. $\mathbf{r}'(t) = -2\pi\sin\pi t\mathbf{i} + 2\pi\cos\pi t\mathbf{j} + 3\mathbf{k}$, $\mathbf{r}'(1/3) = -\sqrt{3}\,\pi\mathbf{i} + \pi\mathbf{j} + 3\mathbf{k}$,
$\mathbf{r}(1/3) = \mathbf{i} + \sqrt{3}\,\mathbf{j} + \mathbf{k}$; $x = 1 - \sqrt{3}\,\pi t$, $y = \sqrt{3} + \pi t$, $z = 1 + 3t$

26. $\mathbf{r}'(t) = \frac{1}{t}\mathbf{i} - e^{-t}\mathbf{j} + 3t^2\mathbf{k}$, $\mathbf{r}'(2) = \frac{1}{2}\mathbf{i} - e^{-2}\mathbf{j} + 12\mathbf{k}$,

$\mathbf{r}(2) = \ln 2\mathbf{i} + e^{-2}\mathbf{j} + 8\mathbf{k}$; $x = \ln 2 + \frac{1}{2}t$, $y = e^{-2} - e^{-2}t$, $z = 8 + 12t$

27. $\mathbf{r}'(t) = 2\mathbf{i} + \frac{3}{2\sqrt{3t+4}}\mathbf{j}$, $t = 0$ at P_0 so $\mathbf{r}'(0) = 2\mathbf{i} + \frac{3}{4}\mathbf{j}$,

$\mathbf{r}(0) = -\mathbf{i} + 2\mathbf{j}$; $\mathbf{r} = (-\mathbf{i} + 2\mathbf{j}) + t\left(2\mathbf{i} + \frac{3}{4}\mathbf{j}\right)$

28. $\mathbf{r}'(t) = -4\sin t\mathbf{i} - 3\mathbf{j}$, $t = \pi/3$ at P_0 so $\mathbf{r}'(\pi/3) = -2\sqrt{3}\mathbf{i} - 3\mathbf{j}$,
$\mathbf{r}(\pi/3) = 2\mathbf{i} - \pi\mathbf{j}$; $\mathbf{r} = (2\mathbf{i} - \pi\mathbf{j}) + t(-2\sqrt{3}\mathbf{i} - 3\mathbf{j})$

29. $\mathbf{r}'(t) = 2t\mathbf{i} + \frac{1}{(t+1)^2}\mathbf{j} - 2t\mathbf{k}$, $t = -2$ at P_0 so $\mathbf{r}'(-2) = -4\mathbf{i} + \mathbf{j} + 4\mathbf{k}$,

$\mathbf{r}(-2) = 4\mathbf{i} + \mathbf{j}$; $\mathbf{r} = (4\mathbf{i} + \mathbf{j}) + t(-4\mathbf{i} + \mathbf{j} + 4\mathbf{k})$

30. $\mathbf{r}'(t) = \cos t\mathbf{i} + \sinh t\mathbf{j} + \frac{1}{1+t^2}\mathbf{k}$, $t = 0$ at P_0 so $\mathbf{r}'(0) = \mathbf{i} + \mathbf{k}$, $\mathbf{r}(0) = \mathbf{j}$; $\mathbf{r} = t\mathbf{i} + \mathbf{j} + t\mathbf{k}$

31. **(a)** $\lim_{t\to 0}(\mathbf{r}(t) - \mathbf{r}'(t)) = \mathbf{i} - \mathbf{j} + \mathbf{k}$

(b) $\lim_{t\to 0}(\mathbf{r}(t) \times \mathbf{r}'(t)) = \lim_{t\to 0}(-\cos t\mathbf{i} - \sin t\mathbf{j} + k) = -\mathbf{i} + \mathbf{k}$

(c) $\lim_{t\to 0}(\mathbf{r}(t) \cdot \mathbf{r}'(t)) = 0$

32. $\mathbf{r}(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}''(t)) = \begin{vmatrix} t & t^2 & t^3 \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = 2t^3$, so $\lim_{t\to 1}\mathbf{r}(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}''(t)) = 2$

33. **(a)** $\mathbf{r}_1' = 2\mathbf{i} + 6t\mathbf{j} + 3t^2\mathbf{k}$, $\mathbf{r}_2' = 4t^3\mathbf{k}$, $\mathbf{r}_1 \cdot \mathbf{r}_2 = t^7$; $\frac{d}{dt}(\mathbf{r}_1 \cdot \mathbf{r}_2) = 7t^6 = \mathbf{r}_1 \cdot \mathbf{r}_2' + \mathbf{r}_1' \cdot \mathbf{r}_2$

(b) $\mathbf{r}_1 \times \mathbf{r}_2 = 3t^6\mathbf{i} - 2t^5\mathbf{j}$, $\frac{d}{dt}(\mathbf{r}_1 \times \mathbf{r}_2) = 18t^5\mathbf{i} - 10t^4\mathbf{j} = \mathbf{r}_1 \times \mathbf{r}_2' + \mathbf{r}_1' \times \mathbf{r}_2$

34. **(a)** $\mathbf{r}_1' = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}$, $\mathbf{r}_2' = \mathbf{k}$, $\mathbf{r}_1 \cdot \mathbf{r}_2 = \cos t + t^2$; $\frac{d}{dt}(\mathbf{r}_1 \cdot \mathbf{r}_2) = -\sin t + 2t = \mathbf{r}_1 \cdot \mathbf{r}_2' + \mathbf{r}_1' \cdot \mathbf{r}_2$

(b) $\mathbf{r}_1 \times \mathbf{r}_2 = t\sin t\mathbf{i} + t(1 - \cos t)\mathbf{j} - \sin t\mathbf{k}$,

$\frac{d}{dt}(\mathbf{r}_1 \times \mathbf{r}_2) = (\sin t + t\cos t)\mathbf{i} + (1 + t\sin t - \cos t)\mathbf{j} - \cos t\mathbf{k} = \mathbf{r}_1 \times \mathbf{r}_2' + \mathbf{r}_1' \times \mathbf{r}_2$

35. $3t\mathbf{i} + 2t^2\mathbf{j} + \mathbf{C}$

36. $(\sin t)\mathbf{i} - (\cos t)\mathbf{j} + \mathbf{C}$

37. $(-t\cos t + \sin t)\mathbf{i} + t\mathbf{j} + \mathbf{C}$

38. $\langle (t-1)e^t, t(\ln t - 1)\rangle + \mathbf{C}$

39. $(t^3/3)\mathbf{i} - t^2\mathbf{j} + \ln|t|\mathbf{k} + \mathbf{C}$

40. $\langle -e^{-t}, e^t, t^3\rangle + \mathbf{C}$

41. $\left\langle \frac{1}{3}\sin 3t, \frac{1}{3}\cos 3t \right\rangle\Big]_0^{\pi/3} = \langle 0, -2/3\rangle$

42. $\left(\frac{1}{3}t^3\mathbf{i} + \frac{1}{4}t^4\mathbf{j}\right)\Big]_0^1 = \frac{1}{3}\mathbf{i} + \frac{1}{4}\mathbf{j}$

43. $\displaystyle\int_0^2 \sqrt{t^2+t^4}dt = \int_0^2 t(1+t^2)^{1/2}dt = \frac{1}{3}\left(1+t^2\right)^{3/2}\Big]_0^2 = (5\sqrt{5}-1)/3$

44. $\displaystyle\left\langle -\frac{2}{5}(3-t)^{5/2}, \frac{2}{5}(3+t)^{5/2}, t\right\rangle\Big]_{-3}^{3} = \langle 72\sqrt{6}/5, 72\sqrt{6}/5, 6\rangle$

45. $\displaystyle\left(\frac{2}{3}t^{3/2}\mathbf{i} + 2t^{1/2}\mathbf{j}\right)\Big]_1^9 = \frac{52}{3}\mathbf{i} + 4\mathbf{j}$

46. $\displaystyle\frac{1}{2}(e^2-1)\mathbf{i} + (1-e^{-1})\mathbf{j} + \frac{1}{2}\mathbf{k}$

47. $\mathbf{y}(t) = \int \mathbf{y}'(t)dt = \frac{1}{3}t^3\mathbf{i} + t^2\mathbf{j} + \mathbf{C}, \mathbf{y}(0) = \mathbf{C} = \mathbf{i} + \mathbf{j}, \mathbf{y}(t) = (\frac{1}{3}t^3 + 1)\mathbf{i} + (t^2+1)\mathbf{j}$

48. $\mathbf{y}(t) = \int \mathbf{y}'(t)dt = (\sin t)\mathbf{i} - (\cos t)\mathbf{j} + \mathbf{C}$,
$\mathbf{y}(0) = -\mathbf{j} + \mathbf{C} = \mathbf{i} - \mathbf{j}$ so $\mathbf{C} = \mathbf{i}$ and $\mathbf{y}(t) = (1+\sin t)\mathbf{i} - (\cos t)\mathbf{j}$.

49. $\mathbf{y}'(t) = \int \mathbf{y}''(t)dt = t\mathbf{i} + e^t\mathbf{j} + \mathbf{C}_1, \mathbf{y}'(0) = \mathbf{j} + \mathbf{C}_1 = \mathbf{j}$ so $\mathbf{C}_1 = \mathbf{0}$ and $\mathbf{y}'(t) = t\mathbf{i} + e^t\mathbf{j}$.

$\displaystyle\mathbf{y}(t) = \int \mathbf{y}'(t)dt = \frac{1}{2}t^2\mathbf{i} + e^t\mathbf{j} + \mathbf{C}_2,\ \mathbf{y}(0) = \mathbf{j} + \mathbf{C}_2 = 2\mathbf{i}$ so $\mathbf{C}_2 = 2\mathbf{i} - \mathbf{j}$ and

$\displaystyle\mathbf{y}(t) = \left(\frac{1}{2}t^2 + 2\right)\mathbf{i} + (e^t-1)\mathbf{j}$

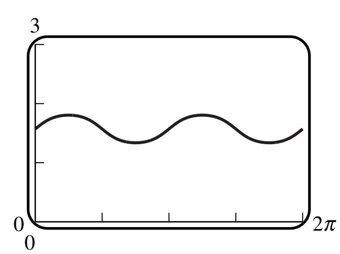
50. $\mathbf{y}'(t) = \int \mathbf{y}''(t)dt = 4t^3\mathbf{i} - t^2\mathbf{j} + \mathbf{C}_1,\ \mathbf{y}'(0) = \mathbf{C}_1 = \mathbf{0},\ \mathbf{y}'(t) = 4t^3\mathbf{i} - t^2\mathbf{j}$

$\displaystyle\mathbf{y}(t) = \int \mathbf{y}'(t)dt = t^4\mathbf{i} - \frac{1}{3}t^3\mathbf{j} + \mathbf{C}_2,\ \mathbf{y}(0) = \mathbf{C}_2 = 2\mathbf{i} - 4\mathbf{j},\ \mathbf{y}(t) = (t^4+2)\mathbf{i} - (\frac{1}{3}t^3 + 4)\mathbf{j}$

51. $\mathbf{r}'(t) = -4\sin t\mathbf{i} + 3\cos t\mathbf{j},\ \mathbf{r}(t)\cdot\mathbf{r}'(t) = -7\cos t\sin t$, so $\mathbf{r}$ and $\mathbf{r}'$ are perpendicular for $t = 0, \pi/2, \pi, 3\pi/2, 2\pi$. Since

$\|\mathbf{r}(t)\| = \sqrt{16\cos^2 t + 9\sin^2 t}, \|\mathbf{r}'(t)\| = \sqrt{16\sin^2 t + 9\cos^2 t}$,

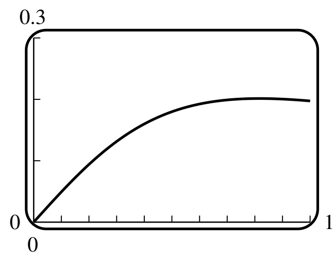
$\displaystyle\|\mathbf{r}\|\|\mathbf{r}'\| = \sqrt{144 + 337\sin^2 t\cos^2 t},\quad \theta = \cos^{-1}\left[\frac{-7\sin t\cos t}{\sqrt{144+337\sin^2 t\cos^2 t}}\right]$, with the graph



From the graph it appears that θ is bounded away from 0 and π, meaning that $\mathbf{r}$ and $\mathbf{r}'$ are never parallel. We can check this by considering them as vectors in 3-space, and then $\mathbf{r}\times\mathbf{r}' = 12\,\mathbf{k} \neq \mathbf{0}$, so they are never parallel.

52. $\mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j},\ \mathbf{r}(t)\cdot\mathbf{r}'(t) = 2t^3 + 3t^5 = 0$ only for $t = 0$ since $2 + 3t^2 > 0$.

$\|\mathbf{r}(t)\| = t^2\sqrt{1+t^2}, \|\mathbf{r}'(t)\| = t\sqrt{4+9t^2},\ \ \theta = \cos^{-1}\left[\dfrac{2+3t^2}{\sqrt{1+t^2}\sqrt{4+9t^2}}\right]$ with the graph



θ appears to be bounded away from π and is zero only for $t = 0$, at which point $\mathbf{r} = \mathbf{r}' = \mathbf{0}$.

53. **(a)** $2t - t^2 - 3t = -2,\ t^2 + t - 2 = 0,\ (t+2)(t-1) = 0$ so $t = -2, 1$. The points of intersection are $(-2, 4, 6)$ and $(1, 1, -3)$.

(b) $\mathbf{r}' = \mathbf{i} + 2t\mathbf{j} - 3\mathbf{k}$; $\mathbf{r}'(-2) = \mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$, $\mathbf{r}'(1) = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, and $\mathbf{n} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ is normal to the plane. Let θ be the acute angle, then
for $t = -2$: $\cos\theta = |\mathbf{n}\cdot\mathbf{r}'|/(\|\mathbf{n}\|\ \|\mathbf{r}'\|) = 3/\sqrt{156}$, $\theta \approx 76^\circ$;
for $t = 1$: $\cos\theta = |\mathbf{n}\cdot\mathbf{r}'|/(\|\mathbf{n}\|\ \|\mathbf{r}'\|) = 3/\sqrt{84}$, $\theta \approx 71^\circ$.

54. $\mathbf{r}' = -2e^{-2t}\mathbf{i} - \sin t\mathbf{j} + 3\cos t\mathbf{k}$, $t = 0$ at the point $(1, 1, 0)$ so $\mathbf{r}'(0) = -2\mathbf{i} + 3\mathbf{k}$ and hence the tangent line is $x = 1 - 2t,\ y = 1,\ z = 3t$. But $x = 0$ in the yz-plane so $1 - 2t = 0,\ t = 1/2$. The point of intersection is $(0, 1, 3/2)$.

55. $\mathbf{r}_1(1) = \mathbf{r}_2(2) = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$ so the graphs intersect at P; $\mathbf{r}_1'(t) = 2t\mathbf{i} + \mathbf{j} + 9t^2\mathbf{k}$ and $\mathbf{r}_2'(t) = \mathbf{i} + \dfrac{1}{2}t\mathbf{j} - \mathbf{k}$ so $\mathbf{r}_1'(1) = 2\mathbf{i} + \mathbf{j} + 9\mathbf{k}$ and $\mathbf{r}_2'(2) = \mathbf{i} + \mathbf{j} - \mathbf{k}$ are tangent to the graphs at P, thus $\cos\theta = \dfrac{\mathbf{r}_1'(1)\cdot\mathbf{r}_2'(2)}{\|\mathbf{r}_1'(1)\|\ \|\mathbf{r}_2'(2)\|} = -\dfrac{6}{\sqrt{86}\sqrt{3}}$, $\theta = \cos^{-1}(6/\sqrt{258}) \approx 68^\circ$.

56. $\mathbf{r}_1(0) = \mathbf{r}_2(-1) = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ so the graphs intersect at P; $\mathbf{r}_1'(t) = -2e^{-t}\mathbf{i} - (\sin t)\mathbf{j} + 2t\mathbf{k}$ and $\mathbf{r}_2'(t) = -\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$ so $\mathbf{r}_1'(0) = -2\mathbf{i}$ and $\mathbf{r}_2'(-1) = -\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ are tangent to the graphs at P, thus $\cos\theta = \dfrac{\mathbf{r}_1'(0)\cdot\mathbf{r}_2'(-1)}{\|\mathbf{r}_1'(0)\|\ \|\mathbf{r}_2'(-1)\|} = \dfrac{1}{\sqrt{14}}$, $\theta \approx 74^\circ$.

57. $\dfrac{d}{dt}[\mathbf{r}(t)\times\mathbf{r}'(t)] = \mathbf{r}(t)\times\mathbf{r}''(t) + \mathbf{r}'(t)\times\mathbf{r}'(t) = \mathbf{r}(t)\times\mathbf{r}''(t) + \mathbf{0} = \mathbf{r}(t)\times\mathbf{r}''(t)$

58.
$$\frac{d}{dt}[\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})] = \mathbf{u}\cdot\frac{d}{dt}[\mathbf{v}\times\mathbf{w}] + \frac{d\mathbf{u}}{dt}\cdot[\mathbf{v}\times\mathbf{w}] = \mathbf{u}\cdot\left(\mathbf{v}\times\frac{d\mathbf{w}}{dt} + \frac{d\mathbf{v}}{dt}\times\mathbf{w}\right) + \frac{d\mathbf{u}}{dt}\cdot[\mathbf{v}\times\mathbf{w}]$$
$$= \mathbf{u}\cdot\left[\mathbf{v}\times\frac{d\mathbf{w}}{dt}\right] + \mathbf{u}\cdot\left[\frac{d\mathbf{v}}{dt}\times\mathbf{w}\right] + \frac{d\mathbf{u}}{dt}\cdot[\mathbf{v}\times\mathbf{w}]$$

59. In Exercise 58, write each scalar triple product as a determinant.

60. Let $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j}$, $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, $\mathbf{r}_1(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j}$, $\mathbf{r}_2(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j}$ and use properties of derivatives.

61. Let $\mathbf{r}_1(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$ and $\mathbf{r}_2(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$, in both (6) and (7); show that the left and right members of the equalities are the same.

62. **(a)** $$\int k\mathbf{r}(t)\,dt = \int k(x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k})\,dt$$
$$= k\int x(t)\,dt\,\mathbf{i} + k\int y(t)\,dt\,\mathbf{j} + k\int z(t)\,dt\,\mathbf{k} = k\int \mathbf{r}(t)\,dt$$

(b) Similar to Part (a) **(c)** Use Part (a) on Part (b) with $k = -1$

EXERCISE SET 13.3

1. **(a)** The tangent vector reverses direction at the four cusps.

(b) $\mathbf{r}'(t) = -3\cos^2 t\sin t\mathbf{i} + 3\sin^2 t\cos t\mathbf{j} = \mathbf{0}$ when $t = 0, \pi/2, \pi, 3\pi/2, 2\pi$.

2. $\mathbf{r}'(t) = \cos t\mathbf{i} + 2\sin t\cos t\mathbf{j} = \mathbf{0}$ when $t = \pi/2, 3\pi/2$. The tangent vector reverses direction at $(1, 1)$ and $(-1, 1)$.

3. $\mathbf{r}'(t) = 3t^2\mathbf{i} + (6t - 2)\mathbf{j} + 2t\mathbf{k}$; smooth

4. $\mathbf{r}'(t) = -2t\sin(t^2)\mathbf{i} + 2t\cos(t^2)\mathbf{j} - e^{-t}\mathbf{k}$; smooth

5. $\mathbf{r}'(t) = (1 - t)e^{-t}\mathbf{i} + (2t - 2)\mathbf{j} - \pi\sin(\pi t)\mathbf{k}$; not smooth, $\mathbf{r}'(1) = \mathbf{0}$

6. $\mathbf{r}'(t) = \pi\cos(\pi t)\mathbf{i} + (2 - 1/t)\mathbf{j} + (2t - 1)\mathbf{k}$; not smooth, $\mathbf{r}'(1/2) = \mathbf{0}$

7. $(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2 = (-3\cos^2 t\sin t)^2 + (3\sin^2 t\cos t)^2 + 0^2 = 9\sin^2 t\cos^2 t$,
$$L = \int_0^{\pi/2} 3\sin t\cos t\,dt = 3/2$$

8. $(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2 = (-3\sin t)^2 + (3\cos t)^2 + 16 = 25$, $L = \int_0^{\pi} 5dt = 5\pi$

9. $\mathbf{r}'(t) = \langle e^t, -e^{-t}, \sqrt{2}\rangle$, $\|\mathbf{r}'(t)\| = e^t + e^{-t}$, $L = \int_0^1 (e^t + e^{-t})dt = e - e^{-1}$

10. $(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2 = 1/4 + (1 - t)/4 + (1 + t)/4 = 3/4$, $L = \int_{-1}^1 (\sqrt{3}/2)dt = \sqrt{3}$

11. $\mathbf{r}'(t) = 3t^2\mathbf{i} + \mathbf{j} + \sqrt{6}\,t\mathbf{k}$, $\|\mathbf{r}'(t)\| = 3t^2 + 1$, $L = \int_1^3 (3t^2 + 1)dt = 28$

12. $\mathbf{r}'(t) = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\|\mathbf{r}'(t)\| = \sqrt{14}$, $L = \int_3^4 \sqrt{14}\,dt = \sqrt{14}$

13. $\mathbf{r}'(t) = -3\sin t\mathbf{i} + 3\cos t\mathbf{j} + \mathbf{k}$, $\|\mathbf{r}'(t)\| = \sqrt{10}$, $L = \int_0^{2\pi} \sqrt{10}\,dt = 2\pi\sqrt{10}$

14. $\mathbf{r}'(t)=2t\mathbf{i}+t\cos t\mathbf{j}+t\sin t\mathbf{k}$, $\|\mathbf{r}'(t)\|=\sqrt{5}t$, $L=\int_0^{\pi}\sqrt{5}t\,dt=\pi^2\sqrt{5}/2$

15. $(d\mathbf{r}/dt)(dt/d\tau)=(\mathbf{i}+2t\mathbf{j})(4)=4\mathbf{i}+8t\mathbf{j}=4\mathbf{i}+8(4\tau+1)\mathbf{j}$;
$\mathbf{r}(\tau)=(4\tau+1)\mathbf{i}+(4\tau+1)^2\,\mathbf{j}$, $\mathbf{r}'(\tau)=4\mathbf{i}+2(4)(4\tau+1)\,\mathbf{j}$

16. $(d\mathbf{r}/dt)(dt/d\tau)=\langle -3\sin t, 3\cos t\rangle(\pi)=\langle -3\pi\sin\pi\tau,\ 3\pi\cos\pi\tau\rangle$;
$\mathbf{r}(\tau)=\langle 3\cos\pi\tau, 3\sin\pi\tau\rangle$, $\mathbf{r}'(\tau)=\langle -3\pi\sin\pi\tau, 3\pi\cos\pi\tau\rangle$

17. $(d\mathbf{r}/dt)(dt/d\tau)=(e^t\mathbf{i}-4e^{-t}\mathbf{j})(2\tau)=2\tau e^{\tau^2}\mathbf{i}-8\tau e^{-\tau^2}\mathbf{j}$;
$\mathbf{r}(\tau)=e^{\tau^2}\mathbf{i}+4e^{-\tau^2}\mathbf{j}$, $\mathbf{r}'(\tau)=2\tau e^{\tau^2}\mathbf{i}-4(2)\tau e^{-\tau^2}\mathbf{j}$

18. $(d\mathbf{r}/dt)(dt/d\tau)=\left(\frac{9}{2}t^{1/2}\mathbf{j}+\mathbf{k}\right)(-1/\tau^2)=-\frac{9}{2\tau^{5/2}}\mathbf{j}-\frac{1}{\tau^2}\mathbf{k}$;

$\mathbf{r}(\tau)=\mathbf{i}+3\tau^{-3/2}\mathbf{j}+\frac{1}{\tau}\mathbf{k}$, $\mathbf{r}'(\tau)=-\frac{9}{2}\tau^{-5/2}\mathbf{j}-\frac{1}{\tau^2}\mathbf{k}$

19. **(a)** $\|\mathbf{r}'(t)\|=\sqrt{2}$, $s=\int_0^t\sqrt{2}\,dt=\sqrt{2}t$; $\mathbf{r}=\frac{s}{\sqrt{2}}\mathbf{i}+\frac{s}{\sqrt{2}}\mathbf{j}$, $x=\frac{s}{\sqrt{2}}$, $y=\frac{s}{\sqrt{2}}$

(b) Similar to Part (a), $x=y=z=\frac{s}{\sqrt{3}}$

20. **(a)** $x=-\frac{s}{\sqrt{2}}$, $y=-\frac{s}{\sqrt{2}}$ **(b)** $x=-\frac{s}{\sqrt{3}}$, $y=-\frac{s}{\sqrt{3}}$, $z=-\frac{s}{\sqrt{3}}$

21. **(a)** $\mathbf{r}(t)=\langle 1,3,4\rangle$ when $t=0$,
so $s=\int_0^t\sqrt{1+4+4}\,du=3t$, $x=1+s/3$, $y=3-2s/3$, $z=4+2s/3$

(b) $\mathbf{r}\Big]_{s=25}=\langle 28/3,-41/3,62/3\rangle$

22. **(a)** $\mathbf{r}(t)=\langle -5,0,1\rangle$ when $t=0$, so $s=\int_0^t\sqrt{9+4+1}\,du=\sqrt{14}t$,
$x=-5+3s/\sqrt{14}$, $y=2s/\sqrt{14}$, $z=5+s/\sqrt{14}$

(b) $\mathbf{r}(s)\Big]_{s=10}=\langle -5+30/\sqrt{14}, 20/\sqrt{14}, 5+10/\sqrt{14}\rangle$

23. $x=3+\cos t$, $y=2+\sin t$, $(dx/dt)^2+(dy/dt)^2=1$,
$s=\int_0^t du=t$ so $t=s$, $x=3+\cos s$, $y=2+\sin s$ for $0\le s\le 2\pi$.

24. $x=\cos^3 t$, $y=\sin^3 t$, $(dx/dt)^2+(dy/dt)^2=9\sin^2 t\cos^2 t$,
$s=\int_0^t 3\sin u\cos u\,du=\frac{3}{2}\sin^2 t$ so $\sin t=(2s/3)^{1/2}$, $\cos t=(1-2s/3)^{1/2}$,
$x=(1-2s/3)^{3/2}$, $y=(2s/3)^{3/2}$ for $0\le s\le 3/2$

25. $x = t^3/3$, $y = t^2/2$, $(dx/dt)^2 + (dy/dt)^2 = t^2(t^2 + 1)$,

$$s = \int_0^t u(u^2+1)^{1/2}du = \frac{1}{3}[(t^2+1)^{3/2} - 1] \text{ so } t = [(3s+1)^{2/3} - 1]^{1/2},$$

$$x = \frac{1}{3}[(3s+1)^{2/3} - 1]^{3/2},\ y = \frac{1}{2}[(3s+1)^{2/3} - 1] \text{ for } s \ge 0$$

26. $x = (1+t)^2$, $y = (1+t)^3$, $(dx/dt)^2 + (dy/dt)^2 = (1+t)^2[4 + 9(1+t)^2]$,

$$s = \int_0^t (1+u)[4+9(1+u)^2]^{1/2}du = \frac{1}{27}([4+9(1+t)^2]^{3/2} - 13\sqrt{13}) \text{ so}$$

$$1 + t = \frac{1}{3}[(27s + 13\sqrt{13})^{2/3} - 4]^{1/2},\ x = \frac{1}{9}[(27s + 13\sqrt{13})^{2/3} - 4],$$

$$y = \frac{1}{27}[(27s + 13\sqrt{13})^{2/3} - 4]^{3/2} \text{ for } 0 \le s \le (80\sqrt{10} - 13\sqrt{13})/27$$

27. $x = e^t \cos t$, $y = e^t \sin t$, $(dx/dt)^2 + (dy/dt)^2 = 2e^{2t}$, $s = \displaystyle\int_0^t \sqrt{2}\, e^u\, du = \sqrt{2}(e^t - 1)$ so

$t = \ln(s/\sqrt{2} + 1)$, $x = (s/\sqrt{2} + 1)\cos[\ln(s/\sqrt{2} + 1)]$, $y = (s/\sqrt{2} + 1)\sin[\ln(s/\sqrt{2} + 1)]$
for $0 \le s \le \sqrt{2}(e^{\pi/2} - 1)$

28. $x = \sin(e^t)$, $y = \cos(e^t)$, $z = \sqrt{3}e^t$,

$$(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2 = 4e^{2t},\ s = \int_0^t 2e^u\, du = 2(e^t - 1) \text{ so}$$

$e^t = 1 + s/2$; $x = \sin(1 + s/2)$, $y = \cos(1 + s/2)$, $z = \sqrt{3}(1 + s/2)$ for $s \ge 0$

29. $dx/dt = -a\sin t$, $dy/dt = a\cos t$, $dz/dt = c$,

$$s(t_0) = L = \int_0^{t_0} \sqrt{a^2\sin^2 t + a^2\cos^2 t + c^2}\, dt = \int_0^{t_0} \sqrt{a^2 + c^2}\, dt = t_0\sqrt{a^2 + c^2}$$

30. From Exercise 29, $s(t_0) = t_0\sqrt{a^2 + c^2} = wt_0$, so $s(t) = wt$ and

$$\mathbf{r} = a\cos\frac{s}{w}\mathbf{i} + \sin\frac{s}{w}\mathbf{j} + \frac{bs}{w}\mathbf{k}.$$

31. $x = at - a\sin t$, $y = a - a\cos t$, $(dx/dt)^2 + (dy/dt)^2 = 4a^2\sin^2(t/2)$,

$$s = \int_0^t 2a\sin(u/2)du = 4a[1 - \cos(t/2)] \text{ so } \cos(t/2) = 1 - s/(4a),\ t = 2\cos^{-1}[1 - s/(4a)],$$

$\cos t = 2\cos^2(t/2) - 1 = 2[1 - s/(4a)]^2 - 1$,

$\sin t = 2\sin(t/2)\cos(t/2) = 2(1 - [1 - s/(4a)]^2)^{1/2}(2[1 - s/(4a)]^2 - 1)$,

$x = 2a\cos^{-1}[1 - s/(4a)] - 2a(1 - [1 - s/(4a)]^2)^{1/2}(2[1 - s/(4a)]^2 - 1)$,

$$y = \frac{s(8a - s)}{8a} \text{ for } 0 \le s \le 8a$$

32. $\dfrac{dx}{dt} = \cos\theta\dfrac{dr}{dt} - r\sin\theta\dfrac{d\theta}{dt}$, $\dfrac{dy}{dt} = \sin\theta\dfrac{dr}{dt} + r\cos\theta\dfrac{d\theta}{dt}$,

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$$

33. **(a)** $(dr/dt)^2 + r^2(d\theta/dt)^2 + (dz/dt)^2 = 9e^{4t}$, $L = \int_0^{\ln 2} 3e^{2t}dt = \frac{3}{2}e^{2t}\Big]_0^{\ln 2} = 9/2$

(b) $(dr/dt)^2 + r^2(d\theta/dt)^2 + (dz/dt)^2 = 5t^2 + t^4 = t^2(5 + t^2)$,

$L = \int_1^2 t(5 + t^2)^{1/2}dt = 9 - 2\sqrt{6}$

34. $\frac{dx}{dt} = \sin\phi\cos\theta\frac{d\rho}{dt} + \rho\cos\phi\cos\theta\frac{d\phi}{dt} - \rho\sin\phi\sin\theta\frac{d\theta}{dt}$,

$\frac{dy}{dt} = \sin\phi\sin\theta\frac{d\rho}{dt} + \rho\cos\phi\sin\theta\frac{d\phi}{dt} + \rho\sin\phi\cos\theta\frac{d\theta}{dt}$, $\frac{dz}{dt} = \cos\phi\frac{d\rho}{dt} - \rho\sin\phi\frac{d\phi}{dt}$,

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = \left(\frac{d\rho}{dt}\right)^2 + \rho^2\sin^2\phi\left(\frac{d\theta}{dt}\right)^2 + \rho^2\left(\frac{d\phi}{dt}\right)^2$$

35. **(a)** $(d\rho/dt)^2 + \rho^2\sin^2\phi(d\theta/dt)^2 + \rho^2(d\phi/dt)^2 = 3e^{-2t}$, $L = \int_0^2 \sqrt{3}e^{-t}dt = \sqrt{3}(1 - e^{-2})$

(b) $(d\rho/dt)^2 + \rho^2\sin^2\phi(d\theta/dt)^2 + \rho^2(d\phi/dt)^2 = 5$, $L = \int_1^5 \sqrt{5}dt = 4\sqrt{5}$

36. **(a)** $\frac{d}{dt}\mathbf{r}(t) = \mathbf{i} + 2t\mathbf{j}$ is never zero, but $\frac{d}{d\tau}\mathbf{r}(\tau^3) = \frac{d}{d\tau}(\tau^3\mathbf{i} + \tau^6\mathbf{j}) = 3\tau^2\mathbf{i} + 6\tau^5\mathbf{j}$ is zero at $\tau = 0$.

(b) $\frac{d\mathbf{r}}{d\tau} = \frac{d\mathbf{r}}{dt}\frac{dt}{d\tau}$, and since $t = \tau^3$, $\frac{dt}{d\tau} = 0$ when $\tau = 0$.

37. **(a)** $g(\tau) = \pi\tau$ **(b)** $g(\tau) = \pi(1 - \tau)$

38. $t = 1 - \tau$

39. Represent the helix by $x = a\cos t$, $y = a\sin t$, $z = ct$ with $a = 6.25$ and $c = 10/\pi$, so that the radius of the helix is the distance from the axis of the cylinder to the center of the copper cable, and the helix makes one turn in a distance of 20 in. ($t = 2\pi$). From Exercise 29 the length of the helix is $2\pi\sqrt{6.25^2 + (10/\pi)^2} \approx 44$ in.

40. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t^{3/2}\mathbf{k}$, $\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + \frac{3}{2}t^{1/2}\mathbf{k}$

(a) $\|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 9t/4} = \frac{1}{2}\sqrt{4 + 9t}$

(b) $\frac{ds}{dt} = \frac{1}{2}\sqrt{4 + 9t}$

(c) $\int_0^2 \frac{1}{2}\sqrt{4 + 9t}\,dt = \frac{2}{27}(11\sqrt{22} - 4)$

41. $\mathbf{r}'(t) = (1/t)\mathbf{i} + 2\mathbf{j} + 2t\mathbf{k}$

(a) $\|\mathbf{r}'(t)\| = \sqrt{1/t^2 + 4 + 4t^2} = \sqrt{(2t + 1/t)^2} = 2t + 1/t$

(b) $\frac{ds}{dt} = 2t + 1/t$

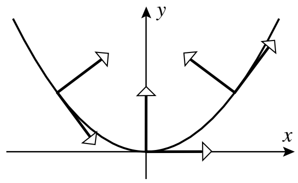
(c) $\int_1^3 (2t + 1/t)dt = 8 + \ln 3$

42. If $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is smooth, then $\|\mathbf{r}'(t)\|$ is continuous and nonzero. Thus the angle between $\mathbf{r}'(t)$ and $\mathbf{i}$, given by $\cos^{-1}(x'(t)/\|\mathbf{r}'(t)\|)$, is a continuous function of t. Similarly, the angles between $\mathbf{r}'(t)$ and the vectors $\mathbf{j}$ and $\mathbf{k}$ are continuous functions of t.

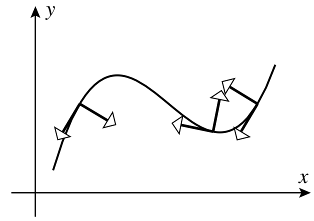
43. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ and use the chain rule.

EXERCISE SET 13.4

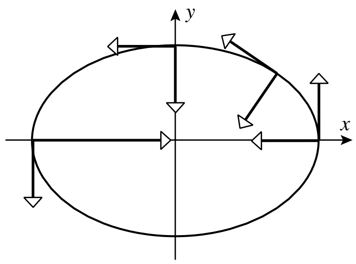
1. **(a)**



(b)



2.



3. $\mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$, $\|\mathbf{r}'(t)\| = \sqrt{4t^2+1}$, $\mathbf{T}(t) = (4t^2+1)^{-1/2}(2t\mathbf{i}+\mathbf{j})$,
$\mathbf{T}'(t) = (4t^2+1)^{-1/2}(2\mathbf{i}) - 4t(4t^2+1)^{-3/2}(2t\mathbf{i}+\mathbf{j})$;
$\mathbf{T}(1) = \dfrac{2}{\sqrt{5}}\mathbf{i} + \dfrac{1}{\sqrt{5}}\mathbf{j}$, $\mathbf{T}'(1) = \dfrac{2}{5\sqrt{5}}(\mathbf{i}-2\mathbf{j})$, $\mathbf{N}(1) = \dfrac{1}{\sqrt{5}}\mathbf{i} - \dfrac{2}{\sqrt{5}}\mathbf{j}$.

4. $\mathbf{r}'(t) = t\mathbf{i} + t^2\mathbf{j}$, $\mathbf{T}(t) = (t^2+t^4)^{-1/2}(t\mathbf{i}+t^2\mathbf{j})$,
$\mathbf{T}'(t) = (t^2+t^4)^{-1/2}(\mathbf{i}+2t\mathbf{j}) - (t+2t^3)(t^2+t^4)^{-3/2}(t\mathbf{i}+t^2\mathbf{j})$;
$\mathbf{T}(1) = \dfrac{1}{\sqrt{2}}\mathbf{i} + \dfrac{1}{\sqrt{2}}\mathbf{j}$, $\mathbf{T}'(1) = \dfrac{1}{2\sqrt{2}}(-\mathbf{i}+\mathbf{j})$, $\mathbf{N}(1) = -\dfrac{1}{\sqrt{2}}\mathbf{i} + \dfrac{1}{\sqrt{2}}\mathbf{j}$

5. $\mathbf{r}'(t) = -5\sin t\mathbf{i} + 5\cos t\mathbf{j}$, $\|\mathbf{r}'(t)\| = 5$, $\mathbf{T}(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$, $\mathbf{T}'(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}$;
$\mathbf{T}(\pi/3) = -\dfrac{\sqrt{3}}{2}\mathbf{i} + \dfrac{1}{2}\mathbf{j}$, $\mathbf{T}'(\pi/3) = -\dfrac{1}{2}\mathbf{i} - \dfrac{\sqrt{3}}{2}\mathbf{j}$, $\mathbf{N}(\pi/3) = -\dfrac{1}{2}\mathbf{i} - \dfrac{\sqrt{3}}{2}\mathbf{j}$

6. $\mathbf{r}'(t) = \dfrac{1}{t}\mathbf{i} + \mathbf{j}$, $\|\mathbf{r}'(t)\| = \dfrac{\sqrt{1+t^2}}{t}$, $\mathbf{T}(t) = (1+t^2)^{-1/2}(\mathbf{i}+t\mathbf{j})$,
$\mathbf{T}'(t) = (1+t^2)^{-1/2}(\mathbf{j}) - t(1+t^2)^{-3/2}(\mathbf{i}+t\mathbf{j})$; $\mathbf{T}(e) = \dfrac{1}{\sqrt{1+e^2}}\mathbf{i} + \dfrac{e}{\sqrt{1+e^2}}\mathbf{j}$,
$\mathbf{T}'(e) = \dfrac{1}{(1+e^2)^{3/2}}(-e\mathbf{i}+\mathbf{j})$, $\mathbf{N}(e) = -\dfrac{e}{\sqrt{1+e^2}}\mathbf{i} + \dfrac{1}{\sqrt{1+e^2}}\mathbf{j}$

7. $\mathbf{r}'(t) = -4\sin t\mathbf{i} + 4\cos t\mathbf{j} + \mathbf{k}$, $\mathbf{T}(t) = \dfrac{1}{\sqrt{17}}(-4\sin t\mathbf{i} + 4\cos t\mathbf{j} + \mathbf{k})$,
$\mathbf{T}'(t) = \dfrac{1}{\sqrt{17}}(-4\cos t\mathbf{i} - 4\sin t\mathbf{j})$, $\mathbf{T}(\pi/2) = -\dfrac{4}{\sqrt{17}}\mathbf{i} + \dfrac{1}{\sqrt{17}}\mathbf{k}$
$\mathbf{T}'(\pi/2) = -\dfrac{4}{\sqrt{17}}\mathbf{j}$, $\mathbf{N}(\pi/2) = -\mathbf{j}$

8. $\mathbf{r}'(t) = \mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$, $\mathbf{T}(t) = (1+t^2+t^4)^{-1/2}(\mathbf{i}+t\mathbf{j}+t^2\mathbf{k})$,
$\mathbf{T}'(t) = (1+t^2+t^4)^{-1/2}(\mathbf{j}+2t\mathbf{k}) - (t+2t^3)(1+t^2+t^4)^{-3/2}(\mathbf{i}+t\mathbf{j}+t^2\mathbf{k})$,
$\mathbf{T}(0) = \mathbf{i}$, $\mathbf{T}'(0) = \mathbf{j} = \mathbf{N}(0)$

9. $\mathbf{r}'(t) = e^t[(\cos t - \sin t)\mathbf{i} + (\cos t + \sin t)\mathbf{j} + \mathbf{k}]$, $\mathbf{T}(t) = \dfrac{1}{\sqrt{3}}[(\cos t - \sin t)\mathbf{i} + (\cos t + \sin t)\mathbf{j} + \mathbf{k}]$,

$\mathbf{T}'(t) = \dfrac{1}{\sqrt{3}}[(-\sin t - \cos t)\mathbf{i} + (-\sin t + \cos t)\mathbf{j}]$,

$\mathbf{T}(0) = \dfrac{1}{\sqrt{3}}\mathbf{i} + \dfrac{1}{\sqrt{3}}\mathbf{j} + \dfrac{1}{\sqrt{3}}\mathbf{k}$, $\mathbf{T}'(0) = \dfrac{1}{\sqrt{3}}(-\mathbf{i} + \mathbf{j})$, $\mathbf{N}(0) = -\dfrac{1}{\sqrt{2}}\mathbf{i} + \dfrac{1}{\sqrt{2}}\mathbf{j}$

10. $\mathbf{r}'(t) = \sinh t\mathbf{i} + \cosh t\mathbf{j} + \mathbf{k}$, $\|\mathbf{r}'(t)\| = \sqrt{\sinh^2 t + \cosh^2 t + 1} = \sqrt{2}\cosh t$,

$\mathbf{T}(t) = \dfrac{1}{\sqrt{2}}(\tanh t\mathbf{i} + \mathbf{j} + \operatorname{sech} t\mathbf{k})$, $\mathbf{T}'(t) = \dfrac{1}{\sqrt{2}}(\operatorname{sech}^2 t\mathbf{i} - \operatorname{sech} t \tanh t\mathbf{k})$, at $t = \ln 2$,

$\tanh(\ln 2) = \dfrac{3}{5}$ and $\operatorname{sech}(\ln 2) = \dfrac{4}{5}$ so $\mathbf{T}(\ln 2) = \dfrac{3}{5\sqrt{2}}\mathbf{i} + \dfrac{1}{\sqrt{2}}\mathbf{j} + \dfrac{4}{5\sqrt{2}}\mathbf{k}$,

$\mathbf{T}'(\ln 2) = \dfrac{4}{25\sqrt{2}}(4\mathbf{i} - 3\mathbf{k})$, $\mathbf{N}(\ln 2) = \dfrac{4}{5}\mathbf{i} - \dfrac{3}{5}\mathbf{k}$

11. From the remark, the line is parametrized by normalizing $\mathbf{v}$, but $\mathbf{T}(t_0) = \mathbf{v}/\|\mathbf{v}\|$, so $\mathbf{r} = \mathbf{r}(t_0) + t\mathbf{v}$ becomes $\mathbf{r} = \mathbf{r}(t_0) + s\mathbf{T}(t_0)$.

12. $\mathbf{r}'(t)\Big]_{t=1} = \langle 1, 2t\rangle\Big]_{t=1} = \langle 1, 2\rangle$, and $\mathbf{T}(1) = \langle \dfrac{1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}}\rangle$, so the tangent line can be parametrized as

$\mathbf{r} = \langle 1, 1\rangle + s\left\langle \dfrac{1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}}\right\rangle$, so $x = 1 + \dfrac{s}{\sqrt{5}}, y = 1 + \dfrac{2s}{\sqrt{5}}$.

13. $\mathbf{r}'(t) = \cos t\mathbf{i} - \sin t\mathbf{j} + t\mathbf{k}$, $\mathbf{r}'(0) = \mathbf{i}$, $\mathbf{r}(0) = \mathbf{j}$, $\mathbf{T}(0) = \mathbf{i}$, so the tangent line has the parametrization $x = s, y = 1$.

14. $\mathbf{r}(1) = \mathbf{i} + \mathbf{j} + \sqrt{8}\mathbf{k}$, $\mathbf{r}'(t) = \mathbf{i} + \mathbf{j} - \dfrac{t}{\sqrt{9 - t^2}}\mathbf{k}$, $\mathbf{r}'(1) = \mathbf{i} + \mathbf{j} - \dfrac{1}{\sqrt{8}}\mathbf{k}$, $\|\mathbf{r}'(1)\| = \dfrac{\sqrt{17}}{\sqrt{8}}$, so the tangent line has parametrizations $\mathbf{r} = \mathbf{i} + \mathbf{j} + \sqrt{8}\mathbf{k} + t\left(\mathbf{i} + \mathbf{j} - \dfrac{1}{\sqrt{8}}\mathbf{k}\right) = \mathbf{i} + \mathbf{j} + \sqrt{8}\mathbf{k} + \dfrac{s\sqrt{8}}{\sqrt{17}}\left(\mathbf{i} + \mathbf{j} - \dfrac{1}{\sqrt{8}}\mathbf{k}\right)$.

15. $\mathbf{T} = \dfrac{3}{5}\cos t\,\mathbf{i} - \dfrac{3}{5}\sin t\,\mathbf{j} + \dfrac{4}{5}\,\mathbf{k}$, $\mathbf{N} = -\sin t\,\mathbf{i} - \cos t\,\mathbf{j}$, $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \dfrac{4}{5}\cos t\,\mathbf{i} - \dfrac{4}{5}\sin t\,\mathbf{j} - \dfrac{3}{5}\,\mathbf{k}$. Check:
$\mathbf{r}' = 3\cos t\,\mathbf{i} - 3\sin t\,\mathbf{j} + 4\,\mathbf{k}$, $\mathbf{r}'' = -3\sin t\,\mathbf{i} - 3\cos t\,\mathbf{j}$, $\mathbf{r}' \times \mathbf{r}'' = 12\cos t\,\mathbf{i} - 12\sin t\,\mathbf{j} - 9\,\mathbf{k}$,
$\|\mathbf{r}' \times \mathbf{r}''\| = 15$, $(\mathbf{r}' \times \mathbf{r}'')/\|\mathbf{r}' \times \mathbf{r}''\| = \frac{4}{5}\cos t\,\mathbf{i} - \frac{4}{5}\sin t\,\mathbf{j} - \frac{3}{5}\,\mathbf{k} = \mathbf{B}$.

16. $\mathbf{T}'(t) = \dfrac{1}{\sqrt{2}}[(\cos t + \sin t)\,\mathbf{i} + (-\sin t + \cos t)\,\mathbf{j}], \mathbf{N} = \dfrac{1}{\sqrt{2}}[(-\sin t + \cos t)\,\mathbf{i} - (\cos t + \sin t)\,\mathbf{j}]$,
$\mathbf{B} = \mathbf{T} \times \mathbf{N} = -\mathbf{k}$. Check: $\mathbf{r}' = e^t(\cos t + \sin t)\,\mathbf{i} + e^t(\cos t - \sin t)\,\mathbf{j}$,
$\mathbf{r}'' = 2e^t\cos t\,\mathbf{i} - 2e^t\sin t\,\mathbf{j}$, $\mathbf{r}' \times \mathbf{r}'' = -2e^{2t}\,\mathbf{k}$, $\|\mathbf{r}' \times \mathbf{r}''\| = 2e^{2t}$, $(\mathbf{r}' \times \mathbf{r}'')/\|\mathbf{r}' \times \mathbf{r}''\| = -\mathbf{k} = \mathbf{B}$.

17. $\mathbf{r}'(t) = t\sin t\mathbf{i} + t\cos t\mathbf{j}, \|\mathbf{r}'\| = t, \mathbf{T} = \sin t\mathbf{i} + \cos t\mathbf{j}, \mathbf{N} = \cos t\mathbf{i} - \sin t\mathbf{j}, \mathbf{B} = \mathbf{T} \times \mathbf{N} = -\mathbf{k}$. Check:
$\mathbf{r}' = t\sin t\,\mathbf{i} + t\cos t\,\mathbf{j}$, $\mathbf{r}'' = (\sin t + t\cos t)\,\mathbf{i} + (\cos t - t\sin t)\,\mathbf{j}$, $\mathbf{r}' \times \mathbf{r}'' = -2e^{2t}\,\mathbf{k}$,
$\|\mathbf{r}' \times \mathbf{r}''\| = 2e^{2t}$, $(\mathbf{r}' \times \mathbf{r}'')/\|\mathbf{r}' \times \mathbf{r}''\| = -\mathbf{k} = \mathbf{B}$.

18. $\mathbf{T} = (-a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j} + c\,\mathbf{k})/\sqrt{a^2 + c^2}$, $\mathbf{N} = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$,
$\mathbf{B} = \mathbf{T} \times \mathbf{N} = (c\sin t\,\mathbf{i} - c\cos t\,\mathbf{j} + a\,\mathbf{k})/\sqrt{a^2 + c^2}$. Check:
$\mathbf{r}' = -a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j} + c\,\mathbf{k}$, $\mathbf{r}'' = -a\cos t\,\mathbf{i} - a\sin t\,\mathbf{j}$, $\mathbf{r}' \times \mathbf{r}'' = ca\sin t\,\mathbf{i} - ca\cos t\,\mathbf{j} + a^2\,\mathbf{k}$,
$\|\mathbf{r}' \times \mathbf{r}''\| = a\sqrt{a^2 + c^2}$, $(\mathbf{r}' \times \mathbf{r}'')/\|\mathbf{r}' \times \mathbf{r}''\| = \mathbf{B}$.

19. $\mathbf{r}(\pi/4)=\frac{\sqrt{2}}{2}\mathbf{i}+\frac{\sqrt{2}}{2}\mathbf{j}+\mathbf{k}, \mathbf{T}=-\sin t\mathbf{i}+\cos t\mathbf{j}=\frac{\sqrt{2}}{2}(-\mathbf{i}+\mathbf{j}), \mathbf{N}=-(\cos t\mathbf{i}+\sin t\mathbf{j})=-\frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j})$, $\mathbf{B}=\mathbf{k}$; the rectifying, osculating, and normal planes are given (respectively) by $x+y=\sqrt{2}$, $z=1, -x+y=0$.

20. $\mathbf{r}(0)=\mathbf{i}+\mathbf{j}, \mathbf{T}=\frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k}), \mathbf{N}=\frac{1}{\sqrt{2}}(-\mathbf{j}+\mathbf{k}), \mathbf{B}=\frac{1}{\sqrt{6}}(2\mathbf{i}-\mathbf{j}-\mathbf{k})$; the rectifying, osculating, and normal planes are given (respectively) by $-y+z=-1, 2x-y-z=1, x+y+z=2$.

21. **(a)** By formulae (1) and (11), $\mathbf{N}(t)=\mathbf{B}(t)\times\mathbf{T}(t)=\dfrac{\mathbf{r}'(t)\times\mathbf{r}''(t)}{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}\times\dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$.

(b) Since $\mathbf{r}'$ is perpendicular to $\mathbf{r}'\times\mathbf{r}''$ it follows from Lagrange's Identity (Exercise 32 of Section 12.4) that $\|(\mathbf{r}'(t)\times\mathbf{r}''(t))\times\mathbf{r}'(t)\|=\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|\|\mathbf{r}'(t)\|$, and the result follows.

(c) From Exercise 39 of Section 12.4,
$(\mathbf{r}'(t)\times\mathbf{r}''(t))\times\mathbf{r}'(t)=\|\mathbf{r}'(t)\|^2\mathbf{r}''(t)-(\mathbf{r}'(t)\cdot\mathbf{r}''(t))\mathbf{r}'(t)=\mathbf{u}(t)$, so $\mathbf{N}(t)=\mathbf{u}(t)/\|\mathbf{u}(t)\|$

22. **(a)** $\mathbf{r}'(t)=2t\mathbf{i}+\mathbf{j}, \mathbf{r}'(1)=2\mathbf{i}+\mathbf{j}, \mathbf{r}''(t)=2\mathbf{i}, \mathbf{u}=2\mathbf{i}-4\mathbf{j}, \mathbf{N}=\frac{1}{\sqrt{5}}\mathbf{i}-\frac{2}{\sqrt{5}}\mathbf{j}$

(b) $\mathbf{r}'(t)=-4\sin t\mathbf{i}+4\cos t\mathbf{j}+\mathbf{k},\ \mathbf{r}'(\frac{\pi}{2})=-4\mathbf{i}+\mathbf{k},\ \mathbf{r}''(t)=-4\cos t\mathbf{i}-4\sin t\mathbf{j}$,
$\mathbf{r}''(\frac{\pi}{2})=-4\mathbf{j}, \mathbf{u}=17(-4\mathbf{j}), \mathbf{N}=-\mathbf{j}$

23. $\mathbf{r}'(t)=\cos t\mathbf{i}-\sin t\mathbf{j}+\mathbf{k}, \mathbf{r}''(t)=-\sin t\mathbf{i}-\cos t\mathbf{j}, \mathbf{u}=-2(\sin t\mathbf{i}+\cos t\mathbf{j}), \|\mathbf{u}\|=2, \mathbf{N}=-\sin t\mathbf{i}-\cos t\mathbf{j}$

24. $\mathbf{r}'(t)=\mathbf{i}+2t\mathbf{j}+3t^2\mathbf{k}, \mathbf{r}''(t)=2\mathbf{j}+6t\mathbf{k}, \mathbf{u}(t)=-(4t+18t^3)\mathbf{i}+(2-18t^4)\mathbf{j}+(6t+12t^3)\mathbf{k}$,
$$\mathbf{N}=\frac{1}{2\sqrt{81t^8+117t^6+54t^4+13t^2+1}}\left(-(4t+18t^3)\mathbf{i}+(2-18t^4)\mathbf{j}+(6t+12t^3)\mathbf{k}\right)$$

EXERCISE SET 13.5

1. $\kappa\approx\frac{1}{0.5}=2$

2. $\kappa\approx\frac{1}{4/3}=\frac{3}{4}$

3. $\mathbf{r}'(t)=2t\mathbf{i}+3t^2\mathbf{j},\ \mathbf{r}''(t)=2\mathbf{i}+6t\mathbf{j},\ \kappa=\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|/\|\mathbf{r}'(t)\|^3=\dfrac{6}{t(4+9t^2)^{3/2}}$

4. $\mathbf{r}'(t)=-4\sin t\mathbf{i}+\cos t\mathbf{j},\ \mathbf{r}''(t)=-4\cos t\mathbf{i}-\sin t\mathbf{j},\ \kappa=\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|/\|\mathbf{r}'(t)\|^3=\dfrac{4}{(16\sin^2 t+\cos^2 t)^{3/2}}$

5. $\mathbf{r}'(t)=3e^{3t}\mathbf{i}-e^{-t}\mathbf{j},\ \mathbf{r}''(t)=9e^{3t}\mathbf{i}+e^{-t}\mathbf{j},\ \kappa=\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|/\|\mathbf{r}'(t)\|^3=\dfrac{12e^{2t}}{(9e^{6t}+e^{-2t})^{3/2}}$

6. $\mathbf{r}'(t)=-3t^2\mathbf{i}+(1-2t)\mathbf{j},\ \mathbf{r}''(t)=-6t\mathbf{i}-2\mathbf{j},\ \kappa=\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|/\|\mathbf{r}'(t)\|^3=\dfrac{6|t^2-t|}{(9t^4+4t^2-4t+1)^{3/2}}$

7. $\mathbf{r}'(t)=-4\sin t\mathbf{i}+4\cos t\mathbf{j}+\mathbf{k},\ \mathbf{r}''(t)=-4\cos t\mathbf{i}-4\sin t\mathbf{j}$,
$\kappa=\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|/\|\mathbf{r}'(t)\|^3=4/17$

8. $\mathbf{r}'(t)=\mathbf{i}+t\mathbf{j}+t^2\mathbf{k},\ \mathbf{r}''(t)=\mathbf{j}+2t\mathbf{k},\ \kappa=\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|/\|\mathbf{r}'(t)\|^3=\dfrac{\sqrt{t^4+4t^2+1}}{(t^4+t^2+1)^{3/2}}$

9. $\mathbf{r}'(t) = \sinh t\mathbf{i} + \cosh t\mathbf{j} + \mathbf{k}$, $\mathbf{r}''(t) = \cosh t\mathbf{i} + \sinh t\mathbf{j}$, $\kappa = \|\mathbf{r}'(t) \times \mathbf{r}''(t)\|/\|\mathbf{r}'(t)\|^3 = \dfrac{1}{2\cosh^2 t}$

10. $\mathbf{r}'(t) = \mathbf{j} + 2t\mathbf{k}$, $\mathbf{r}''(t) = 2\mathbf{k}$, $\kappa = \|\mathbf{r}'(t) \times \mathbf{r}''(t)\|/\|\mathbf{r}'(t)\|^3 = \dfrac{2}{(4t^2+1)^{3/2}}$

11. $\mathbf{r}'(t) = -3\sin t\mathbf{i} + 4\cos t\mathbf{j} + \mathbf{k}$, $\mathbf{r}''(t) = -3\cos t\mathbf{i} - 4\sin t\mathbf{j}$,
$\mathbf{r}'(\pi/2) = -3\mathbf{i} + \mathbf{k}$, $\mathbf{r}''(\pi/2) = -4\mathbf{j}$; $\kappa = \|4\mathbf{i} + 12\mathbf{k}\|/\| - 3\mathbf{i} + \mathbf{k}\|^3 = 2/5, \rho = 5/2$

12. $\mathbf{r}'(t) = e^t\mathbf{i} - e^{-t}\mathbf{j} + \mathbf{k}$, $\mathbf{r}''(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$,
$\mathbf{r}'(0) = \mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{r}''(0) = \mathbf{i} + \mathbf{j}$; $\kappa = \| - \mathbf{i} + \mathbf{j} + 2\mathbf{k}\|/\|\mathbf{i} - \mathbf{j} + \mathbf{k}\|^3 = \sqrt{2}/3, \rho = 3/\sqrt{2}$

13. $\mathbf{r}'(t) = e^t(\cos t - \sin t)\mathbf{i} + e^t(\cos t + \sin t)\mathbf{j} + e^t\mathbf{k}$,
$\mathbf{r}''(t) = -2e^t \sin t\mathbf{i} + 2e^t\cos t\mathbf{j} + e^t\mathbf{k}$, $\mathbf{r}'(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$,
$\mathbf{r}''(0) = 2\mathbf{j} + \mathbf{k}$; $\kappa = \| - \mathbf{i} - \mathbf{j} + 2\mathbf{k}\|/\|\mathbf{i} + \mathbf{j} + \mathbf{k}\|^3 = \sqrt{2}/3, \rho = 3\sqrt{2}/2$

14. $\mathbf{r}'(t) = \cos t\mathbf{i} - \sin t\mathbf{j} + t\mathbf{k}$, $\mathbf{r}''(t) = -\sin t\mathbf{i} - \cos t\mathbf{j} + \mathbf{k}$,
$\mathbf{r}'(0) = \mathbf{i}$, $\mathbf{r}''(0) = -\mathbf{j} + \mathbf{k}$; $\kappa = \| - \mathbf{j} - \mathbf{k}\|/\|\mathbf{i}\|^3 = \sqrt{2}, \rho = \sqrt{2}/2$

15. $\mathbf{r}'(s) = \dfrac{1}{2}\cos\left(1 + \dfrac{s}{2}\right)\mathbf{i} - \dfrac{1}{2}\sin\left(1 + \dfrac{s}{2}\right)\mathbf{j} + \dfrac{\sqrt{3}}{2}\mathbf{k}$, $\|\mathbf{r}'(s)\| = 1$, so
$\dfrac{d\mathbf{T}}{ds} = -\dfrac{1}{4}\sin\left(1 + \dfrac{s}{2}\right)\mathbf{i} - \dfrac{1}{4}\cos\left(1 + \dfrac{s}{2}\right)\mathbf{j}$, $\kappa = \left\|\dfrac{d\mathbf{T}}{ds}\right\| = \dfrac{1}{4}$

16. $\mathbf{r}'(s) = -\sqrt{\dfrac{3-2s}{3}}\mathbf{i} + \sqrt{\dfrac{2s}{3}}\mathbf{j}$, $\|\mathbf{r}'(s)\| = 1$, so

$$\frac{d\mathbf{T}}{ds} = \frac{1}{\sqrt{9-6s}}\mathbf{i} + \frac{1}{\sqrt{6s}}\mathbf{j}, \kappa = \left\|\frac{d\mathbf{T}}{ds}\right\| = \sqrt{\frac{1}{9-6s} + \frac{1}{6s}} = \sqrt{\frac{3}{2s(9-6s)}}$$

17. **(a)** $\mathbf{r}' = x'\mathbf{i} + y'\mathbf{j}, \mathbf{r}'' = x''\mathbf{i} + y''\mathbf{j}$, $\|\mathbf{r}' \times \mathbf{r}''\| = |x'y'' - x''y'|$, $\kappa = \dfrac{|x'y'' - y'x''|}{(x'^2 + y'^2)^{3/2}}$

(b) Set $x = t$, $y = f(x) = f(t)$, $x' = 1$, $x'' = 0$, $y' = \dfrac{dy}{dx}$, $y'' = \dfrac{d^2y}{dx^2}, \kappa = \dfrac{|d^2y/dx^2|}{(1 + (dy/dx)^2)^{3/2}}$

18. $\dfrac{dy}{dx} = \tan\phi$, $(1 + \tan^2\phi)^{3/2} = (\sec^2\phi)^{3/2} = |\sec\phi|^3$, $\kappa(x) = \dfrac{|y''|}{|\sec\phi|^3} = |y''\cos^3\phi|$

19. $\kappa(x) = \dfrac{|\sin x|}{(1 + \cos^2 x)^{3/2}}$, $\kappa(\pi/2) = 1$

20. $\kappa(x) = \dfrac{2|x|}{(1 + x^4)^{3/2}}$, $\kappa(0) = 0$

21. $\kappa(x) = \dfrac{2|x|^3}{(x^4 + 1)^{3/2}}$, $\kappa(1) = 1/\sqrt{2}$

22. $\kappa(x) = \dfrac{e^{-x}}{(1 + e^{-2x})^{3/2}}$, $\kappa(1) = \dfrac{e^{-1}}{(1 + e^{-2})^{3/2}}$

23. $\kappa(x) = \dfrac{2\sec^2 x|\tan x|}{(1 + \sec^4 x)^{3/2}}$, $\kappa(\pi/4) = 4/(5\sqrt{5})$

24. By implicit differentiation, $dy/dx = 4x/y$, $d^2y/dx^2 = 36/y^3$ so $\kappa = \dfrac{36/|y|^3}{(1 + 16x^2/y^2)^{3/2}}$;
if $(x, y) = (2, 5)$ then $\kappa = \dfrac{36/125}{(1 + 64/25)^{3/2}} = \dfrac{36}{89\sqrt{89}}$

25. $x'(t) = 2t,\ y'(t) = 3t^2,\ x''(t) = 2,\ y''(t) = 6t,$
$x'(1/2) = 1,\ y'(1/2) = 3/4,\ x''(1/2) = 2,\ y''(1/2) = 3;\ \kappa = 96/125$

26. $x'(t) = -4\sin t,\ y'(t) = \cos t,\ x''(t) = -4\cos t,\ y''(t) = -\sin t,$
$x'(\pi/2) = -4,\ y'(\pi/2) = 0,\ x''(\pi/2) = 0,\ y''(\pi/2) = -1;\ \kappa = 1/16$

27. $x'(t) = 3e^{3t},\ y'(t) = -e^{-t},\ x''(t) = 9e^{3t},\ y''(t) = e^{-t},$
$x'(0) = 3,\ y'(0) = -1,\ x''(0) = 9,\ y''(0) = 1;\ \kappa = 6/(5\sqrt{10})$

28. $x'(t) = -3t^2,\ y'(t) = 1 - 2t,\ x''(t) = -6t,\ y''(t) = -2,$
$x'(1) = -3,\ y'(1) = -1,\ x''(1) = -6,\ y''(1) = -2;\ \kappa = 0$

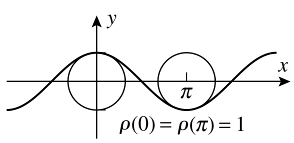
29. $x'(t) = 1, y'(t) = -1/t^2, x''(t) = 0, y''(t) = 2/t^3$
$x'(1) = 1, y'(1) = -1, x''(1) = 0, y''(1) = 2; \kappa = 1/\sqrt{2}$

30. $x'(t) = 4\cos 2t, y'(t) = 3\cos t, x''(t) = -8\sin 2t, y''(t) = -3\sin t,$
$x'(\pi/2) = -4, y'(\pi/2) = 0, x''(\pi/2) = 0, y''(\pi/2) = -3, \kappa = 12/4^{3/2} = 3/2$

31. **(a)** $\kappa(x) = \dfrac{|\cos x|}{(1+\sin^2 x)^{3/2}},$

$\rho(x) = \dfrac{(1+\sin^2 x)^{3/2}}{|\cos x|}$

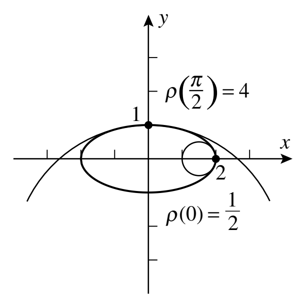
$\rho(0) = \rho(\pi) = 1.$



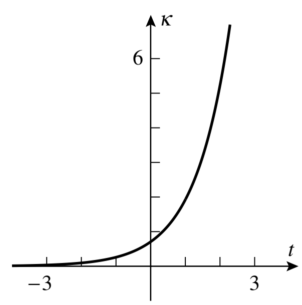
(b) $\kappa(t) = \dfrac{2}{(4\sin^2 t + \cos^2 t)^{3/2}},$

$\rho(t) = \dfrac{1}{2}(4\sin^2 t + \cos^2 t)^{3/2},$

$\rho(0) = 1/2,\ \rho(\pi/2) = 4$



32. $x'(t) = -e^{-t}(\cos t + \sin t),$
$y'(t) = e^{-t}(\cos t - \sin t),$
$x''(t) = 2e^{-t}\sin t,$
$y''(t) = -2e^{-t}\cos t;$
using the formula of Exercise 17(a),
$\kappa = \dfrac{1}{\sqrt{2}}e^t.$



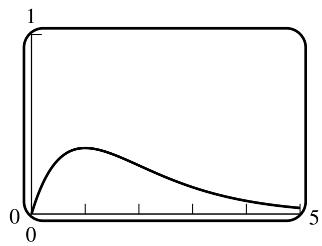
33. **(a)** At $x = 0$ the curvature of I has a large value, yet the value of II there is zero, so II is not the curvature of I; hence I is the curvature of II.

(b) I has points of inflection where the curvature is zero, but II is not zero there, and hence is not the curvature of I; so I is the curvature of II.

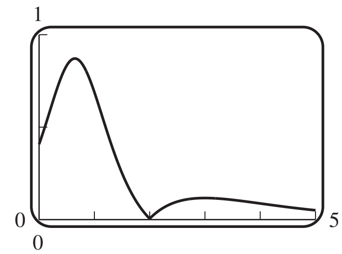
34. **(a)** II takes the value zero at $x = 0$, yet the curvature of I is large there; hence I is the curvature of II.

(b) I has constant zero curvature; II has constant, positive curvature; hence I is the curvature of II.

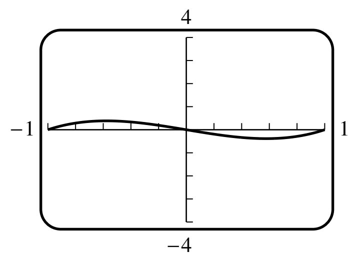
35. **(a)**



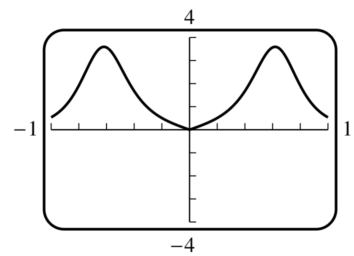
(b)



36. **(a)**

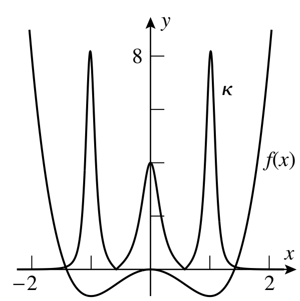


(b)



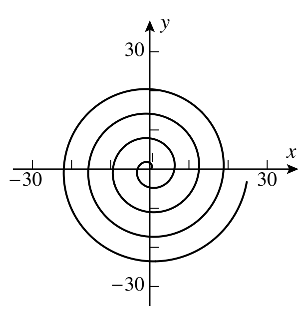
37. **(a)** $\kappa = \dfrac{|12x^2 - 4|}{(1 + (4x^3 - 4x)^2)^{3/2}}$

(b)



(c) $f'(x) = 4x^3 - 4x = 0$ at $x = 0, \pm 1$, $f''(x) = 12x^2 - 4$, so extrema at $x = 0, \pm 1$, and $\rho = 1/4$ for $x = 0$ and $\rho = 1/8$ when $x = \pm 1$.

38. **(a)**



(c) $\kappa(t) = \dfrac{t^2 + 2}{(t^2 + 1)^{3/2}}$

(d) $\lim_{t \to +\infty} \kappa(t) = 0$

39. $\mathbf{r}'(\theta) = \left(-r\sin\theta + \cos\theta\dfrac{dr}{d\theta}\right)\mathbf{i} + \left(r\cos\theta + \sin\theta\dfrac{dr}{d\theta}\right)\mathbf{j}$;

$\mathbf{r}''(\theta) = \left(-r\cos\theta - 2\sin\theta\dfrac{dr}{d\theta} + \cos\theta\dfrac{d^2r}{d\theta^2}\right)\mathbf{i} + \left(-r\sin\theta + 2\cos\theta\dfrac{dr}{d\theta} + \sin\theta\dfrac{d^2r}{d\theta^2}\right)\mathbf{j}$;

$$\kappa = \frac{\left|r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}\right|}{\left[r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right]^{3/2}}.$$

40. Let $r = a$ be the circle, so that $dr/d\theta = 0$, and $\kappa(\theta) = \dfrac{1}{r} = \dfrac{1}{a}$

41. $\kappa(\theta) = \dfrac{3}{2\sqrt{2}(1+\cos\theta)^{1/2}}$, $\kappa(\pi/2) = \dfrac{3}{2\sqrt{2}}$

42. $\kappa(\theta) = \dfrac{1}{\sqrt{5}e^{2\theta}}$, $\kappa(1) = \dfrac{1}{\sqrt{5}e^2}$

43. $\kappa(\theta) = \dfrac{10 + 8\cos^2 3\theta}{(1 + 8\cos^2\theta)^{3/2}}$, $\kappa(0) = \dfrac{2}{3}$

44. $\kappa(\theta) = \dfrac{\theta^2 + 2}{(\theta^2+1)^{3/2}}$, $\kappa(1) = \dfrac{3}{2\sqrt{2}}$

45. The radius of curvature is zero when $\theta = \pi$, so there is a cusp there.

46. $\dfrac{dr}{d\theta} = -\sin\theta$, $\dfrac{d^2r}{d\theta^2} = -\cos\theta$, $\kappa(\theta) = \dfrac{3}{2^{3/2}\sqrt{1+\cos\theta}}$

47. Let $y = t$, then $x = \dfrac{t^2}{4p}$ and $\kappa(t) = \dfrac{1/|2p|}{[t^2/(4p^2)+1]^{3/2}}$;
$t = 0$ when $(x, y) = (0, 0)$ so $\kappa(0) = 1/|2p|$, $\rho = 2|p|$.

48. $\kappa(x) = \dfrac{e^x}{(1+e^{2x})^{3/2}}$, $\kappa'(x) = \dfrac{e^x(1-2e^{2x})}{(1+e^{2x})^{5/2}}$; $\kappa'(x) = 0$ when $e^{2x} = 1/2$, $x = -(\ln 2)/2$. By the first derivative test, $\kappa(-\dfrac{1}{2}\ln 2)$ is maximum so the point is $(-\dfrac{1}{2}\ln 2, 1/\sqrt{2})$.

49. Let $x = 3\cos t$, $y = 2\sin t$ for $0 \le t < 2\pi$, $\kappa(t) = \dfrac{6}{(9\sin^2 t + 4\cos^2 t)^{3/2}}$ so
$\rho(t) = \dfrac{1}{6}(9\sin^2 t + 4\cos^2 t)^{3/2} = \dfrac{1}{6}(5\sin^2 t + 4)^{3/2}$ which, by inspection, is minimum when $t = 0$ or π. The radius of curvature is minimum at $(3, 0)$ and $(-3, 0)$.

50. $\kappa(x) = \dfrac{6x}{(1+9x^4)^{3/2}}$ for $x > 0$, $\kappa'(x) = \dfrac{6(1-45x^4)}{(1+9x^4)^{5/2}}$; $\kappa'(x) = 0$ when $x = 45^{-1/4}$ which, by the first derivative test, yields the maximum.

51. $\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} - \sin t\mathbf{k}$, $\mathbf{r}''(t) = -\cos t\mathbf{i} - \sin t\mathbf{j} - \cos t\mathbf{k}$,
$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \|-\mathbf{i} + \mathbf{k}\| = \sqrt{2}$, $\|\mathbf{r}'(t)\| = (1 + \sin^2 t)^{1/2}$; $\kappa(t) = \sqrt{2}/(1+\sin^2 t)^{3/2}$,
$\rho(t) = (1+\sin^2 t)^{3/2}/\sqrt{2}$. The minimum value of ρ is $1/\sqrt{2}$; the maximum value is 2.

52. $\mathbf{r}'(t) = e^t\mathbf{i} - e^{-t}\mathbf{j} + \sqrt{2}\mathbf{k}$, $\mathbf{r}''(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$;
$\kappa(t) = \dfrac{\sqrt{2}}{e^{2t} + e^{-2t} + 2}$, $\rho(t) = \dfrac{1}{\sqrt{2}}(e^t + e^{-t})^2 = 2\sqrt{2}\cosh^2 t$. The minimum value of ρ is $2\sqrt{2}$.

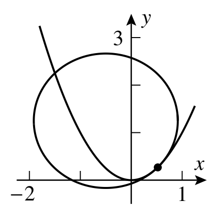
53. From Exercise 39: $dr/d\theta = ae^{a\theta} = ar$, $d^2r/d\theta^2 = a^2e^{a\theta} = a^2r$; $\kappa = 1/[\sqrt{1+a^2}\, r]$.

54. Use implicit differentiation on $r^2 = a^2\cos 2\theta$ to get $2r\dfrac{dr}{d\theta} = -2a^2\sin 2\theta$, $r\dfrac{dr}{d\theta} = -a^2\sin 2\theta$, and again to get $r\dfrac{d^2r}{d\theta^2} + \left(\dfrac{dr}{d\theta}\right)^2 = -2a^2\cos 2\theta$ so $r\dfrac{d^2r}{d\theta^2} = -\left(\dfrac{dr}{d\theta}\right)^2 - 2a^2\cos 2\theta = -\left(\dfrac{dr}{d\theta}\right)^2 - 2r^2$, thus
$\left|r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}\right| = 3\left[r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right]$, $\kappa = \dfrac{3}{[r^2 + (dr/d\theta)^2]^{1/2}}$; $\dfrac{dr}{d\theta} = -\dfrac{a^2\sin 2\theta}{r}$ so
$r^2 + \left(\dfrac{dr}{d\theta}\right)^2 = r^2 + \dfrac{a^4\sin^2 2\theta}{r^2} = \dfrac{r^4 + a^4\sin^2 2\theta}{r^2} = \dfrac{a^4\cos^2 2\theta + a^4\sin^2 2\theta}{r^2} = \dfrac{a^4}{r^2}$, hence $\kappa = \dfrac{3r}{a^2}$.

55. **(a)** $d^2y/dx^2 = 2$, $\kappa(\phi) = |2\cos^3\phi|$

(b) $dy/dx = \tan\phi = 1, \phi = \pi/4$, $\kappa(\pi/4) = |2\cos^3(\pi/4)| = 1/\sqrt{2}$, $\rho = \sqrt{2}$

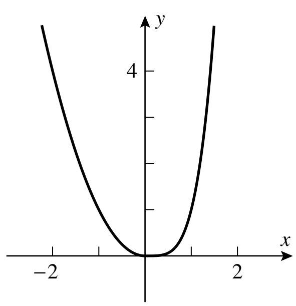
(c)



56. **(a)** $\left(\dfrac{5}{3}, 0\right), \left(0, -\dfrac{5}{2}\right)$ **(b)** clockwise **(c)** it is a point, namely the center of the circle

57. $\kappa = 0$ along $y = 0$; along $y = x^2$, $\kappa(x) = 2/(1 + 4x^2)^{3/2}$, $\kappa(0) = 2$. Along $y = x^3$, $\kappa(x) = 6|x|/(1 + 9x^4)^{3/2}$, $\kappa(0) = 0$.

58. **(a)**



(b) For $y = x^2$, $\kappa(x) = \dfrac{2}{(1 + 4x^2)^{3/2}}$ so $\kappa(0) = 2$; for $y = x^4$,
$\kappa(x) = \dfrac{12x^2}{(1 + 16x^6)^{3/2}}$ so $\kappa(0) = 0$.
κ is not continuous at $x = 0$.

59. $\kappa = 1/r$ along the circle; along $y = ax^2$, $\kappa(x) = 2a/(1 + 4a^2x^2)^{3/2}$, $\kappa(0) = 2a$ so $2a = 1/r$, $a = 1/(2r)$.

60. $\kappa(x) = \dfrac{|y''|}{(1 + y'^2)^{3/2}}$ so the transition will be smooth if the values of y are equal, the values of y' are equal, and the values of y'' are equal at $x = 0$. If $y = e^x$, then $y' = y'' = e^x$; if $y = ax^2 + bx + c$, then $y' = 2ax + b$ and $y'' = 2a$. Equate y, y', and y'' at $x = 0$ to get $c = 1$, $b = 1$, and $a = 1/2$.

61. The result follows from the definitions $\mathbf{N} = \dfrac{\mathbf{T}'(s)}{\|\mathbf{T}'(s)\|}$ and $\kappa = \|\mathbf{T}'(s)\|$.

62. **(a)** $\mathbf{B} \cdot \dfrac{d\mathbf{B}}{ds} = 0$ because $\|\mathbf{B}(s)\| = 1$ so $\dfrac{d\mathbf{B}}{ds}$ is perpendicular to $\mathbf{B}(s)$.

(b) $\mathbf{B}(s) \cdot \mathbf{T}(s) = 0$, $\mathbf{B}(s) \cdot \dfrac{d\mathbf{T}}{ds} + \dfrac{d\mathbf{B}}{ds} \cdot \mathbf{T}(s) = 0$, but $\dfrac{d\mathbf{T}}{ds} = \kappa\mathbf{N}(s)$ so $\kappa\mathbf{B}(s) \cdot \mathbf{N}(s) + \dfrac{d\mathbf{B}}{ds} \cdot \mathbf{T}(s) = 0$, $\dfrac{d\mathbf{B}}{ds} \cdot \mathbf{T}(s) = 0$ because $\mathbf{B}(s) \cdot \mathbf{N}(s) = 0$; thus $\dfrac{d\mathbf{B}}{ds}$ is perpendicular to $\mathbf{T}(s)$.

(c) $\dfrac{d\mathbf{B}}{ds}$ is perpendicular to both $\mathbf{B}(s)$ and $\mathbf{T}(s)$ but so is $\mathbf{N}(s)$, thus $\dfrac{d\mathbf{B}}{ds}$ is parallel to $\mathbf{N}(s)$ and hence a scalar multiple of $\mathbf{N}(s)$.

(d) If C lies in a plane, then $\mathbf{T}(s)$ and $\mathbf{N}(s)$ also lie in the plane; $\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s)$ so $\mathbf{B}(s)$ is always perpendicular to the plane and hence $d\mathbf{B}/ds = \mathbf{0}$, thus $\tau = 0$.

63. $\dfrac{d\mathbf{N}}{ds} = \mathbf{B} \times \dfrac{d\mathbf{T}}{ds} + \dfrac{d\mathbf{B}}{ds} \times \mathbf{T} = \mathbf{B} \times (\kappa\mathbf{N}) + (-\tau\mathbf{N}) \times \mathbf{T} = \kappa\mathbf{B} \times \mathbf{N} - \tau\mathbf{N} \times \mathbf{T}$, but $\mathbf{B} \times \mathbf{N} = -\mathbf{T}$ and $\mathbf{N} \times \mathbf{T} = -\mathbf{B}$ so $\dfrac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} + \tau\mathbf{B}$

64. $\mathbf{r}''(s) = d\mathbf{T}/ds = \kappa\mathbf{N}$ so $\mathbf{r}'''(s) = \kappa d\mathbf{N}/ds + (d\kappa/ds)\mathbf{N}$ but $d\mathbf{N}/ds = -\kappa\mathbf{T} + \tau\mathbf{B}$ so
$\mathbf{r}'''(s) = -\kappa^2\mathbf{T} + (d\kappa/ds)\mathbf{N} + \kappa\tau\mathbf{B}$, $\mathbf{r}'(s) \times \mathbf{r}''(s) = \mathbf{T} \times (\kappa\mathbf{N}) = \kappa\mathbf{T} \times \mathbf{N} = \kappa\mathbf{B}$,
$[\mathbf{r}'(s) \times \mathbf{r}''(s)] \cdot \mathbf{r}'''(s) = -\kappa^3\mathbf{B} \cdot \mathbf{T} + \kappa(d\kappa/ds)\mathbf{B} \cdot \mathbf{N} + \kappa^2\tau\mathbf{B} \cdot \mathbf{B} = \kappa^2\tau$,
$\tau = [\mathbf{r}'(s) \times \mathbf{r}''(s)] \cdot \mathbf{r}'''(s)/\kappa^2 = [\mathbf{r}'(s) \times \mathbf{r}''(s)] \cdot \mathbf{r}'''(s)/\|\mathbf{r}''(s)\|^2$ and
$\mathbf{B} = \mathbf{T} \times \mathbf{N} = [\mathbf{r}'(s) \times \mathbf{r}''(s)]/\|\mathbf{r}''(s)\|$

65. $\mathbf{r} = a\cos(s/w)\mathbf{i} + a\sin(s/w)\mathbf{j} + (cs/w)\mathbf{k}$, $\mathbf{r}' = -(a/w)\sin(s/w)\mathbf{i} + (a/w)\cos(s/w)\mathbf{j} + (c/w)\mathbf{k}$,
$\mathbf{r}'' = -(a/w^2)\cos(s/w)\mathbf{i} - (a/w^2)\sin(s/w)\mathbf{j}$, $\mathbf{r}''' = (a/w^3)\sin(s/w)\mathbf{i} - (a/w^3)\cos(s/w)\mathbf{j}$,
$\mathbf{r}' \times \mathbf{r}'' = (ac/w^3)\sin(s/w)\mathbf{i} - (ac/w^3)\cos(s/w)\mathbf{j} + (a^2/w^3)\mathbf{k}$, $(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}''' = a^2c/w^6$,
$\|\mathbf{r}''(s)\| = a/w^2$, so $\tau = c/w^2$ and $\mathbf{B} = (c/w)\sin(s/w)\mathbf{i} - (c/w)\cos(s/w)\mathbf{j} + (a/w)\mathbf{k}$

66. **(a)** $\mathbf{T}' = \dfrac{d\mathbf{T}}{dt} = \dfrac{d\mathbf{T}}{ds}\dfrac{ds}{dt} = (\kappa\mathbf{N})s' = \kappa s'\mathbf{N}$,

$\mathbf{N}' = \dfrac{d\mathbf{N}}{dt} = \dfrac{d\mathbf{N}}{ds}\dfrac{ds}{dt} = (-\kappa\mathbf{T} + \tau\mathbf{B})s' = -\kappa s'\mathbf{T} + \tau s'\mathbf{B}$.

(b) $\|\mathbf{r}'(t)\| = s'$ so $\mathbf{r}'(t) = s'\mathbf{T}$ and $\mathbf{r}''(t) = s''\mathbf{T} + s'\mathbf{T}' = s''\mathbf{T} + s'(\kappa s'\mathbf{N}) = s''\mathbf{T} + \kappa(s')^2\mathbf{N}$.

(c)
$$\begin{aligned}\mathbf{r}'''(t) &= s''\mathbf{T}' + s'''\mathbf{T} + \kappa(s')^2\mathbf{N}' + [2\kappa s's'' + \kappa'(s')^2]\mathbf{N} \\ &= s''(\kappa s'\mathbf{N}) + s'''\mathbf{T} + \kappa(s')^2(-\kappa s'\mathbf{T} + \tau s'\mathbf{B}) + [2\kappa s's'' + \kappa'(s')^2]\mathbf{N} \\ &= [s''' - \kappa^2(s')^3]\mathbf{T} + [3\kappa s's'' + \kappa'(s')^2]\mathbf{N} + \kappa\tau(s')^3\mathbf{B}.\end{aligned}$$

(d) $\mathbf{r}'(t) \times \mathbf{r}''(t) = s's''\mathbf{T} \times \mathbf{T} + \kappa(s')^3\mathbf{T} \times \mathbf{N} = \kappa(s')^3\mathbf{B}$, $[\mathbf{r}'(t) \times \mathbf{r}''(t)] \cdot \mathbf{r}'''(t) = \kappa^2\tau(s')^6$ so
$\tau = \dfrac{[\mathbf{r}'(t) \times \mathbf{r}''(t)] \cdot \mathbf{r}'''(t)}{\kappa^2(s')^6} = \dfrac{[\mathbf{r}'(t) \times \mathbf{r}''(t)] \cdot \mathbf{r}'''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|^2}$

67. $\mathbf{r}' = 2\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k}$, $\mathbf{r}'' = 2\mathbf{j} + 2t\mathbf{k}$, $\mathbf{r}''' = 2\mathbf{k}$, $\mathbf{r}' \times \mathbf{r}'' = 2t^2\mathbf{i} - 4t\mathbf{j} + 4\mathbf{k}$, $\|\mathbf{r}' \times \mathbf{r}''\| = 2(t^2 + 2)$,
$\tau = 8/[2(t^2 + 2)]^2 = 2/(t^2 + 2)^2$

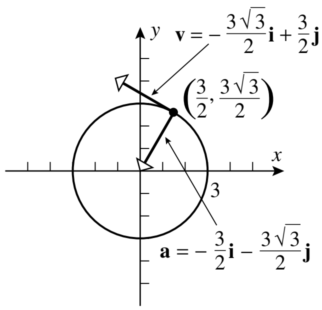
68. $\mathbf{r}' = -a\sin t\mathbf{i} + a\cos t\mathbf{j} + c\mathbf{k}$, $\mathbf{r}'' = -a\cos t\mathbf{i} - a\sin t\mathbf{j}$, $\mathbf{r}''' = a\sin t\mathbf{i} - a\cos t\mathbf{j}$,
$\mathbf{r}' \times \mathbf{r}'' = ac\sin t\mathbf{i} - ac\cos t\mathbf{j} + a^2\mathbf{k}$, $\|\mathbf{r}' \times \mathbf{r}''\| = \sqrt{a^2(a^2+c^2)}$,
$\tau = a^2c/[a^2(a^2+c^2)] = c/(a^2+c^2)$

69. $\mathbf{r}' = e^t\mathbf{i} - e^{-t}\mathbf{j} + \sqrt{2}\mathbf{k}$, $\mathbf{r}'' = e^t\mathbf{i} + e^{-t}\mathbf{j}$, $\mathbf{r}''' = e^t\mathbf{i} - e^{-t}\mathbf{j}$, $\mathbf{r}' \times \mathbf{r}'' = -\sqrt{2}e^{-t}\mathbf{i} + \sqrt{2}e^t\mathbf{j} + 2\mathbf{k}$,
$\|\mathbf{r}' \times \mathbf{r}''\| = \sqrt{2}(e^t + e^{-t})$, $\tau = (-2\sqrt{2})/[2(e^t+e^{-t})^2] = -\sqrt{2}/(e^t+e^{-t})^2$

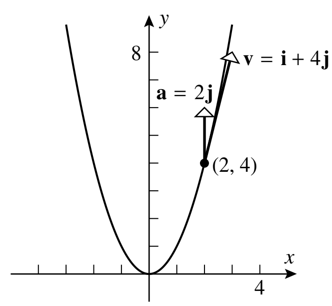
70. $\mathbf{r}' = (1-\cos t)\mathbf{i} + \sin t\mathbf{j} + \mathbf{k}$, $\mathbf{r}'' = \sin t\mathbf{i} + \cos t\mathbf{j}$, $\mathbf{r}''' = \cos t\mathbf{i} - \sin t\mathbf{j}$,
$\mathbf{r}' \times \mathbf{r}'' = -\cos t\mathbf{i} + \sin t\mathbf{j} + (\cos t - 1)\mathbf{k}$,
$\|\mathbf{r}' \times \mathbf{r}''\| = \sqrt{\cos^2 t + \sin^2 t + (\cos t - 1)^2} = \sqrt{1 + 4\sin^4(t/2)}$, $\tau = -1/[1 + 4\sin^4(t/2)]$

EXERCISE SET 13.6

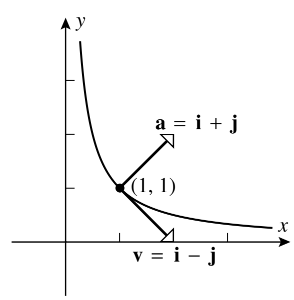
1. $\mathbf{v}(t) = -3\sin t\mathbf{i} + 3\cos t\mathbf{j}$
$\mathbf{a}(t) = -3\cos t\mathbf{i} - 3\sin t\mathbf{j}$
$\|\mathbf{v}(t)\| = \sqrt{9\sin^2 t + 9\cos^2 t} = 3$
$\mathbf{r}(\pi/3) = (3/2)\mathbf{i} + (3\sqrt{3}/2)\mathbf{j}$
$\mathbf{v}(\pi/3) = -(3\sqrt{3}/2)\mathbf{i} + (3/2)\mathbf{j}$
$\mathbf{a}(\pi/3) = -(3/2)\mathbf{i} - (3\sqrt{3}/2)\mathbf{j}$



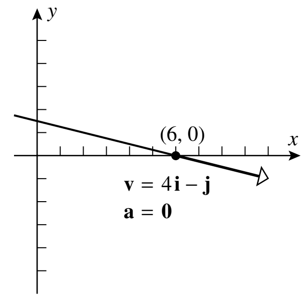
2. $\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j}$
$\mathbf{a}(t) = 2\mathbf{j}$
$\|\mathbf{v}(t)\| = \sqrt{1+4t^2}$
$\mathbf{r}(2) = 2\mathbf{i} + 4\mathbf{j}$
$\mathbf{v}(2) = \mathbf{i} + 4\mathbf{j}$
$\mathbf{a}(2) = 2\mathbf{j}$



3. $\mathbf{v}(t) = e^t\mathbf{i} - e^{-t}\mathbf{j}$
$\mathbf{a}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$
$\|\mathbf{v}(t)\| = \sqrt{e^{2t} + e^{-2t}}$
$\mathbf{r}(0) = \mathbf{i} + \mathbf{j}$
$\mathbf{v}(0) = \mathbf{i} - \mathbf{j}$
$\mathbf{a}(0) = \mathbf{i} + \mathbf{j}$



4. $\mathbf{v}(t) = 4\mathbf{i} - \mathbf{j}$
$\mathbf{a}(t) = \mathbf{0}$
$\|\mathbf{v}(t)\| = \sqrt{17}$
$\mathbf{r}(1) = 6\mathbf{i}$
$\mathbf{v}(1) = 4\mathbf{i} - \mathbf{j}$
$\mathbf{a}(1) = \mathbf{0}$



5. $\mathbf{v} = \mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$, $\mathbf{a} = \mathbf{j} + 2t\mathbf{k}$; at $t = 1$, $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\|\mathbf{v}\| = \sqrt{3}$, $\mathbf{a} = \mathbf{j} + 2\mathbf{k}$

6. $\mathbf{r} = (1 + 3t)\mathbf{i} + (2 - 4t)\mathbf{j} + (7 + t)\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$,
$\mathbf{a} = \mathbf{0}$; at $t = 2$, $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$, $\|\mathbf{v}\| = \sqrt{26}$, $\mathbf{a} = \mathbf{0}$

7. $\mathbf{v} = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \mathbf{k}$, $\mathbf{a} = -2\cos t\mathbf{i} - 2\sin t\mathbf{j}$;
at $t = \pi/4$, $\mathbf{v} = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} + \mathbf{k}$, $\|\mathbf{v}\| = \sqrt{5}$, $\mathbf{a} = -\sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j}$

8. $\mathbf{v} = e^t(\cos t + \sin t)\mathbf{i} + e^t(\cos t - \sin t)\mathbf{j} + \mathbf{k}$, $\mathbf{a} = 2e^t\cos t\mathbf{i} - 2e^t\sin t\mathbf{j}$; at $t = \pi/2$,
$\mathbf{v} = e^{\pi/2}\mathbf{i} - e^{\pi/2}\mathbf{j} + \mathbf{k}$, $\|\mathbf{v}\| = (1 + 2e^{\pi})^{1/2}$, $\mathbf{a} = -2e^{\pi/2}\mathbf{j}$

9. **(a)** $\mathbf{v} = -a\omega\sin\omega t\mathbf{i} + b\omega\cos\omega t\mathbf{j}$, $\mathbf{a} = -a\omega^2\cos\omega t\mathbf{i} - b\omega^2\sin\omega t\mathbf{j} = -\omega^2\mathbf{r}$

(b) From Part (a), $\|\mathbf{a}\| = \omega^2\|\mathbf{r}\|$

10. **(a)** $\mathbf{v} = 16\pi\cos\pi t\mathbf{i} - 8\pi\sin 2\pi t\mathbf{j}$, $\mathbf{a} = -16\pi^2\sin\pi t\mathbf{i} - 16\pi^2\cos 2\pi t\mathbf{j}$;
at $t = 1$, $\mathbf{v} = -16\pi\mathbf{i}$, $\|\mathbf{v}\| = 16\pi$, $\mathbf{a} = -16\pi^2\mathbf{j}$

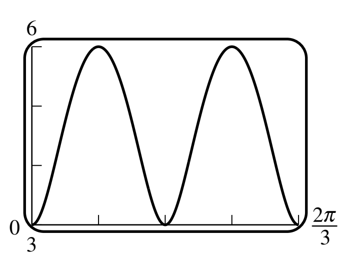
(b) $x = 16\sin\pi t, y = 4\cos 2\pi t = 4\cos^2\pi t - 4\sin^2\pi t = 4 - 8\sin^2\pi t$, $y = 4 - x^2/32$

(c) Both $x(t)$ and $y(t)$ are periodic and have period 2, so after 2 s the particle retraces its path.

11. $\mathbf{v} = (6/\sqrt{t})\mathbf{i} + (3/2)t^{1/2}\mathbf{j}$, $\|\mathbf{v}\| = \sqrt{36/t + 9t/4}, d\|\mathbf{v}\|/dt = (-36/t^2 + 9/4)/(2\sqrt{36/t + 9t/4}) = 0$ if $t = 4$ which yields a minimum by the first derivative test. The minimum speed is $3\sqrt{2}$ when $\mathbf{r} = 24\mathbf{i} + 8\mathbf{j}$.

12. $\mathbf{v} = (1 - 2t)\mathbf{i} - 2t\mathbf{j}$, $\|\mathbf{v}\| = \sqrt{(1 - 2t)^2 + 4t^2} = \sqrt{8t^2 - 4t + 1}$,
$\dfrac{d}{dt}\|\mathbf{v}\| = \dfrac{8t - 2}{\sqrt{8t^2 - 4t + 1}} = 0$ if $t = \dfrac{1}{4}$ which yields a minimum by the first derivative test. The minimum speed is $1/\sqrt{2}$ when the particle is at $\mathbf{r} = \dfrac{3}{16}\mathbf{i} - \dfrac{1}{16}\mathbf{j}$.

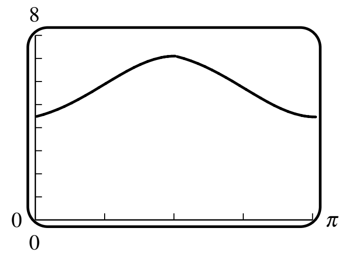
13. **(a)**



(b) $\mathbf{v} = 3\cos 3t\mathbf{i} + 6\sin 3t\mathbf{j}$, $\|\mathbf{v}\| = \sqrt{9\cos^2 3t + 36\sin^2 3t} = 3\sqrt{1+3\sin^2 3t}$; by inspection, maximum speed is 6 and minimum speed is 3

(d) $\dfrac{d}{dt}\|\mathbf{v}\| = \dfrac{27\sin 6t}{2\sqrt{1+3\sin^2 3t}} = 0$ when $t = 0, \pi/6, \pi/3, \pi/2, 2\pi/3$; the maximum speed is 6 which occurs first when $\sin 3t = 1, t = \pi/6$.

14. **(a)**



(d) $\mathbf{v} = -6\sin 2t\mathbf{i} + 2\cos 2t\mathbf{j} + 4\mathbf{k}$, $\|\mathbf{v}\| = \sqrt{36\sin^2 2t + 4\cos^2 2t + 16} = 2\sqrt{8\sin^2 t + 5}$; by inspection the maximum speed is $2\sqrt{13}$ when $t = \pi/2$, the minimum speed is $2\sqrt{5}$ when $t = 0$ or π.

15. $\mathbf{v}(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{C}_1$, $\mathbf{v}(0) = \mathbf{j} + \mathbf{C}_1 = \mathbf{i}$, $\mathbf{C}_1 = \mathbf{i} - \mathbf{j}$, $\mathbf{v}(t) = (1 - \sin t)\mathbf{i} + (\cos t - 1)\mathbf{j}$;
$\mathbf{r}(t) = (t + \cos t)\mathbf{i} + (\sin t - t)\mathbf{j} + \mathbf{C}_2$, $\mathbf{r}(0) = \mathbf{i} + \mathbf{C}_2 = \mathbf{j}$,
$\mathbf{C}_2 = -\mathbf{i} + \mathbf{j}$ so $\mathbf{r}(t) = (t + \cos t - 1)\mathbf{i} + (\sin t - t + 1)\mathbf{j}$

16. $\mathbf{v}(t) = t\mathbf{i} - e^{-t}\mathbf{j} + \mathbf{C}_1$, $\mathbf{v}(0) = -\mathbf{j} + \mathbf{C}_1 = 2\mathbf{i} + \mathbf{j}$; $\mathbf{C}_1 = 2\mathbf{i} + 2\mathbf{j}$ so
$\mathbf{v}(t) = (t+2)\mathbf{i} + (2 - e^{-t})\mathbf{j}$; $\mathbf{r}(t) = (t^2/2 + 2t)\mathbf{i} + (2t + e^{-t})\mathbf{j} + \mathbf{C}_2$
$\mathbf{r}(0) = \mathbf{j} + \mathbf{C}_2 = \mathbf{i} - \mathbf{j}$, $\mathbf{C}_2 = \mathbf{i} - 2\mathbf{j}$ so $\mathbf{r}(t) = (t^2/2 + 2t + 1)\mathbf{i} + (2t + e^{-t} - 2)\mathbf{j}$

17. $\mathbf{v}(t) = -\cos t\mathbf{i} + \sin t\mathbf{j} + e^t\mathbf{k} + \mathbf{C}_1$, $\mathbf{v}(0) = -\mathbf{i} + \mathbf{k} + \mathbf{C}_1 = \mathbf{k}$ so
$\mathbf{C}_1 = \mathbf{i}$, $\mathbf{v}(t) = (1 - \cos t)\mathbf{i} + \sin t\mathbf{j} + e^t\mathbf{k}$; $\mathbf{r}(t) = (t - \sin t)\mathbf{i} - \cos t\mathbf{j} + e^t\mathbf{k} + \mathbf{C}_2$,
$\mathbf{r}(0) = -\mathbf{j} + \mathbf{k} + \mathbf{C}_2 = -\mathbf{i} + \mathbf{k}$ so $\mathbf{C}_2 = -\mathbf{i} + \mathbf{j}$, $\mathbf{r}(t) = (t - \sin t - 1)\mathbf{i} + (1 - \cos t)\mathbf{j} + e^t\mathbf{k}$.

18. $\mathbf{v}(t) = -\dfrac{1}{t+1}\mathbf{j} + \dfrac{1}{2}e^{-2t}\mathbf{k} + \mathbf{C}_1$, $\mathbf{v}(0) = -\mathbf{j} + \dfrac{1}{2}\mathbf{k} + \mathbf{C}_1 = 3\mathbf{i} - \mathbf{j}$ so
$\mathbf{C}_1 = 3\mathbf{i} - \dfrac{1}{2}\mathbf{k}$, $\mathbf{v}(t) = 3\mathbf{i} - \dfrac{1}{t+1}\mathbf{j} + \left(\dfrac{1}{2}e^{-2t} - \dfrac{1}{2}\right)\mathbf{k}$;
$\mathbf{r}(t) = 3t\mathbf{i} - \ln(t+1)\mathbf{j} - \left(\dfrac{1}{4}e^{-2t} + \dfrac{1}{2}t\right)\mathbf{k} + \mathbf{C}_2$,
$\mathbf{r}(0) = -\dfrac{1}{4}\mathbf{k} + \mathbf{C}_2 = 2\mathbf{k}$ so $\mathbf{C}_2 = \dfrac{9}{4}\mathbf{k}$, $\mathbf{r}(t) = 3t\mathbf{i} - \ln(t+1)\mathbf{j} + \left(\dfrac{9}{4} - \dfrac{1}{4}e^{-2t} - \dfrac{1}{2}t\right)\mathbf{k}$.

19. If $\mathbf{a} = \mathbf{0}$ then $x''(t) = y''(t) = z''(t) = 0$, so $x(t) = x_1 t + x_0, y(t) = y_1 t + y_0, z(t) = z_1 t + z_0$, the motion is along a straight line and has constant speed.

20. **(a)** If $\|\mathbf{r}\|$ is constant then so is $\|\mathbf{r}\|^2$, but then $x^2 + y^2 = c^2$ (2-space) or $x^2 + y^2 + z^2 = c^2$ (3-space), so the motion is along a circle or a sphere of radius c centered at the origin, and the velocity vector is always perpendicular to the position vector.

(b) If $\|\mathbf{v}\|$ is constant then by the Theorem, $\mathbf{v}(t) \cdot \mathbf{a}(t) = 0$, so the velocity is always perpendicular to the acceleration.

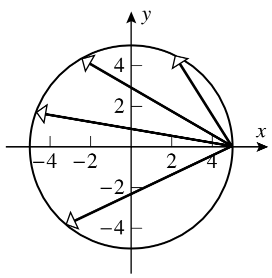
21. $\mathbf{v} = 3t^2\mathbf{i} + 2t\mathbf{j}$, $\mathbf{a} = 6t\mathbf{i} + 2\mathbf{j}$; $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{a} = 6\mathbf{i} + 2\mathbf{j}$ when $t = 1$ so $\cos\theta = (\mathbf{v} \cdot \mathbf{a})/(\|\mathbf{v}\|\,\|\mathbf{a}\|) = 11/\sqrt{130}$, $\theta \approx 15°$.

22. $\mathbf{v} = e^t(\cos t - \sin t)\mathbf{i} + e^t(\cos t + \sin t)\mathbf{j}$, $\mathbf{a} = -2e^t \sin t\mathbf{i} + 2e^t \cos t\mathbf{j}$, $\mathbf{v} \cdot \mathbf{a} = 2e^{2t}$, $\|\mathbf{v}\| = \sqrt{2}e^t$, $\|\mathbf{a}\| = 2e^t$, $\cos\theta = (\mathbf{v} \cdot \mathbf{a})/(\|\mathbf{v}\|\,\|\mathbf{a}\|) = 1/\sqrt{2}$, $\theta = 45°$.

23. **(a)** displacement $= \mathbf{r}_1 - \mathbf{r}_0 = 0.7\mathbf{i} + 2.7\mathbf{j} - 3.4\mathbf{k}$

(b) $\Delta\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_0$, so $\mathbf{r}_0 = \mathbf{r}_1 - \Delta\mathbf{r} = -0.7\mathbf{i} - 2.9\mathbf{j} + 4.8\mathbf{k}$.

24. **(a)**



(b) one revolution, or 10π

25. $\Delta\mathbf{r} = \mathbf{r}(3) - \mathbf{r}(1) = 8\mathbf{i} + (26/3)\mathbf{j}$; $\mathbf{v} = 2t\mathbf{i} + t^2\mathbf{j}$, $s = \int_1^3 t\sqrt{4 + t^2}dt = (13\sqrt{13} - 5\sqrt{5})/3$.

26. $\Delta\mathbf{r} = \mathbf{r}(3\pi/2) - \mathbf{r}(0) = 3\mathbf{i} - 3\mathbf{j}$; $\mathbf{v} = -3\cos t\mathbf{i} - 3\sin t\mathbf{j}$, $s = \int_0^{3\pi/2} 3dt = 9\pi/2$.

27. $\Delta\mathbf{r} = \mathbf{r}(\ln 3) - \mathbf{r}(0) = 2\mathbf{i} - (2/3)\mathbf{j} + \sqrt{2}(\ln 3)\mathbf{k}$; $\mathbf{v} = e^t\mathbf{i} - e^{-t}\mathbf{j} + \sqrt{2}\,\mathbf{k}$, $s = \int_0^{\ln 3} (e^t + e^{-t})dt = 8/3$.

28. $\Delta\mathbf{r} = \mathbf{r}(\pi) - \mathbf{r}(0) = \mathbf{0}$; $\mathbf{v} = -2\sin 2t\mathbf{i} + 2\sin 2t\mathbf{j} - \sin 2t\mathbf{k}$,
$\|\mathbf{v}\| = 3|\sin 2t|$, $s = \int_0^{\pi} 3|\sin 2t|dt = 6\int_0^{\pi/2} \sin 2t\,dt = 6$.

29. In both cases, the equation of the path in rectangular coordinates is $x^2 + y^2 = 4$, the particles move counterclockwise around this circle; $\mathbf{v}_1 = -6\sin 3t\mathbf{i} + 6\cos 3t\mathbf{j}$ and $\mathbf{v}_2 = -4t\sin(t^2)\mathbf{i} + 4t\cos(t^2)\mathbf{j}$ so $\|\mathbf{v}_1\| = 6$ and $\|\mathbf{v}_2\| = 4t$.

30. Let $u = 1 - t^3$ in $\mathbf{r}_2$ to get
$\mathbf{r}_1(u) = (3 + 2(1 - t^3))\mathbf{i} + (1 - t^3)\mathbf{j} + (1 - (1 - t^3))\mathbf{k} = (5 - 2t^3)\mathbf{i} + (1 - t^3)\mathbf{j} + t^3\mathbf{k} = \mathbf{r}_2(t)$
so both particles move along the same path; $\mathbf{v}_1 = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{v}_2 = -6t^2\mathbf{i} - 3t^2\mathbf{j} + 3t^2\mathbf{k}$ so $\|\mathbf{v}_1\| = \sqrt{6}$ and $\|\mathbf{v}_2\| = 3\sqrt{6}t^2$.

31. **(a)** $\mathbf{v} = -e^{-t}\mathbf{i} + e^t\mathbf{j}$, $\mathbf{a} = e^{-t}\mathbf{i} + e^t\mathbf{j}$; when $t = 0$, $\mathbf{v} = -\mathbf{i} + \mathbf{j}$, $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\|\mathbf{v}\| = \sqrt{2}$, $\mathbf{v} \cdot \mathbf{a} = 0$, $\mathbf{v} \times \mathbf{a} = -2\mathbf{k}$ so $a_T = 0$, $a_N = \sqrt{2}$.

(b) $a_T\mathbf{T} = \mathbf{0}, a_N\mathbf{N} = \mathbf{a} - a_T\mathbf{T} = \mathbf{i} + \mathbf{j}$

(c) $\kappa = 1/\sqrt{2}$

32. **(a)** $\mathbf{v} = -2t\sin(t^2)\mathbf{i} + 2t\cos(t^2)\mathbf{j}$, $\mathbf{a} = [-4t^2\cos(t^2) - 2\sin(t^2)]\mathbf{i} + [-4t^2\sin(t^2) + 2\cos(t^2)]\mathbf{j}$; when $t = \sqrt{\pi}/2$, $\mathbf{v} = -\sqrt{\pi/2}\mathbf{i} + \sqrt{\pi/2}\mathbf{j}$, $\mathbf{a} = (-\pi/\sqrt{2} - \sqrt{2})\mathbf{i} + (-\pi/\sqrt{2} + \sqrt{2})\mathbf{j}$, $\|\mathbf{v}\| = \sqrt{\pi}$, $\mathbf{v} \cdot \mathbf{a} = 2\sqrt{\pi}$, $\mathbf{v} \times \mathbf{a} = \pi^{3/2}\mathbf{k}$ so $a_T = 2$, $a_N = \pi$

(b) $a_T\mathbf{T} = -\sqrt{2}(\mathbf{i} - \mathbf{j})$, $a_N\mathbf{N} = \mathbf{a} - a_T\mathbf{T} = -(\pi/\sqrt{2})(\mathbf{i} + \mathbf{j})$

(c) $\kappa = 1$

33. **(a)** $\mathbf{v} = (3t^2 - 2)\mathbf{i} + 2t\mathbf{j}$, $\mathbf{a} = 6t\mathbf{i} + 2\mathbf{j}$; when $t = 1$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$, $\mathbf{a} = 6\mathbf{i} + 2\mathbf{j}$, $\|\mathbf{v}\| = \sqrt{5}$, $\mathbf{v} \cdot \mathbf{a} = 10$, $\mathbf{v} \times \mathbf{a} = -10\mathbf{k}$ so $a_T = 2\sqrt{5}$, $a_N = 2\sqrt{5}$

(b) $a_T\mathbf{T} = \dfrac{2\sqrt{5}}{\sqrt{5}}(\mathbf{i} + 2\mathbf{j}) = 2\mathbf{i} + 4\mathbf{j}$, $a_N\mathbf{N} = \mathbf{a} - a_T\mathbf{T} = 4\mathbf{i} - 2\mathbf{j}$

(c) $\kappa = 2/\sqrt{5}$

34. **(a)** $\mathbf{v} = e^t(-\sin t + \cos t)\mathbf{i} + e^t(\cos t + \sin t)\mathbf{j}$, $\mathbf{a} = -2e^t \sin t\mathbf{i} + 2e^t \cos t\mathbf{j}$; when $t = \pi/4$, $\mathbf{v} = \sqrt{2}e^{\pi/4}\mathbf{j}$, $\mathbf{a} = -\sqrt{2}e^{\pi/4}\mathbf{i} + \sqrt{2}e^{\pi/4}\mathbf{j}$, $\|\mathbf{v}\| = \sqrt{2}e^{\pi/4}$, $\mathbf{v} \cdot \mathbf{a} = 2e^{\pi/2}$, $\mathbf{v} \times \mathbf{a} = 2e^{\pi/2}\mathbf{k}$ so $a_T = \sqrt{2}e^{\pi/4}$, $a_N = \sqrt{2}e^{\pi/4}$

(b) $a_T\mathbf{T} = \sqrt{2}e^{\pi/4}\mathbf{j}$, $a_N\mathbf{N} = \mathbf{a} - a_T\mathbf{T} = -\sqrt{2}e^{\pi/4}\mathbf{i}$

(c) $\kappa = \dfrac{1}{\sqrt{2}e^{\pi/4}}$

35. **(a)** $\mathbf{v} = (-1/t^2)\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$, $\mathbf{a} = (2/t^3)\,\mathbf{i} + 2\mathbf{j} + 6t\mathbf{k}$; when $t = 1$, $\mathbf{v} = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{a} = 2\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$, $\|\mathbf{v}\| = \sqrt{14}$, $\mathbf{v} \cdot \mathbf{a} = 20$, $\mathbf{v} \times \mathbf{a} = 6\mathbf{i} + 12\mathbf{j} - 6\mathbf{k}$ so $a_T = 20/\sqrt{14}$, $a_N = 6\sqrt{3}/\sqrt{7}$

(b) $a_T\mathbf{T} = -\dfrac{10}{7}\mathbf{i} + \dfrac{20}{7}\mathbf{j} + \dfrac{30}{7}\mathbf{k}$, $a_N\mathbf{N} = \mathbf{a} - a_T\mathbf{T} = \dfrac{24}{7}\mathbf{i} - \dfrac{6}{7}\mathbf{j} + \dfrac{12}{7}\mathbf{k}$

(c) $\kappa = \dfrac{6\sqrt{6}}{14^{3/2}} = \left(\dfrac{3}{7}\right)^{3/2}$

36. **(a)** $\mathbf{v} = e^t\mathbf{i} - 2e^{-2t}\mathbf{j} + \mathbf{k}$, $\mathbf{a} = e^t\mathbf{i} + 4e^{-2t}\mathbf{j}$; when $t = 0$, $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{a} = \mathbf{i} + 4\mathbf{j}$, $\|\mathbf{v}\| = \sqrt{6}$, $\mathbf{v} \cdot \mathbf{a} = -7$, $\mathbf{v} \times \mathbf{a} = -4\mathbf{i} + \mathbf{j} + 6\mathbf{k}$ so $a_T = -7/\sqrt{6}$, $a_N = \sqrt{53/6}$

(b) $a_T\mathbf{T} = -\dfrac{7}{6}(\mathbf{i} - 2\mathbf{j} + \mathbf{k}), a_N\mathbf{N} = \mathbf{a} - a_T\mathbf{T} = \dfrac{13}{6}\mathbf{i} + \dfrac{19}{3}\mathbf{j} + \dfrac{7}{6}\mathbf{k}$

(c) $\kappa = \dfrac{\sqrt{53}}{6\sqrt{6}}$

37. **(a)** $\mathbf{v} = 3\cos t\mathbf{i} - 2\sin t\mathbf{j} - 2\cos 2t\mathbf{k}$, $\mathbf{a} = -3\sin t\mathbf{i} - 2\cos t\mathbf{j} + 4\sin 2t\mathbf{k}$; when $t = \pi/2$, $\mathbf{v} = -2\mathbf{j} + 2\mathbf{k}$, $\mathbf{a} = -3\mathbf{i}$, $\|\mathbf{v}\| = 2\sqrt{2}$, $\mathbf{v} \cdot \mathbf{a} = 0$, $\mathbf{v} \times \mathbf{a} = -6\mathbf{j} - 6\mathbf{k}$ so $a_T = 0$, $a_N = 3$

(b) $a_T\mathbf{T} = \mathbf{0}, a_N\mathbf{N} = \mathbf{a} = -3\mathbf{i}$

(c) $\kappa = \dfrac{3}{8}$

38. **(a)** $\mathbf{v} = 3t^2\mathbf{j} - (16/t)\mathbf{k}$, $\mathbf{a} = 6t\mathbf{j} + (16/t^2)\mathbf{k}$; when $t = 1$, $\mathbf{v} = 3\mathbf{j} - 16\mathbf{k}$, $\mathbf{a} = 6\mathbf{j} + 16\mathbf{k}$, $\|\mathbf{v}\| = \sqrt{265}$, $\mathbf{v} \cdot \mathbf{a} = -238$, $\mathbf{v} \times \mathbf{a} = 144\mathbf{i}$ so $a_T = -238/\sqrt{265}$, $a_N = 144/\sqrt{265}$

(b) $a_T\mathbf{T} = -\dfrac{714}{265}\mathbf{j} + \dfrac{3808}{265}\mathbf{k}$, $a_N\mathbf{N} = \mathbf{a} - a_T\mathbf{T} = \dfrac{2304}{265}\mathbf{j} + \dfrac{432}{265}\mathbf{k}$

(c) $\kappa = \dfrac{144}{265^{3/2}}$

39. $\|\mathbf{v}\| = 4$, $\mathbf{v} \cdot \mathbf{a} = -12$, $\mathbf{v} \times \mathbf{a} = 8\mathbf{k}$ so $a_T = -3$, $a_N = 2$, $\mathbf{T} = -\mathbf{j}$, $\mathbf{N} = (\mathbf{a} - a_T\mathbf{T})/a_N = \mathbf{i}$

40. $\|\mathbf{v}\| = \sqrt{5}$, $\mathbf{v} \cdot \mathbf{a} = 3$, $\mathbf{v} \times \mathbf{a} = -6\mathbf{k}$ so $a_T = 3/\sqrt{5}$, $a_N = 6/\sqrt{5}$, $\mathbf{T} = (1/\sqrt{5})(\mathbf{i} + 2\mathbf{j})$, $\mathbf{N} = (\mathbf{a} - a_T\mathbf{T})/a_N = (1/\sqrt{5})(2\mathbf{i} - \mathbf{j})$

41. $\|\mathbf{v}\| = 3$, $\mathbf{v} \cdot \mathbf{a} = 4$, $\mathbf{v} \times \mathbf{a} = 4\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$ so $a_T = 4/3$, $a_N = \sqrt{29}/3$, $\mathbf{T} = (1/3)(2\mathbf{i} + 2\mathbf{j} + \mathbf{k})$, $\mathbf{N} = (\mathbf{a} - a_T\mathbf{T})/a_N = (\mathbf{i} - 8\mathbf{j} + 14\mathbf{k})/(3\sqrt{29})$

42. $\|\mathbf{v}\| = 5$, $\mathbf{v}\cdot\mathbf{a} = -5$, $\mathbf{v}\times\mathbf{a} = -4\mathbf{i} - 10\mathbf{j} - 3\mathbf{k}$ so $a_T = -1$, $a_N = \sqrt{5}$, $\mathbf{T} = (1/5)(3\mathbf{i} - 4\mathbf{k})$, $\mathbf{N} = (\mathbf{a} - a_T\mathbf{T})/a_N = (8\mathbf{i} - 5\mathbf{j} + 6\mathbf{k})/(5\sqrt{5})$

43. $a_T = \dfrac{d^2s}{dt^2} = \dfrac{d}{dt}\sqrt{3t^2+4} = 3t/\sqrt{3t^2+4}$ so when $t = 2$, $a_T = 3/2$.

44. $a_T = \dfrac{d^2s}{dt^2} = \dfrac{d}{dt}\sqrt{t^2+e^{-3t}} = (2t - 3e^{-3t})/[2\sqrt{t^2+e^{-3t}}]$ so when $t = 0$, $a_T = -3/2$.

45. $a_T = \dfrac{d^2s}{dt^2} = \dfrac{d}{dt}\sqrt{(4t-1)^2+\cos^2\pi t} = [4(4t-1) - \pi\cos\pi t\sin\pi t]/\sqrt{(4t-1)^2+\cos^2\pi t}$ so when $t = 1/4$, $a_T = -\pi/\sqrt{2}$.

46. $a_T = \dfrac{d^2s}{dt^2} = \dfrac{d}{dt}\sqrt{t^4+5t^2+3} = (2t^3+5t)/\sqrt{t^4+5t^2+3}$ so when $t = 1$, $a_T = 7/3$.

47. $a_N = \kappa(ds/dt)^2 = (1/\rho)(ds/dt)^2 = (1/1)(2.9\times 10^5)^2 = 8.41\times 10^{10}$ km/s^2

48. $\mathbf{a} = (d^2s/dt^2)\mathbf{T} + \kappa(ds/dt)^2\mathbf{N}$ where $\kappa = \dfrac{|d^2y/dx^2|}{[1+(dy/dx)^2]^{3/2}}$. If $d^2y/dx^2 = 0$, then $\kappa = 0$ and $\mathbf{a} = (d^2s/dt^2)\mathbf{T}$ so $\mathbf{a}$ is tangent to the curve.

49. $a_N = \kappa(ds/dt)^2 = [2/(1+4x^2)^{3/2}](3)^2 = 18/(1+4x^2)^{3/2}$

50. $y = e^x$, $a_N = \kappa(ds/dt)^2 = [e^x/(1+e^{2x})^{3/2}](2)^2 = 4e^x/(1+e^{2x})^{3/2}$

51. $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$; by the Pythagorean Theorem $a_N = \sqrt{\|\mathbf{a}\|^2 - a_T^2} = \sqrt{9-9} = 0$

52. As in Exercise 51, $\|\mathbf{a}\|^2 = a_T^2 + a_N^2, 81 = 9 + a_N^2, a_N = \sqrt{72} = 6\sqrt{2}$.

53. Let $c = ds/dt, a_N = \kappa\left(\dfrac{ds}{dt}\right)^2, a_N = \dfrac{1}{1000}c^2$, so $c^2 = 1000a_N$, $c \le 10\sqrt{10}\sqrt{1.5} \approx 38.73$ m/s.

54. 10 km/h is the same as $\dfrac{100}{36}$ m/s, so $\|\mathbf{F}\| = 500\dfrac{1}{15}\left(\dfrac{100}{36}\right)^2 \approx 257.20$ N.

55. **(a)** $v_0 = 320$, $\alpha = 60°$, $s_0 = 0$ so $x = 160t$, $y = 160\sqrt{3}t - 16t^2$.

(b) $dy/dt = 160\sqrt{3} - 32t$, $dy/dt = 0$ when $t = 5\sqrt{3}$ so
$y_{\max} = 160\sqrt{3}(5\sqrt{3}) - 16(5\sqrt{3})^2 = 1200$ ft.

(c) $y = 16t(10\sqrt{3} - t)$, $y = 0$ when $t = 0$ or $10\sqrt{3}$ so $x_{\max} = 160(10\sqrt{3}) = 1600\sqrt{3}$ ft.

(d) $\mathbf{v}(t) = 160\mathbf{i} + (160\sqrt{3} - 32t)\mathbf{j}$, $\mathbf{v}(10\sqrt{3}) = 160(\mathbf{i} - \sqrt{3}\mathbf{j})$, $\|\mathbf{v}(10\sqrt{3})\| = 320$ ft/s.

56. **(a)** $v_0 = 980$, $\alpha = 45°$, $s_0 = 0$ so $x = 490\sqrt{2}\,t$, $y = 490\sqrt{2}t - 4.9t^2$

(b) $dy/dt = 490\sqrt{2} - 9.8t$, $dy/dt = 0$ when $t = 50\sqrt{2}$ so
$y_{\max} = 490\sqrt{2}(50\sqrt{2}) - 4.9(50\sqrt{2})^2 = 24,500$ m.

(c) $y = 4.9t(100\sqrt{2} - t)$, $y = 0$ when $t = 0$ or $100\sqrt{2}$ so
$x_{\max} = 490\sqrt{2}(100\sqrt{2}) = 98,000$ m.

(d) $\mathbf{v}(t) = 490\sqrt{2}\,\mathbf{i} + (490\sqrt{2} - 9.8t)\mathbf{j}$, $\mathbf{v}(100\sqrt{2}) = 490\sqrt{2}(\mathbf{i} - \mathbf{j})$, $\|\mathbf{v}(100\sqrt{2})\| = 980$ m/s.

57. $v_0 = 80$, $\alpha = -60°$, $s_0 = 168$ so $x = 40t$, $y = 168 - 40\sqrt{3}\,t - 16t^2$; $y = 0$ when $t = -7\sqrt{3}/2$ (invalid) or $t = \sqrt{3}$ so $x(\sqrt{3}) = 40\sqrt{3}$ ft.

58. $v_0 = 80$, $\alpha = 0°$, $s_0 = 168$ so $x = 80t$, $y = 168 - 16t^2$; $y = 0$ when $t = -\sqrt{42}/2$ (invalid) or $t = \sqrt{42}/2$ so $x(\sqrt{42}/2) = 40\sqrt{42}$ ft.

59. $\alpha = 30°$, $s_0 = 0$ so $x = \sqrt{3}v_0t/2$, $y = v_0t/2 - 16t^2$; $dy/dt = v_0/2 - 32t$, $dy/dt = 0$ when $t = v_0/64$ so $y_{\max} = v_0^2/256 = 2500$, $v_0 = 800$ ft/s.

60. $\alpha = 45°$, $s_0 = 0$ so $x = \sqrt{2}\,v_0t/2$, $y = \sqrt{2}v_0t/2 - 4.9t^2$; $y = 0$ when $t = 0$ or $\sqrt{2}v_0/9.8$ so $x_{\max} = v_0^2/9.8 = 24,500$, $v_0 = 490$ m/s.

61. $v_0 = 800$, $s_0 = 0$ so $x = (800\cos\alpha)t$, $y = (800\sin\alpha)t - 16t^2 = 16t(50\sin\alpha - t)$; $y = 0$ when $t = 0$ or $50\sin\alpha$ so $x_{\max} = 40,000\sin\alpha\cos\alpha = 20,000\sin 2\alpha = 10,000$, $2\alpha = 30°$ or $150°$, $\alpha = 15°$ or $75°$.

62. **(a)** $v_0 = 5$, $\alpha = 0°$, $s_0 = 4$ so $x = 5t$, $y = 4 - 16t^2$; $y = 0$ when $t = -1/2$ (invalid) or $1/2$ so it takes the ball $1/2$ s to hit the floor.

(b) $\mathbf{v}(t) = 5\mathbf{i} - 32t\mathbf{j}$, $\mathbf{v}(1/2) = 5\mathbf{i} - 16\mathbf{j}$, $\|\mathbf{v}(1/2)\| = \sqrt{281}$ so the ball hits the floor with a speed of $\sqrt{281}$ ft/s.

(c) $v_0 = 0$, $\alpha = -90°$, $s_0 = 4$ so $x = 0$, $y = 4 - 16t^2$; $y = 0$ when $t = 1/2$ so both balls would hit the ground at the same instant.

63. **(a)** Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ with $\mathbf{j}$ pointing up. Then $\mathbf{a} = -32\mathbf{j} = x''(t)\mathbf{i} + y''(t)\mathbf{j}$, so $x(t) = At + B, y(t) = -16t^2 + Ct + D$. Next, $x(0) = 0, y(0) = 4$ so $x(t) = At, y(t) = -16t^2 + Ct + 4$; $y'(0)/x'(0) = \tan 60° = \sqrt{3}$, so $C = \sqrt{3}A$; and $40 = v_0 = \sqrt{x'(0)^2 + y'(0)^2} = \sqrt{A^2 + 3A^2}, A = 20$, thus $\mathbf{r}(t) = 20t\,\mathbf{i} + (-16t^2 + 20\sqrt{3}t + 4)\,\mathbf{j}$. When $x = 15$, $t = \dfrac{3}{4}$, and $y = 4 + 20\sqrt{3}\dfrac{3}{4} - 16\left(\dfrac{3}{4}\right)^2 \approx 20.98$ ft, so the water clears the corner point A with 0.98 ft to spare.

(b) $y = 20$ when $-16t^2 + 20\sqrt{3}t - 16 = 0, t = 0.668$ (reject) or $1.497, x(1.497) \approx 29.942$ ft, so the water hits the roof.

(c) about $29.942 - 15 = 14.942$ ft

64. $x = (v_0/2)t, y = 4 + (v_0\sqrt{3}/2)t - 16t^2$, solve $x = 15, y = 20$ simultaneously for v_0 and t,
$v_0/2 = 15/t, t^2 = \dfrac{15}{16}\sqrt{3} - 1, t \approx 0.7898, v_0 \approx 30/0.7898 \approx 37.98$ ft/s.

65. **(a)** $x = (35\sqrt{2}/2)t$, $y = (35\sqrt{2}/2)t - 4.9t^2$, from Exercise 17a in Section 13.5

$$\kappa = \frac{|x'y'' - x''y'|}{[(x')^2 + (y')^2]^{3/2}},\ \kappa(0) = \frac{9.8}{35^2\sqrt{2}} = 0.004\sqrt{2} \approx 0.00565685; \rho = 1/\kappa \approx 176.78 \text{ m}$$

(b) $y'(t) = 0$ when $t = \dfrac{25}{14}\sqrt{2}, y = \dfrac{125}{4}$ m

66. (a) $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}, a_T = \dfrac{d^2s}{dt^2} = -7.5 \text{ ft/s}^2,\ a_N = \kappa\left(\dfrac{ds}{dt}\right)^2 = \dfrac{1}{\rho}(132)^2 = \dfrac{132^2}{3000} \text{ ft/s}^2,$

$$\|\mathbf{a}\| = \sqrt{a_T^2 + a_N^2} = \sqrt{(7.5)^2 + \left(\frac{132^2}{3000}\right)^2} \approx 9.49 \text{ ft/s}^2$$

(b) $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{T}}{\|\mathbf{a}\|\|\mathbf{T}\|} = \dfrac{a_T}{\|\mathbf{a}\|} \approx -\dfrac{7.5}{9.49} \approx -0.79, \theta \approx 2.48 \text{ radians} \approx 142°$

67. $s_0 = 0$ so $x = (v_0\cos\alpha)t,\ y = (v_0\sin\alpha)t - gt^2/2$

(a) $dy/dt = v_0\sin\alpha - gt$ so $dy/dt = 0$ when $t = (v_0\sin\alpha)/g,\ y_{\max} = (v_0\sin\alpha)^2/(2g)$

(b) $y = 0$ when $t = 0$ or $(2v_0\sin\alpha)/g$, so $x = R = (2v_0^2\sin\alpha\cos\alpha)/g = (v_0^2\sin 2\alpha)/g$ when $t = (2v_0\sin\alpha)/g$; R is maximum when $2\alpha = 90°$, $\alpha = 45°$, and the maximum value of R is v_0^2/g.

68. The range is $(v_0^2\sin 2\alpha)/g$ and the maximum range is v_0^2/g so $(v_0^2\sin 2\alpha)/g = (3/4)v_0^2/g$, $\sin 2\alpha = 3/4,\ \alpha = (1/2)\sin^{-1}(3/4) \approx 24.3°$ or $\alpha = (1/2)[180° - \sin^{-1}(3/4)] \approx 65.7°$.

69. $v_0 = 80,\ \alpha = 30°,\ s_0 = 5$ so $x = 40\sqrt{3}t,\ y = 5 + 40t - 16t^2$

(a) $y = 0$ when $t = (-40 \pm \sqrt{(40)^2 - 4(-16)(5)})/(-32) = (5 \pm \sqrt{30})/4$, reject $(5 - \sqrt{30})/4$ to get $t = (5 + \sqrt{30})/4 \approx 2.62$ s.

(b) $x \approx 40\sqrt{3}(2.62) \approx 181.5\,\text{ft}$.

70. (a) $v_0 = v,\ s_0 = h$ so $x = (v\cos\alpha)t,\ y = h + (v\sin\alpha)t - \dfrac{1}{2}gt^2$. If $x = R$, then $(v\cos\alpha)t = R$, $t = \dfrac{R}{v\cos\alpha}$ but $y = 0$ for this value of t so $h + (v\sin\alpha)[R/(v\cos\alpha)] - \dfrac{1}{2}g[R/(v\cos\alpha)]^2 = 0$, $h + (\tan\alpha)R - g(\sec^2\alpha)R^2/(2v^2) = 0,\ g(\sec^2\alpha)R^2 - 2v^2(\tan\alpha)R - 2v^2h = 0$.

(b) $2g\sec^2\alpha\tan\alpha R^2 + 2g\sec^2\alpha R\dfrac{dR}{d\alpha} - 2v^2\sec^2\alpha R - 2v^2\tan\alpha\dfrac{dR}{d\alpha} = 0$; if $\dfrac{dR}{d\alpha} = 0$ and $\alpha = \alpha_0$ when $R = R_0$, then $2g\sec^2\alpha_0\tan\alpha_0 R_0^2 - 2v^2\sec^2\alpha_0 R_0 = 0,\ g\tan\alpha_0 R_0 - v^2 = 0$, $\tan\alpha_0 = v^2/(gR_0)$.

(c) If $\alpha = \alpha_0$ and $R = R_0$, then from Part (a) $g(\sec^2\alpha_0)R_0^2 - 2v^2(\tan\alpha_0)R_0 - 2v^2h = 0$, but from Part (b) $\tan\alpha_0 = v^2/(gR_0)$ so $\sec^2\alpha_0 = 1 + \tan^2\alpha_0 = 1 + v^4/(gR_0)^2$ thus
$g[1 + v^4/(gR_0)^2]R_0^2 - 2v^2[v^2/(gR_0)]R_0 - 2v^2h = 0,\ gR_0^2 - v^4/g - 2v^2h = 0$,
$R_0^2 = v^2(v^2 + 2gh)/g^2,\ R_0 = (v/g)\sqrt{v^2 + 2gh}$ and
$\tan\alpha_0 = v^2/(v\sqrt{v^2 + 2gh}) = v/\sqrt{v^2 + 2gh},\ \alpha_0 = \tan^{-1}(v/\sqrt{v^2 + 2gh})$.

71. (a) $v_0(\cos\alpha)(2.9) = 259\cos 23°$ so $v_0\cos\alpha \approx 82.21061,\ v_0(\sin\alpha)(2.9) - 16(2.9)^2 = -259\sin 23°$ so $v_0\sin\alpha \approx 11.50367$; divide $v_0\sin\alpha$ by $v_0\cos\alpha$ to get $\tan\alpha \approx 0.139929$, thus $\alpha \approx 8°$ and $v_0 \approx 82.21061/\cos 8° \approx 83$ ft/s.

(b) From Part (a), $x \approx 82.21061t$ and $y \approx 11.50367t - 16t^2$ for $0 \le t \le 2.9$; the distance traveled is $\displaystyle\int_0^{2.9}\sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt \approx 268.76$ ft.

EXERCISE SET 13.7

1. The results follow from formulae (1) and (7) of Section 11.6.

2. **(a)** $(r_{\max} - r_{\min})/(r_{\max} + r_{\min}) = 2ae/(2a) = e$

(b) $r_{\max}/r_{\min} = (1+e)/(1-e)$, and the result follows.

3. **(a)** From (15) and (6), at $t = 0$,
$\mathbf{C} = \mathbf{v}_0 \times \mathbf{b}_0 - GM\mathbf{u} = v_0\mathbf{j} \times r_0v_0\mathbf{k} - GM\mathbf{u} = r_0v_0^2\mathbf{i} - GM\mathbf{i} = (r_0v_0^2 - GM)\mathbf{i}$

(b) From (22), $r_0v_0^2 - GM = GMe$, so from (7) and (17), $\mathbf{v} \times \mathbf{b} = GM(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) + GMe\mathbf{i}$, and the result follows.

(c) From (10) it follows that $\mathbf{b}$ is perpendicular to $\mathbf{v}$, and the result follows.

(d) From Part (c) and (10), $\|\mathbf{v} \times \mathbf{b}\| = \|\mathbf{v}\|\|\mathbf{b}\| = vr_0v_0$. From Part (b),
$\|\mathbf{v} \times \mathbf{b}\| = GM\sqrt{(e+\cos\theta)^2 + \sin^2\theta} = GM\sqrt{e^2 + 2e\cos\theta + 1}$. By (10) and Part (c), $\|\mathbf{v} \times \mathbf{b}\| = \|\mathbf{v}\|\|\mathbf{b}\| = v(r_0v_0)$ thus $v = \dfrac{GM}{r_0v_0}\sqrt{e^2 + 2e\cos\theta + 1}$. From (22),
$r_0v_0^2/(GM) = 1 + e$, $GM/(r_0v_0) = v_0/(1+e)$ so $v = \dfrac{v_0}{1+e}\sqrt{e^2 + 2e\cos\theta + 1}$.

4. At the end of the minor axis, $\cos\theta = -c/a = -e$ so
$v = \dfrac{v_0}{1+e}\sqrt{e^2 + 2e(-e) + 1} = \dfrac{v_0}{1+e}\sqrt{1-e^2} = v_0\sqrt{\dfrac{1-e}{1+e}}$.

5. $v_{\max}$ occurs when $\theta = 0$ so $v_{\max} = v_0$; $v_{\min}$ occurs when $\theta = \pi$ so
$v_{\min} = \dfrac{v_0}{1+e}\sqrt{e^2 - 2e + 1} = v_{\max}\dfrac{1-e}{1+e}$, thus $v_{\max} = v_{\min}\dfrac{1+e}{1-e}$.

6. If the orbit is a circle then $e = 0$ so from Part (d) of Exercise 3, $v = v_0$ at all points on the orbit. Use (22) with $e = 0$ to get $v_0 = \sqrt{GM/r_0}$ so $v = \sqrt{GM/r_0}$.

7. $r_0 = 6440 + 200 = 6640$ km so $v = \sqrt{3.99 \times 10^5/6640} \approx 7.75$ km/s.

8. From Example 1, the orbit is 22,250 mi above the Earth, thus $v \approx \sqrt{\dfrac{1.24 \times 10^{12}}{26{,}250}} \approx 6873$ mi/h.

9. From (23) with $r_0 = 6440 + 300 = 6740$ km, $v_{\text{esc}} = \sqrt{\dfrac{2(3.99) \times 10^5}{6740}} \approx 10.88$ km/s.

10. From (29), $T = \dfrac{2\pi}{\sqrt{GM}}a^{3/2}$. But $T = 1$ yr $= 365 \cdot 24 \cdot 3600$ s, thus $M = \dfrac{4\pi^2a^3}{GT^2} \approx 1.99 \times 10^{30}$ kg.

11. **(a)** At perigee, $r = r_{\min} = a(1-e) = 238{,}900\ (1 - 0.055) \approx 225{,}760$ mi; at apogee,
$r = r_{\max} = a(1+e) = 238{,}900(1 + 0.055) \approx 252{,}040$ mi. Subtract the sum of the radius of the Moon and the radius of the Earth to get
minimum distance $= 225{,}760 - 5080 = 220{,}680$ mi,
and maximum distance $= 252{,}040 - 5080 = 246{,}960$ mi.

(b) $T = 2\pi\sqrt{a^3/(GM)} = 2\pi\sqrt{(238{,}900)^3/(1.24 \times 10^{12})} \approx 659$ hr ≈ 27.5 days.

12. **(a)** $r_{\min} = 6440 + 649 = 7{,}089$ km, $r_{\max} = 6440 + 4{,}340 = 10{,}780$ km so $a = (r_{\min} + r_{\max})/2 = 8934.5$ km.

(b) $e = (10{,}780 - 7{,}089)/(10{,}780 + 7{,}089) \approx 0.207$.

(c) $T = 2\pi\sqrt{a^3/(GM)} = 2\pi\sqrt{(8934.5)^3/(3.99 \times 10^5)} \approx 8400 \text{ s} \approx 140 \text{ min}$

13. **(a)** $r_0 = 4000 + 180 = 4180$ mi, $v = \sqrt{\dfrac{GM}{r_0}} = \sqrt{1.24 \times 10^{12}/4180} \approx 17{,}224$ mi/h

(b) $r_0 = 4180$ mi, $v_0 = \sqrt{\dfrac{GM}{r_0}} + 600$; $e = \dfrac{r_0 v_0^2}{GM} - 1 = 1200\sqrt{\dfrac{r_0}{GM}} + (600)^2\dfrac{r_0}{GM} \approx 0.071$; $r_{\max} = 4180(1 + 0.071)/(1 - 0.071) \approx 4819$ mi; the apogee altitude is $4819 - 4000 = 819$ mi.

14. By equation (20), $r = \dfrac{k}{1 + e\cos\theta}$, where $k > 0$. By assumption, r is minimal when $\theta = 0$, hence $e \geq 0$.

CHAPTER 13 SUPPLEMENTARY EXERCISES

2. **(a)** the line through the tips of $\mathbf{r}_0$ and $\mathbf{r}_1$

(b) the line segment connecting the tips of $\mathbf{r}_0$ and $\mathbf{r}_1$

(c) the line through the tip of $\mathbf{r}_0$ which is parallel to $\mathbf{r}'(t_0)$

4. **(a)** speed **(b)** distance traveled **(c)** distance of the particle from the origin

7. **(a)** $\mathbf{r}(t) = \displaystyle\int_0^t \cos\left(\frac{\pi u^2}{2}\right) du\,\mathbf{i} + \int_0^t \sin\left(\frac{\pi u^2}{2}\right) du\,\mathbf{j}$;

$$\left\|\frac{d\mathbf{r}}{dt}\right\|^2 = x'(t)^2 + y'(t)^2 = \cos^2\left(\frac{\pi t^2}{2}\right) + \sin^2\left(\frac{\pi t^2}{2}\right) = 1 \text{ and } \mathbf{r}(0) = \mathbf{0}$$

(b) $\mathbf{r}'(s) = \cos\left(\dfrac{\pi s^2}{2}\right)\mathbf{i} + \sin\left(\dfrac{\pi s^2}{2}\right)\mathbf{j}, \mathbf{r}''(s) = -\pi s\sin\left(\dfrac{\pi s^2}{2}\right)\mathbf{i} + \pi s\cos\left(\dfrac{\pi s^2}{2}\right)\mathbf{j}$, $\kappa = \|\mathbf{r}''(s)\| = \pi|s|$

(c) $\kappa(s) \to +\infty$, so the spiral winds ever tighter.

8. **(a)** The tangent vector to the curve is always tangent to the sphere.

(b) $\|\mathbf{v}\| = \text{const}$, so $\mathbf{v} \cdot \mathbf{a} = 0$; the acceleration vector is always perpendicular to the velocity vector

(c) $\|\mathbf{r}(t)\|^2 = \left(1 - \dfrac{1}{4}\cos^2 t\right)(\cos^2 t + \sin^2 t) + \dfrac{1}{4}\cos^2 t = 1$

9. **(a)** $\|\mathbf{r}(t)\| = 1$, so by Theorem 13.2.9, $\mathbf{r}'(t)$ is always perpendicular to the vector $\mathbf{r}(t)$. Then $\mathbf{v}(t) = R\omega(-\sin\omega t\mathbf{i} + \cos\omega t\mathbf{j}), v = \|\mathbf{v}(t)\| = R\omega$

(b) $\mathbf{a} = -R\omega^2(\cos\omega t\mathbf{i} + \sin\omega t\mathbf{j}), a = \|\mathbf{a}\| = R\omega^2$, and $\mathbf{a} = -\omega^2\mathbf{r}$ is directed toward the origin.

(c) The smallest value of t for which $\mathbf{r}(t) = \mathbf{r}(0)$ satisfies $\omega t = 2\pi$, so $T = t = \dfrac{2\pi}{\omega}$.

10. **(a)** $F = \|\mathbf{F}\| = m\|\mathbf{a}\| = mR\omega^2 = mR\dfrac{v^2}{R^2} = \dfrac{mv^2}{R}$

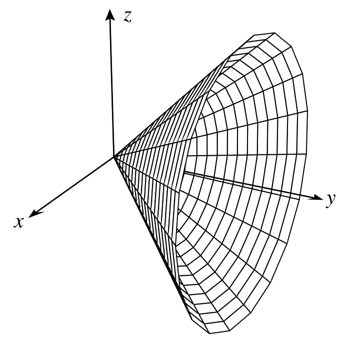
(b) $R = 6440 + 3200 = 9640$ km, $6.43 = v = R\omega = 9640\omega,\ \omega = \dfrac{6.43}{9640} \approx 0.000667,$

$a = R\omega^2 = v\omega = \dfrac{6.43^2}{9640} \approx 0.00429\ \text{km/s}^2$

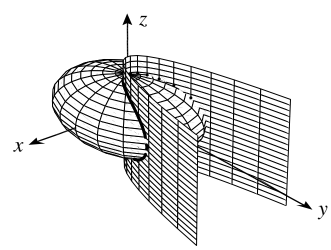
$\mathbf{a} = -a(\cos\omega t\mathbf{i} + \sin\omega t\mathbf{j}) \approx -0.00429[\cos(0.000667t)\mathbf{i} + \sin(0.000667t)\mathbf{j}]$

(c) $F = ma \approx 70(0.00429)\ \text{kg} \cdot \text{km/s}^2 \approx 0.30030\ \text{kN} = 300.30\ \text{N}$

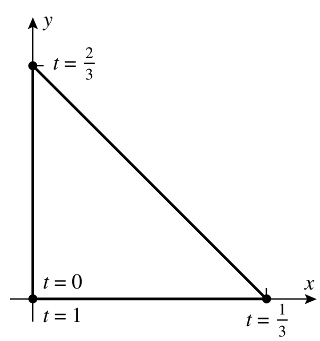
11. **(a)** Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then $x^2 + z^2 = t^2(\sin^2 \pi t + \cos^2 \pi t) = t^2 = y^2$



(b) Let $x = t$, then $y = t^2, z = \pm\sqrt{4 - t^2/3 - t^4/6}$



12.



13. **(a)** $\|\mathbf{e}_r(t)\|^2 = \cos^2\theta + \sin^2\theta = 1$, so $\mathbf{e}_r(t)$ is a unit vector; $\mathbf{r}(t) = r(t)\mathbf{e}(t)$, so they have the same direction if $r(t) > 0$, opposite if $r(t) < 0$. $\mathbf{e}_\theta(t)$ is perpendicular to $\mathbf{e}_r(t)$ since $\mathbf{e}_r(t) \cdot \mathbf{e}_\theta(t) = 0$, and it will result from a counterclockwise rotation of $\mathbf{e}_r(t)$ provided $\mathbf{e}(t) \times \mathbf{e}_\theta(t) = \mathbf{k}$, which is true.

(b) $\frac{d}{dt}\mathbf{e}_r(t) = \frac{d\theta}{dt}(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) = \frac{d\theta}{dt}\mathbf{e}_\theta(t)$ and $\frac{d}{dt}\mathbf{e}_\theta(t) = -\frac{d\theta}{dt}(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) = -\frac{d\theta}{dt}\mathbf{e}_r(t)$, so

$\mathbf{v}(t) = \frac{d}{dt}\mathbf{r}(t) = \frac{d}{dt}(r(t)\mathbf{e}_r(t)) = r'(t)\mathbf{e}_r(t) + r(t)\frac{d\theta}{dt}\mathbf{e}_\theta(t)$

(c) From Part (b), $\mathbf{a} = \frac{d}{dt}\mathbf{v}(t)$

$$= r''(t)\mathbf{e}_r(t) + r'(t)\frac{d\theta}{dt}\mathbf{e}_\theta(t) + r'(t)\frac{d\theta}{dt}\mathbf{e}_\theta(t) + r(t)\frac{d^2\theta}{dt^2}\mathbf{e}_\theta(t) - r(t)\left(\frac{d\theta}{dt}\right)^2\mathbf{e}_r(t)$$

$$= \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{e}_r(t) + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]\mathbf{e}_\theta(t)$$

14. The height $y(t)$ of the rocket satisfies $\tan\theta = y/b, y = b\tan\theta, v = \frac{dy}{dt} = \frac{dy}{d\theta}\frac{d\theta}{dt} = b\sec^2\theta\frac{d\theta}{dt}$.

15. $\mathbf{r} = \mathbf{r}_0 + t\,\overrightarrow{PQ} = (t-1)\mathbf{i} + (4-2t)\mathbf{j} + (3+2t)\mathbf{k}; \left\|\frac{d\mathbf{r}}{dt}\right\| = 3, \mathbf{r}(s) = \frac{s-3}{3}\mathbf{i} + \frac{12-2s}{3}\mathbf{j} + \frac{9+2s}{3}\mathbf{k}$

16. By equation (26) of Section 13.6, $\mathbf{r}(t) = (60\cos\alpha)t\mathbf{i} + ((60\sin\alpha)t - 16t^2 + 4)\mathbf{j}$, and the maximum height of the baseball occurs when $y'(t) = 0$, $60\sin\alpha = 32t$, $t = \frac{15}{8}\sin\alpha$, so the ball clears the ceiling if $y_{\max} = (60\sin\alpha)\frac{15}{8}\sin\alpha - 16\frac{15^2}{8^2}\sin^2\alpha + 4 \le 25, \frac{15^2\sin^2\alpha}{4} \le 21, \sin^2\alpha \le \frac{28}{75}$. The ball hits the wall when $x = 60, t = \sec\alpha$, and $y(\sec\alpha) = 60\sin\alpha\sec\alpha - 16\sec^2\alpha + 4$. Maximize the height $h(\alpha) = y(\sec\alpha) = 60\tan\alpha - 16\sec^2\alpha + 4$, subject to the constraint $\sin^2\alpha \le \frac{28}{75}$. Then $h'(\alpha) = 60\sec^2\alpha - 32\sec^2\alpha\tan\alpha = 0, \tan\alpha = \frac{60}{32} = \frac{15}{8}$, so $\sin\alpha = \frac{15}{\sqrt{8^2+15^2}} = \frac{15}{17}$, but for this value of α the constraint is not satisfied (the ball hits the ceiling). Hence the maximum value of h occurs at one of the endpoints of the α-interval on which the ball clears the ceiling, i.e. $\left[0, \sin^{-1}\sqrt{28/75}\right]$. Since $h'(0) = 60$, it follows that h is increasing throughout the interval, since $h' > 0$ inside the interval. Thus $h_{\max}$ occurs when $\sin^2\alpha = \frac{28}{75}$, $h_{\max} = 60\tan\alpha - 16\sec^2\alpha + 4 = 60\frac{\sqrt{28}}{\sqrt{47}} - 16\frac{75}{47} + 4 = \frac{120\sqrt{329} - 1012}{47} \approx 24.78$ ft. Note: the possibility that the baseball keeps climbing until it hits the wall can be rejected as follows: if so, then $y'(t) = 0$ after the ball hits the wall, i.e. $t = \frac{15}{8}\sin\alpha$ occurs after $t = \sec\alpha$, hence $\frac{15}{8}\sin\alpha \ge \sec\alpha$, $15\sin\alpha\cos\alpha \ge 8$, $15\sin 2\alpha \ge 16$, impossible.

17. $\mathbf{r}'(1) = 3\mathbf{i} + 10\mathbf{j} + 10\mathbf{k}$, so if $\mathbf{r}'(t) = 3t^2\mathbf{i} + 10\mathbf{j} + 10t\mathbf{k}$ is perpendicular to $\mathbf{r}'(1)$, then
$9t^2 + 100 + 100t = 0, t = -10, -10/9$,
so $\mathbf{r} = -1000\mathbf{i} - 100\mathbf{j} + 500\mathbf{k}, -(1000/729)\mathbf{i} - (100/9)\mathbf{j} + (500/81)\mathbf{k}$.

18. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, then $\frac{dx}{dt} = x(t), \frac{dy}{dt} = y(t), x(0) = x_0, y(0) = y_0$, so
$x(t) = x_0e^t, y(t) = y_0e^t$, $\mathbf{r}(t) = e^t\mathbf{r}_0$. If $\mathbf{r}(t)$ is a vector in 3-space then an analogous solution holds.

19. **(a)** $\dfrac{d\mathbf{v}}{dt} = 2t^2\mathbf{i} + \mathbf{j} + \cos 2t\mathbf{k}, \mathbf{v}_0 = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, so $x'(t) = \dfrac{2}{3}t^3 + 1, y'(t) = t + 2, z'(t) = \dfrac{1}{2}\sin 2t - 1$,

$x(t) = \dfrac{1}{6}t^4 + t, y(t) = \dfrac{1}{2}t^2 + 2t, z(t) = -\dfrac{1}{4}\cos 2t - t + \dfrac{1}{4}$, since $\mathbf{r}(0) = \mathbf{0}$. Hence

$$\mathbf{r}(t) = \left(\frac{1}{6}t^4 + t\right)\mathbf{i} + \left(\frac{1}{2}t^2 + 2t\right)\mathbf{j} - \left(\frac{1}{4}\cos 2t + t - \frac{1}{4}\right)\mathbf{k}$$

(b) $\left.\dfrac{ds}{dt}\right]_{t=1} = \|\mathbf{r}'(t)\|\Big]_{t=1} \sqrt{(5/3)^2 + 9 + (1 - (\sin 2)/2)^2} \approx 3.475$

20. $\|\mathbf{v}\|^2 = \mathbf{v}(t) \cdot \mathbf{v}(t),\ 2\|\mathbf{v}\|\dfrac{d}{dt}\|\mathbf{v}\| = 2\mathbf{v} \cdot \mathbf{a},\ \dfrac{d}{dt}(\|\mathbf{v}\|) = \dfrac{1}{\|\mathbf{v}\|}(\mathbf{v} \cdot \mathbf{a})$