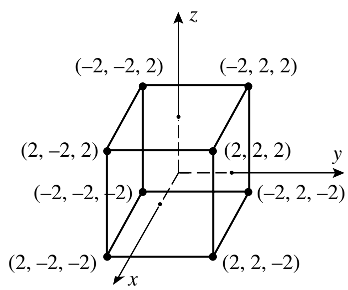
CHAPTER 12
Three-Dimensional Space; Vectors

EXERCISE SET 12.1

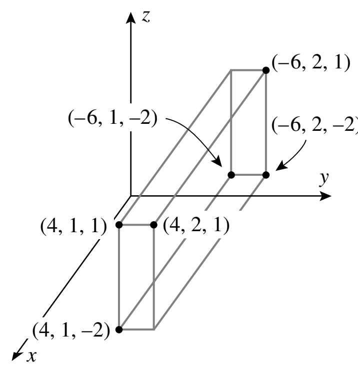
1. (a) $(0,0,0),(3,0,0),(3,5,0),(0,5,0),(0,0,4),(3,0,4),(3,5,4),(0,5,4)$

(b) $(0,1,0),(4,1,0),(4,6,0),(0,6,0),(0,1,-2),(4,1,-2),(4,6,-2),(0,6,-2)$

2. corners: $(2,2,\pm 2)$, $(2,-2,\pm 2)$, $(-2,2,\pm 2)$, $(-2,-2,\pm 2)$



3. corners: $(4,2,-2)$, $(4,2,1)$, $(4,1,1)$, $(4,1,-2)$, $(-6,1,1)$, $(-6,2,1)$, $(-6,2,-2)$, $(-6,1,-2)$



4. (a) $(x_2,y_1,z_1),(x_2,y_2,z_1),(x_1,y_2,z_1)(x_1,y_1,z_2),(x_2,y_1,z_2),(x_1,y_2,z_2)$

(b) The midpoint of the diagonal has coordinates which are the coordinates of the midpoints of the edges. The midpoint of the edge (x_1,y_1,z_1) and (x_2,y_1,z_1) is $\left(\frac{1}{2}(x_1+x_2),y_1,z_1\right)$; the midpoint of the edge (x_2,y_1,z_1) and (x_2,y_2,z_1) is $\left(x_2,\frac{1}{2}(y_1+y_2),z_1\right)$; the midpoint of the edge (x_2,y_2,z_1) and $(x_2,y_2,z_2))$ is $\left(x_2,y_2,\frac{1}{2}(z_1+z_2)\right)$. Thus the coordinates of the midpoint of the diagonal are $\frac{1}{2}(x_1+x_2),\frac{1}{2}(y_1+y_2),\frac{1}{2}(z_1+z_2)$.

5. The diameter is $d=\sqrt{(1-3)^2+(-2-4)^2+(4+12)^2}=\sqrt{296}$, so the radius is $\sqrt{296}/2=\sqrt{74}$. The midpoint $(2,1,-4)$ of the endpoints of the diameter is the center of the sphere.

6. Each side has length $\sqrt{14}$ so the triangle is equilateral.

7. (a) The sides have lengths 7, 14, and $7\sqrt{5}$; it is a right triangle because the sides satisfy the Pythagorean theorem, $(7\sqrt{5})^2=7^2+14^2$.

(b) (2,1,6) is the vertex of the 90° angle because it is opposite the longest side (the hypotenuse).

(c) area = (1/2)(altitude)(base) = (1/2)(7)(14) = 49

8. (a) 3 **(b)** 2 **(c)** 5

(d) $\sqrt{(2)^2+(-3)^2}=\sqrt{13}$ **(e)** $\sqrt{(-5)^2+(-3)^2}=\sqrt{34}$ **(f)** $\sqrt{(-5)^2+(2)^2}=\sqrt{29}$

9. (a) $(x-1)^2+y^2+(z+1)^2=16$

(b) $r=\sqrt{(-1-0)^2+(3-0)^2+(2-0)^2}=\sqrt{14}$, $(x+1)^2+(y-3)^2+(z-2)^2=14$

(c) $r = \dfrac{1}{2}\sqrt{(-1-0)^2 + (2-2)^2 + (1-3)^2} = \dfrac{1}{2}\sqrt{5}$, center $(-1/2, 2, 2)$,
$(x+1/2)^2 + (y-2)^2 + (z-2)^2 = 5/4$

10. $r = |[\text{distance between } (0{,}0{,}0) \text{ and } (3,-2,4)] \pm 1| = \sqrt{29} \pm 1$,
$x^2 + y^2 + z^2 = r^2 = \left(\sqrt{29} \pm 1\right)^2 = 30 \pm 2\sqrt{29}$

11. $(x-2)^2 + (y+1)^2 + (z+3)^2 = r^2$,
(a) $r^2 = 3^2 = 9$ **(b)** $r^2 = 1^2 = 1$ **(c)** $r^2 = 2^2 = 4$

12. **(a)** The sides have length 1, so the radius is $\dfrac{1}{2}$; hence $(x+2)^2 + (y-1)^2 + (z-3)^2 = \dfrac{1}{4}$
(b) The diagonal has length $\sqrt{1+1+1} = \sqrt{3}$ and is a diameter, so $(x+2)^2 + (y-1)^2 + (z-3)^2 = \dfrac{3}{4}$.

13. $(x+5)^2 + (y+2)^2 + (z+1)^2 = 49$; sphere, $C(-5,-2,-1)$, $r = 7$

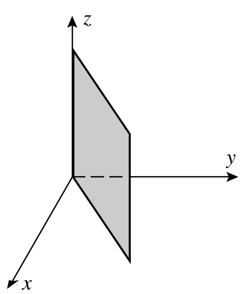
14. $x^2 + (y-1/2)^2 + z^2 = 1/4$; sphere, $C(0, 1/2, 0)$, $r = 1/2$

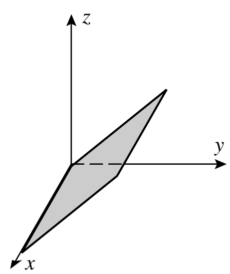
15. $(x-1/2)^2 + (y-3/4)^2 + (z+5/4)^2 = 54/16$; sphere, $C(1/2, 3/4, -5/4)$, $r = 3\sqrt{6}/4$

16. $(x+1)^2 + (y-1)^2 + (z+1)^2 = 0$; the point $(-1, 1, -1)$

17. $(x-3/2)^2 + (y+2)^2 + (z-4)^2 = -11/4$; no graph

18. $(x-1)^2 + (y-3)^2 + (z-4)^2 = 25$; sphere, $C(1,3,4)$, $r = 5$

19. **(a)**


(b)


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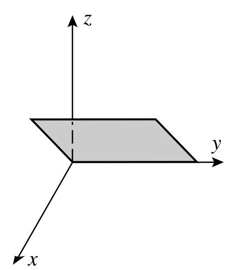
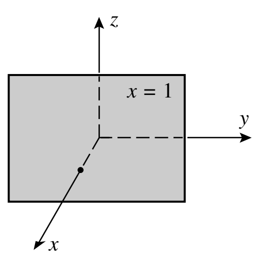


20. **(a)**


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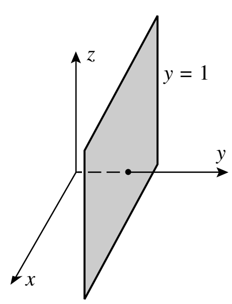
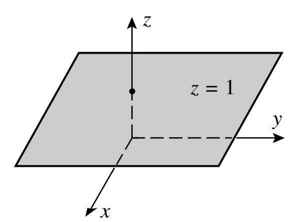
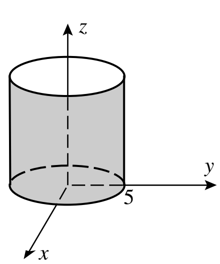


(c)


21. **(a)**


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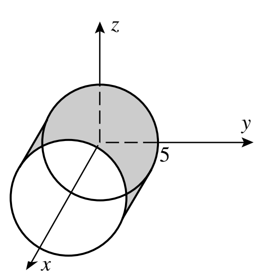
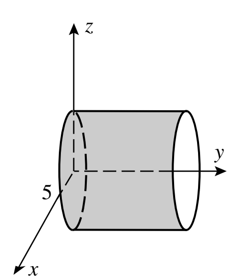
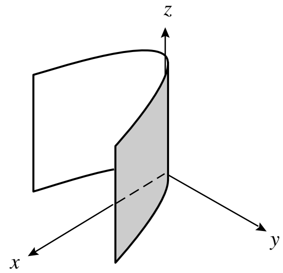
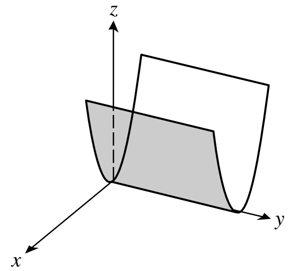


(c)


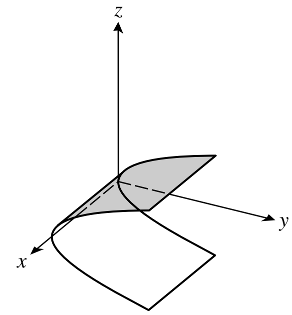
22. **(a)**



(b)



(c)



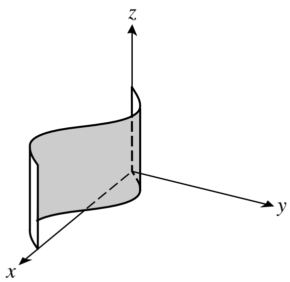
23. **(a)** $-2y + z = 0$ **(b)** $-2x + z = 0$

(c) $(x-1)^2 + (y-1)^2 = 1$ **(d)** $(x-1)^2 + (z-1)^2 = 1$

24. **(a)** $(x-a)^2 + (z-a)^2 = a^2$ **(b)** $(x-a)^2 + (y-a)^2 = a^2$

(c) $(y-a)^2 + (z-a)^2 = a^2$

25.



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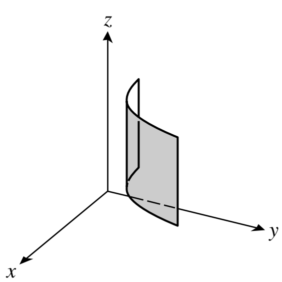
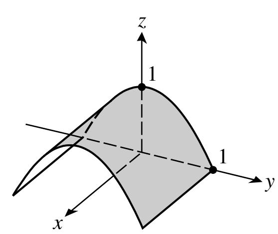


27.



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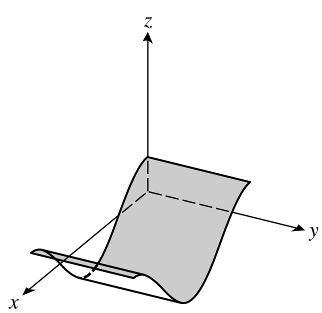

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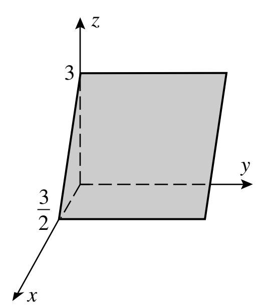

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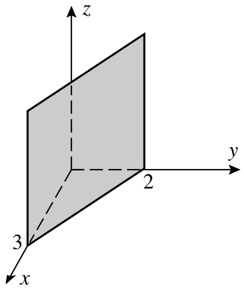

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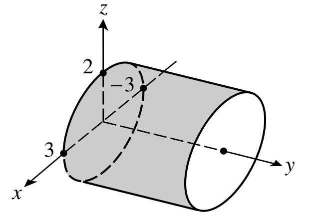
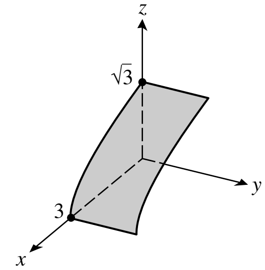


32.



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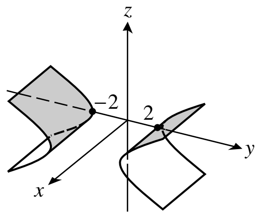
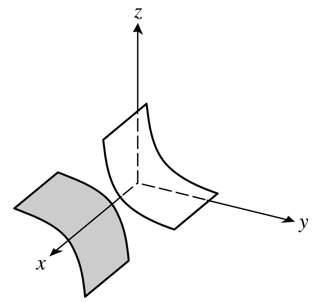
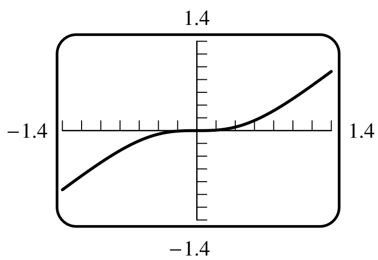


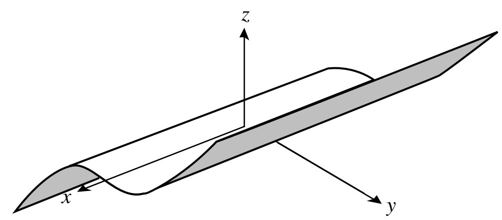
34.



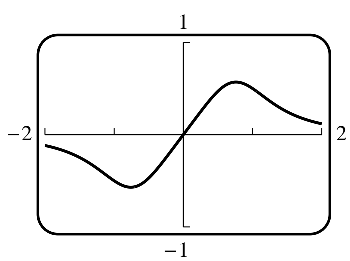
35. **(a)**



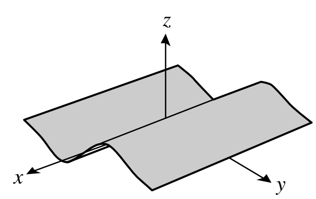
(b)



36. **(a)**



(b)



37. Complete the square to get $(x+1)^2+(y-1)^2+(z-2)^2=9$; center $(-1,1,2)$, radius 3. The distance between the origin and the center is $\sqrt{6}<3$ so the origin is inside the sphere. The largest distance is $3+\sqrt{6}$, the smallest is $3-\sqrt{6}$.

38. $(x-1)^2+y^2+(z+4)^2 \le 25$; all points on and inside the sphere of radius 5 with center at $(1,0,-4)$.

39. $(y+3)^2+(z-2)^2>16$; all points outside the circular cylinder $(y+3)^2+(z-2)^2=16$.

40. $\sqrt{(x-1)^2+(y+2)^2+z^2}=2\sqrt{x^2+(y-1)^2+(z-1)^2}$, square and simplify to get $3x^2+3y^2+3z^2+2x-12y-8z+3=0$, then complete the square to get $(x+1/3)^2+(y-2)^2+(z-4/3)^2=44/9$; center $(-1/3,2,4/3)$, radius $2\sqrt{11}/3$.

41. Let r be the radius of a styrofoam sphere. The distance from the origin to the center of the bowling ball is equal to the sum of the distance from the origin to the center of the styrofoam sphere nearest the origin and the distance between the center of this sphere and the center of the bowling ball so $\sqrt{3}R = \sqrt{3}r + r + R$, $(\sqrt{3}+1)r = (\sqrt{3}-1)R$, $r = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}R = (2-\sqrt{3})R$.

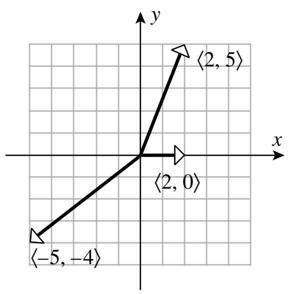
42. **(a)** Complete the square to get $(x+G/2)^2 + (y+H/2)^2 + (z+I/2)^2 = K/4$, so the equation represents a sphere when $K > 0$, a point when $K = 0$, and no graph when $K < 0$.

(b) $C(-G/2, -H/2, -I/2)$, $r = \sqrt{K}/2$

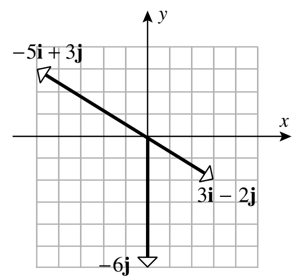
43.
$$\begin{aligned}(a\sin\phi\cos\theta)^2 + (a\sin\phi\sin\theta)^2 + (a\cos\phi)^2 &= a^2\sin^2\phi\cos^2\theta + a^2\sin^2\phi\sin^2\theta + a^2\cos^2\phi \\ &= a^2\sin^2\phi(\cos^2\theta + \sin^2\theta) + a^2\cos^2\phi \\ &= a^2\sin^2\phi + a^2\cos^2\phi = a^2(\sin^2\phi + \cos^2\phi) = a^2\end{aligned}$$

EXERCISE SET 12.2

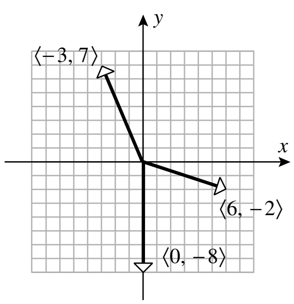
1. **(a–c)**



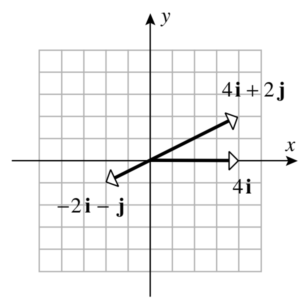
(d–f)



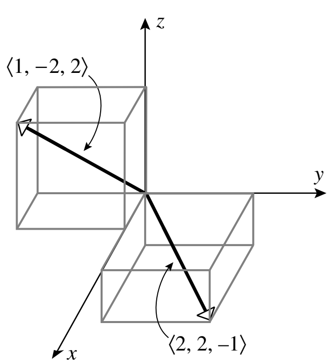
2. **(a–c)**



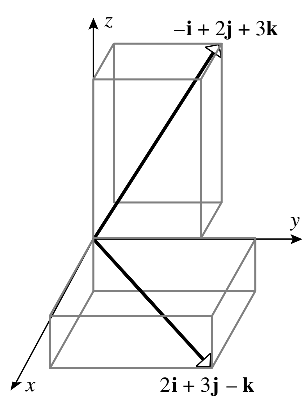
(d–f)



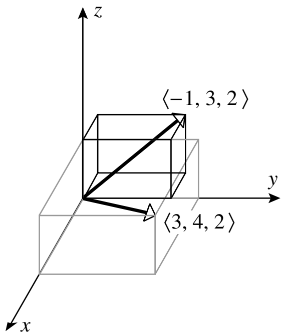
3. **(a–b)**



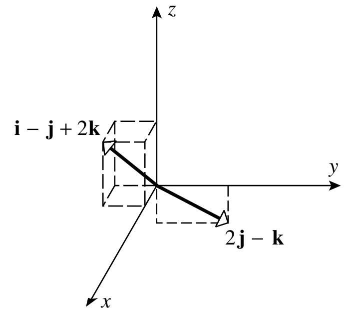
(c–d)



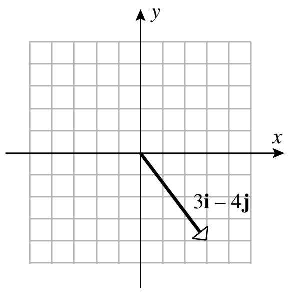
4. **(a–b)**



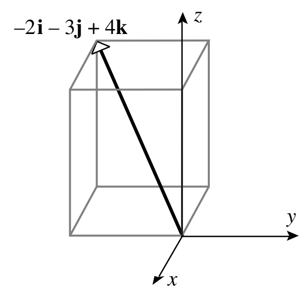
(c–d)



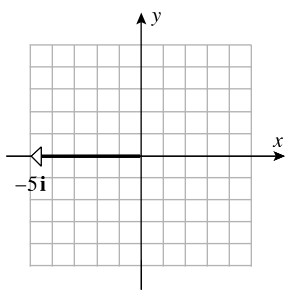
5. **(a)** $\langle 4-1, 1-5\rangle = \langle 3, -4\rangle$



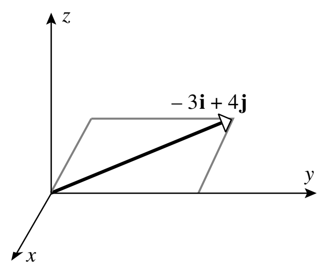
(b) $\langle 0-2, 0-3, 4-0\rangle = \langle -2, -3, 4\rangle$



6. **(a)** $\langle -3-2, 3-3\rangle = \langle -5, 0\rangle$



(b) $\langle 0-3, 4-0, 4-4\rangle = \langle -3, 4, 0\rangle$



7. **(a)** $\langle 2-3,\ 8-5\rangle = \langle -1, 3\rangle$
(b) $\langle 0-7,\ 0-(-2)\rangle = \langle -7, 2\rangle$
(c) $\langle -3, 6, 1\rangle$

8. **(a)** $\langle -4-(-6),\ -1-(-2)\rangle = \langle 2, 1\rangle$
(b) $\langle -1, 6, 1\rangle$
(c) $\langle 5, 0, 0\rangle$

9. **(a)** Let (x, y) be the terminal point, then $x - 1 = 3$, $x = 4$ and $y - (-2) = -2$, $y = -4$. The terminal point is $(4, -4)$.
(b) Let (x, y, z) be the initial point, then $5 - x = -3$, $-y = 1$, and $-1 - z = 2$ so $x = 8$, $y = -1$, and $z = -3$. The initial point is $(8, -1, -3)$.

10. **(a)** Let (x, y) be the terminal point, then $x - 2 = 7$, $x = 9$ and $y - (-1) = 6$, $y = 5$. The terminal point is (9,5).
(b) Let (x, y, z) be the terminal point, then $x + 2 = 1$, $y - 1 = 2$, and $z - 4 = -3$ so $x = -1$, $y = 3$, and $z = 1$. The terminal point is $(-1, 3, 1)$.

11. **(a)** $-\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ **(b)** $18\mathbf{i} + 12\mathbf{j} - 6\mathbf{k}$ **(c)** $-\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}$
(d) $40\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}$ **(e)** $-2\mathbf{i} - 16\mathbf{j} - 18\mathbf{k}$ **(f)** $-\mathbf{i} + 13\mathbf{j} - 2\mathbf{k}$

12. **(a)** $\langle 1,-2,0\rangle$ **(b)** $\langle 28,0,-14\rangle + \langle 3,3,9\rangle = \langle 31,3,-5\rangle$
(c) $\langle 3,-1,-5\rangle$ **(d)** $3(\langle 2,-1,3\rangle - \langle 28,0,-14\rangle) = 3\langle -26,-1,17\rangle = \langle -78,-3,51\rangle$
(e) $\langle -12,0,6\rangle - \langle 8,8,24\rangle = \langle -20,-8,-18\rangle$
(f) $\langle 8,0,-4\rangle - \langle 3,0,6\rangle = \langle 5,0,-10\rangle$

13. **(a)** $\|\mathbf{v}\| = \sqrt{1+1} = \sqrt{2}$ **(b)** $\|\mathbf{v}\| = \sqrt{1+49} = 5\sqrt{2}$
(c) $\|\mathbf{v}\| = \sqrt{21}$ **(d)** $\|\mathbf{v}\| = \sqrt{14}$

14. **(a)** $\|\mathbf{v}\| = \sqrt{9+16} = 5$ **(b)** $\|\mathbf{v}\| = \sqrt{2+7} = 3$
(c) $\|\mathbf{v}\| = 3$ **(d)** $\|\mathbf{v}\| = \sqrt{3}$

15. **(a)** $\|\mathbf{u}+\mathbf{v}\| = \|2\mathbf{i}-2\mathbf{j}+2\mathbf{k}\| = 2\sqrt{3}$ **(b)** $\|\mathbf{u}\| + \|\mathbf{v}\| = \sqrt{14}+\sqrt{2}$
(c) $\|-2\mathbf{u}\| + 2\|\mathbf{v}\| = 2\sqrt{14}+2\sqrt{2}$ **(d)** $\|3\mathbf{u}-5\mathbf{v}+\mathbf{w}\| = \|-12\mathbf{j}+2\mathbf{k}\| = 2\sqrt{37}$
(e) $(1/\sqrt{6})\mathbf{i} + (1/\sqrt{6})\mathbf{j} - (2/\sqrt{6})\mathbf{k}$ **(f)** 1

16. If one vector is a positive multiple of the other, say $\mathbf{u} = \alpha\mathbf{v}$ with $\alpha > 0$, then $\mathbf{u}, \mathbf{v}$ and $\mathbf{u}+\mathbf{v}$ are parallel and $\|\mathbf{u}+\mathbf{v}\| = (1+\alpha)\|\mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$.

17. **(a)** $\|-\mathbf{i}+4\mathbf{j}\| = \sqrt{17}$ so the required vector is $\left(-1/\sqrt{17}\right)\mathbf{i} + \left(4/\sqrt{17}\right)\mathbf{j}$
(b) $\|6\mathbf{i}-4\mathbf{j}+2\mathbf{k}\| = 2\sqrt{14}$ so the required vector is $(-3\mathbf{i}+2\mathbf{j}-\mathbf{k})/\sqrt{14}$
(c) $\overrightarrow{AB} = 4\mathbf{i}+\mathbf{j}-\mathbf{k}$, $\|\overrightarrow{AB}\| = 3\sqrt{2}$ so the required vector is $(4\mathbf{i}+\mathbf{j}-\mathbf{k})/\left(3\sqrt{2}\right)$

18. **(a)** $\|3\mathbf{i}-4\mathbf{j}\| = 5$ so the required vector is $-\frac{1}{5}(3\mathbf{i}-4\mathbf{j}) = -\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$
(b) $\|2\mathbf{i}-\mathbf{j}-2\mathbf{k}\| = 3$ so the required vector is $\frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$
(c) $\overrightarrow{AB} = 4\mathbf{i}-3\mathbf{j}$, $\|\overrightarrow{AB}\| = 5$ so the required vector is $\frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}$

19. **(a)** $-\frac{1}{2}\mathbf{v} = \langle -3/2, 2\rangle$ **(b)** $\|\mathbf{v}\| = \sqrt{85}$, so $\frac{\sqrt{17}}{\sqrt{85}}\mathbf{v} = \frac{1}{\sqrt{5}}\langle 7,0,-6\rangle$ has length $\sqrt{17}$

20. **(a)** $3\mathbf{v} = -6\mathbf{i}+9\mathbf{j}$ **(b)** $-\frac{2}{\|v\|}\mathbf{v} = \frac{6}{\sqrt{26}}\mathbf{i} - \frac{8}{\sqrt{26}}\mathbf{j} - \frac{2}{\sqrt{26}}\mathbf{k}$

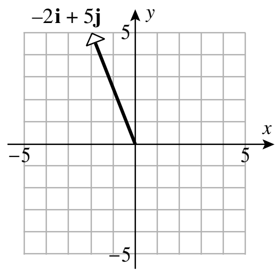
21. **(a)** $\mathbf{v} = \|\mathbf{v}\|\langle\cos(\pi/4), \sin(\pi/4)\rangle = \langle 3\sqrt{2}/2, 3\sqrt{2}/2\rangle$
(b) $\mathbf{v} = \|\mathbf{v}\|\langle\cos 90^\circ, \sin 90^\circ\rangle = \langle 0,2\rangle$
(c) $\mathbf{v} = \|\mathbf{v}\|\langle\cos 120^\circ, \sin 120^\circ\rangle = \langle -5/2, 5\sqrt{3}/2\rangle$
(d) $\mathbf{v} = \|\mathbf{v}\|\langle\cos\pi, \sin\pi\rangle = \langle -1,0\rangle$

22. From (12), $\mathbf{v} = \langle\cos(\pi/6), \sin(\pi/6)\rangle = \langle\sqrt{3}/2, 1/2\rangle$ and $\mathbf{w} = \langle\cos(3\pi/4), \sin(3\pi/4)\rangle = \langle -\sqrt{2}/2, \sqrt{2}/2\rangle$, so $\mathbf{v}+\mathbf{w} = ((\sqrt{3}-\sqrt{2})/2, (1+\sqrt{2})/2$, $\mathbf{v}-\mathbf{w} = ((\sqrt{3}+\sqrt{2})/2, (1-\sqrt{2})/2)$

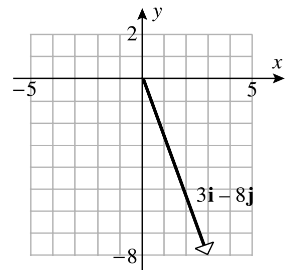
23. From (12), $\mathbf{v} = \langle\cos 30^\circ, \sin 30^\circ\rangle = \langle\sqrt{3}/2, 1/2\rangle$ and $\mathbf{w} = \langle\cos 135^\circ, \sin 135^\circ\rangle = \langle -\sqrt{2}/2, \sqrt{2}/2\rangle$, so $\mathbf{v}+\mathbf{w} = ((\sqrt{3}-\sqrt{2})/2, (1+\sqrt{2})/2)$

24. $\mathbf{w} = \langle 1,0\rangle$, and from (12), $\mathbf{v} = \langle\cos 120^\circ, \sin 120^\circ\rangle = \langle -1/2, \sqrt{3}/2\rangle$, so $\mathbf{v}+\mathbf{w} = \langle 1/2, \sqrt{3}/2\rangle$

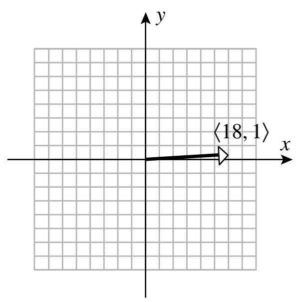
25. **(a)** The initial point of $\mathbf{u}+\mathbf{v}+\mathbf{w}$ is the origin and the endpoint is $(-2,5)$, so $\mathbf{u}+\mathbf{v}+\mathbf{w}=\langle -2,5\rangle$.



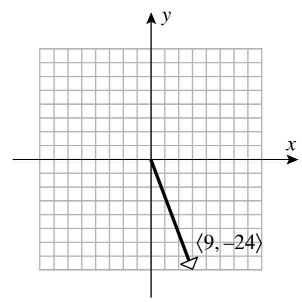
(b) The initial point of $\mathbf{u}+\mathbf{v}+\mathbf{w}$ is $(-5,4)$ and the endpoint is $(-2,-4)$, so $\mathbf{u}+\mathbf{v}+\mathbf{w}=\langle 3,-8\rangle$.



26. **(a)** $\mathbf{v}=\langle -10,2\rangle$ by inspection, so $\mathbf{u}-\mathbf{v}+\mathbf{w}=\mathbf{u}+\mathbf{v}+\mathbf{w}-2\mathbf{v}=\langle -2,5\rangle+\langle 20,-4\rangle=\langle 18,1\rangle$.



(b) $\mathbf{v}=\langle -3,8\rangle$ by inspection, so $\mathbf{u}-\mathbf{v}+\mathbf{w}=\mathbf{u}+\mathbf{v}+\mathbf{w}-2\mathbf{v}=\langle 3,-8\rangle+\langle 6,-16\rangle=\langle 9,-24\rangle$.



27. $6\mathbf{x}=2\mathbf{u}-\mathbf{v}-\mathbf{w}=\langle -4,6\rangle, \mathbf{x}=\langle -2/3,1\rangle$

28. $\mathbf{u}-2\mathbf{x}=\mathbf{x}-\mathbf{w}+3\mathbf{v}$, $3\mathbf{x}=\mathbf{u}+\mathbf{w}-3\mathbf{v}$, $\mathbf{x}=\dfrac{1}{3}(\mathbf{u}+\mathbf{w}-3\mathbf{v})=\langle 2/3,2/3\rangle$

29. $\mathbf{u}=\dfrac{5}{7}\mathbf{i}+\dfrac{2}{7}\mathbf{j}+\dfrac{1}{7}\mathbf{k}$, $\mathbf{v}=\dfrac{8}{7}\mathbf{i}-\dfrac{1}{7}\mathbf{j}-\dfrac{4}{7}\mathbf{k}$

30. $\mathbf{u}=\langle -5,8\rangle, \mathbf{v}=\langle 7,-11\rangle$

31. $\|(\mathbf{i}+\mathbf{j})+(\mathbf{i}-2\mathbf{j})\|=\|2\mathbf{i}-\mathbf{j}\|=\sqrt{5}, \|(\mathbf{i}+\mathbf{j}-(\mathbf{i}-2\mathbf{j})\|=\|3\mathbf{j}\|=3$

32. Let A, B, C be the vertices (0,0), (1,3), (2,4) and D the fourth vertex (x,y). For the parallelogram ABCD, $\overrightarrow{AD}=\overrightarrow{BC}$, $\langle x,y\rangle=\langle 1,1\rangle$ so $x=1$, $y=1$ and D is at (1,1). For the parallelogram ACBD, $\overrightarrow{AD}=\overrightarrow{CB}$, $\langle x,y\rangle=\langle -1,-1\rangle$ so $x=-1$, $y=-1$ and D is at $(-1,-1)$. For the parallelogram $ABDC$, $\overrightarrow{AC}=\overrightarrow{BD}$, $\langle x-1,y-3\rangle=\langle 2,4\rangle$, so $x=3, y=7$ and D is at $(3,7)$.

33. **(a)** $5=\|k\mathbf{v}\|=|k|\|\mathbf{v}\|=\pm 3k$, so k $=\pm 5/3$

(b) $6=\|k\mathbf{v}\|=|k|\|\mathbf{v}\|=2\|\mathbf{v}\|$, so $\|\mathbf{v}\|=3$

34. If $\|k\mathbf{v}\|=0$ then $|k|\|\mathbf{v}\|=0$ so either $k=0$ or $\|\mathbf{v}\|=0$; in the latter case, by (9) or (10), $\mathbf{v}=\mathbf{0}$.

35. **(a)** Choose two points on the line, for example $P_1(0,2)$ and $P_2(1,5)$; then $\overrightarrow{P_1P_2}=\langle 1,3\rangle$ is parallel to the line, $\|\langle 1,3\rangle\|=\sqrt{10}$, so $\langle 1/\sqrt{10},3/\sqrt{10}\rangle$ and $\langle -1/\sqrt{10},-3/\sqrt{10}\rangle$ are unit vectors parallel to the line.

(b) Choose two points on the line, for example $P_1(0,4)$ and $P_2(1,3)$; then $\overrightarrow{P_1P_2} = \langle 1,-1\rangle$ is parallel to the line, $\|\langle 1,-1\rangle\| = \sqrt{2}$ so $\langle 1/\sqrt{2}, -1/\sqrt{2}\rangle$ and $\langle -1/\sqrt{2}, 1/\sqrt{2}\rangle$ are unit vectors parallel to the line.

(c) Pick any line that is perpendicular to the line $y = -5x+1$, for example $y = x/5$; then $P_1(0,0)$ and $P_2(5,1)$ are on the line, so $\overrightarrow{P_1P_2} = \langle 5,1\rangle$ is perpendicular to the line, so $\pm\dfrac{1}{\sqrt{26}}\langle 5,1\rangle$ are unit vectors perpendicular to the line.

36. **(a)** $\pm\mathbf{k}$ **(b)** $\pm\mathbf{j}$ **(c)** $\pm\mathbf{i}$

37. **(a)** the circle of radius 1 about the origin

(b) the closed disk of radius 1 about the origin

(c) all points outside the closed disk of radius 1 about the origin

38. **(a)** the circle of radius 1 about the tip of $\mathbf{r}_0$

(b) the closed disk of radius 1 about the tip of $\mathbf{r}_0$

(c) all points outside the closed disk of radius 1 about the tip of $\mathbf{r}_0$

39. **(a)** the (hollow) sphere of radius 1 about the origin

(b) the closed ball of radius 1 about the origin

(c) all points outside the closed ball of radius 1 about the origin

40. The sum of the distances between (x,y) and the points (x_1,y_1), (x_2,y_2) is the constant k, so the set consists of all points on the ellipse with foci at (x_1,y_1) and (x_2,y_2), and major axis of length k.

41. Since $\phi = \pi/2$, from (14) we get $\|\mathbf{F}_1+\mathbf{F}_2\|^2 = \|\mathbf{F}_1\|^2 + \|\mathbf{F}_2\|^2 = 3600+900$, so $\|\mathbf{F}_1+\mathbf{F}_2\| = 30\sqrt{5}$ lb, and $\sin\alpha = \dfrac{\|\mathbf{F}_2\|}{\|\mathbf{F}_1+\mathbf{F}_2\|}\sin\phi = \dfrac{30}{30\sqrt{5}}, \alpha \approx 26.57°, \theta = \alpha \approx 26.57°$.

42. $\|\mathbf{F}_1+\mathbf{F}_2\|^2 = \|\mathbf{F}_1\|^2 + \|\mathbf{F}_2\|^2 + 2\|\mathbf{F}_1\|\|\mathbf{F}_2\|\cos\phi = 14{,}400 + 10{,}000 + 2(120)(100)\dfrac{1}{2} = 36{,}400$, so $\|\mathbf{F}_1+\mathbf{F}_2\| = 20\sqrt{91}$ N, $\sin\alpha = \dfrac{\|\mathbf{F}_2\|}{\|\mathbf{F}_1+\mathbf{F}_2\|}\sin\phi = \dfrac{100}{20\sqrt{91}}\sin 60° = \dfrac{5\sqrt{3}}{2\sqrt{91}}, \alpha \approx 27.00°$, $\theta = \alpha \approx 27.00°$.

43. $\|\mathbf{F}_1+\mathbf{F}_2\|^2 = \|\mathbf{F}_1\|^2 + \|\mathbf{F}_2\|^2 + 2\|\mathbf{F}_1\|\|\mathbf{F}_2\|\cos\phi = 160{,}000 + 160{,}000 - 2(400)(400)\dfrac{\sqrt{3}}{2}$, so $\|\mathbf{F}_1+\mathbf{F}_2\| \approx 207.06$ N, and $\sin\alpha = \dfrac{\|\mathbf{F}_2\|}{\|\mathbf{F}_1+\mathbf{F}_2\|}\sin\phi \approx \dfrac{400}{207.06}\left(\dfrac{1}{2}\right), \alpha = 75.00°$, $\theta = \alpha - 30° = 45.00°$.

44. $\|\mathbf{F}_1+\mathbf{F}_2\|^2 = \|\mathbf{F}_1\|^2 + \|\mathbf{F}_2\|^2 + 2\|\mathbf{F}_1\|\|\mathbf{F}_2\|\cos\phi = 16 + 4 + 2(4)(2)\cos 77°$, so $\|\mathbf{F}_1+\mathbf{F}_2\| \approx 4.86$ lb, and $\sin\alpha = \dfrac{\|\mathbf{F}_2\|}{\|\mathbf{F}_1+\mathbf{F}_2\|}\sin\phi = \dfrac{2}{4.86}\sin 77°, \alpha \approx 23.64°, \theta = \alpha - 27° \approx -3.36°$.

45. Let $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$ be the forces in the diagram with magnitudes $40, 50, 75$ respectively. Then $\mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3 = (\mathbf{F}_1+\mathbf{F}_2)+\mathbf{F}_3$. Following the examples, $\mathbf{F}_1+\mathbf{F}_2$ has magnitude 45.83 N and makes an angle 79.11° with the positive x-axis. Then $\|(\mathbf{F}_1+\mathbf{F}_2)+\mathbf{F}_3\|^2 \approx 45.83^2 + 75^2 + 2(45.83)(75)\cos 79.11°$, so $\mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3$ has magnitude ≈ 94.995 N and makes an angle $\theta = \alpha \approx 28.28°$ with the positive x-axis.

46. Let $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$ be the forces in the diagram with magnitudes $150, 200, 100$ respectively. Then $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = (\mathbf{F}_1 + \mathbf{F}_2) + \mathbf{F}_3$. Following the examples, $\mathbf{F}_1 + \mathbf{F}_2$ has magnitude 279.34 N and makes an angle $91.24°$ with the positive x-axis. Then
$\|\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3\|^2 \approx 279.34^2 + 100^2 + 2(279.34)(100)\cos(270 - 91.24)°$, and $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$ has magnitude ≈ 179.37 N and makes an angle $91.94°$ with the positive x-axis.

47. Let $\mathbf{F}_1, \mathbf{F}_2$ be the forces in the diagram with magnitudes $8, 10$ respectively. Then $\|\mathbf{F}_1 + \mathbf{F}_2\|$ has magnitude $\sqrt{8^2 + 10^2 + 2 \cdot 8 \cdot 10 \cos 120°} = 2\sqrt{21} \approx 9.165$ lb, and makes an angle
$60° + \sin^{-1} \dfrac{\|\mathbf{F}_1\|}{\|\mathbf{F}_1 + \mathbf{F}_2\|} \sin 120 \approx 109.11°$ with the positive x-axis, so $\mathbf{F}$ has magnitude 9.165 lb and makes an angle $-70.89°$ with the positive x-axis.

48. $\|\mathbf{F}_1 + \mathbf{F}_2\| = \sqrt{120^2 + 150^2 + 2 \cdot 120 \cdot 150 \cos 75°} = 214.98$ N and makes an angle $92.63°$ with the positive x-axis, and $\|\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3\| = 232.90$ N and makes an angle $67.23°$ with the positive x-axis, hence $\mathbf{F}$ has magnitude 232.90 N and makes an angle $-112.77°$ with the positive x-axis.

49. $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F} = \mathbf{0}$, where $\mathbf{F}$ has magnitude 250 and makes an angle $-90°$ with the positive x-axis. Thus $\|\mathbf{F}_1 + \mathbf{F}_2\|^2 = \|\mathbf{F}_1\|^2 + \|\mathbf{F}_2\|^2 + 2\|\mathbf{F}_1\|\|\mathbf{F}_2\| \cos 105° = 250^2$ and

$45° = \alpha = \sin^{-1}\left(\dfrac{\|\mathbf{F}_2\|}{250} \sin 105°\right)$, so $\dfrac{\sqrt{2}}{2} \approx \dfrac{\|\mathbf{F}_2\|}{250} 0.9659$, $\|\mathbf{F}_2\| \approx 183.02$ lb,

$\|\mathbf{F}_1\|^2 + 2(183.02)(-0.2588)\|\mathbf{F}_1\| + (183.02)^2 = 62{,}500$, $\|\mathbf{F}_1\| = 224.13$ lb.

50. Similar to Exercise 49, $\|\mathbf{F}_1\| = 100\sqrt{3}$ N, $\|\mathbf{F}_2\| = 100$ N

51. **(a)** $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = (2c_1 + 4c_2)\,\mathbf{i} + (-c_1 + 2c_2)\,\mathbf{j} = 4\mathbf{j}$, so $2c_1 + 4c_2 = 0$ and $-c_1 + 2c_2 = 4$ which gives $c_1 = -2$, $c_2 = 1$.

(b) $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \langle c_1 - 2c_2, -3c_1 + 6c_2 \rangle = \langle 3, 5 \rangle$, so $c_1 - 2c_2 = 3$ and $-3c_1 + 6c_2 = 5$ which has no solution.

52. **(a)** Equate corresponding components to get the system of equations $c_1 + 3c_2 = -1$, $2c_2 + c_3 = 1$, and $c_1 + c_3 = 5$. Solve to get $c_1 = 2$, $c_2 = -1$, and $c_3 = 3$.

(b) Equate corresponding components to get the system of equations $c_1 + 3c_2 + 4c_3 = 2$, $-c_1 - c_3 = 1$, and $c_2 + c_3 = -1$. From the second and third equations, $c_1 = -1 - c_3$ and $c_2 = -1 - c_3$; substitute these into the first equation to get $-4 = 2$, which is false so the system has no solution.

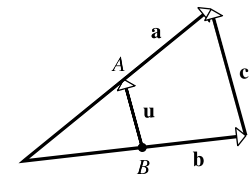
53. Place $\mathbf{u}$ and $\mathbf{v}$ tip to tail so that $\mathbf{u} + \mathbf{v}$ is the vector from the initial point of $\mathbf{u}$ to the terminal point of $\mathbf{v}$. The shortest distance between two points is along the line joining these points so $\|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$.

54. (a): $\mathbf{u} + \mathbf{v} = (u_1\mathbf{i} + u_2\,\mathbf{j}) + (v_1\mathbf{i} + v_2\,\mathbf{j}) = (v_1\mathbf{i} + v_2\,\mathbf{j}) + (u_1\mathbf{i} + u_2\,\mathbf{j}) = \mathbf{v} + \mathbf{u}$
(c): $\mathbf{u} + \mathbf{0} = (u_1\mathbf{i} + u_2\,\mathbf{j}) + 0\mathbf{i} + 0\mathbf{j} = u_1\mathbf{i} + u_2\,\mathbf{j} = \mathbf{u}$
(e): $k(l\mathbf{u}) = k(l(u_1\mathbf{i} + u_2\,\mathbf{j})) = k(lu_1\mathbf{i} + lu_2\,\mathbf{j}) = klu_1\mathbf{i} + klu_2\,\mathbf{j} = (kl)\mathbf{u}$

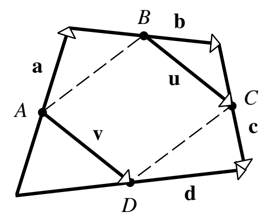
55. (d): $\mathbf{u} + (-\mathbf{u}) = (u_1\mathbf{i} + u_2\,\mathbf{j}) + (-u_1\mathbf{i} - u_2\,\mathbf{j}) = (u_1 - u_1)\mathbf{i} + (u_1 - u_1)\,\mathbf{j} = \mathbf{0}$
(g): $(k + l)\mathbf{u} = (k + l)(u_1\mathbf{i} + u_2\,\mathbf{j}) = ku_1\mathbf{i} + ku_2\,\mathbf{j} + lu_1\mathbf{i} + lu_2\,\mathbf{j} = k\mathbf{u} + l\mathbf{u}$
(h): $1\mathbf{u} = 1(u_1\mathbf{i} + u_2\,\mathbf{j}) = 1u_1\mathbf{i} + 1u_2\,\mathbf{j} = u_1\mathbf{i} + u_2\,\mathbf{j} = \mathbf{u}$

56. Draw the triangles with sides formed by the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$, $k\mathbf{v}$, $k\mathbf{u} + k\mathbf{v}$. By similar triangles, $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$.

57. Let **a**, **b**, **c** be vectors along the sides of the triangle and A,B the midpoints of **a** and **b**, then $\mathbf{u} = \frac{1}{2}\mathbf{a} - \frac{1}{2}\mathbf{b} = \frac{1}{2}(\mathbf{a} - \mathbf{b}) = \frac{1}{2}\mathbf{c}$ so **u** is parallel to **c** and half as long.



58. Let **a**, **b**, **c**, **d** be vectors along the sides of the quadrilateral and A, B, C, D the corresponding midpoints, then $\mathbf{u} = \frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{c}$ and $\mathbf{v} = \frac{1}{2}\mathbf{d} - \frac{1}{2}\mathbf{a}$ but $\mathbf{d} = \mathbf{a} + \mathbf{b} + \mathbf{c}$ so $\mathbf{v} = \frac{1}{2}(\mathbf{a} + \mathbf{b} + \mathbf{c}) - \frac{1}{2}\mathbf{a} = \frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{c} = \mathbf{u}$ thus ABCD is a parallelogram because sides AD and BC are equal and parallel.



EXERCISE SET 12.3

1. **(a)** $(1)(6) + (2)(-8) = -10$; $\cos\theta = (-10)/[(\sqrt{5})(10)] = -1/\sqrt{5}$

(b) $(-7)(0) + (-3)(1) = -3$; $\cos\theta = (-3)/[(\sqrt{58})(1)] = -3/\sqrt{58}$

(c) $(1)(8) + (-3)(-2) + (7)(-2) = 0$; $\cos\theta = 0$

(d) $(-3)(4) + (1)(2) + (2)(-5) = -20$; $\cos\theta = (-20)/[(\sqrt{14})(\sqrt{45})] = -20/(3\sqrt{70})$

2. **(a)** $\mathbf{u} \cdot \mathbf{v} = 1(2)\cos(\pi/6) = \sqrt{3}$ **(b)** $\mathbf{u} \cdot \mathbf{v} = 2(3)\cos 135^\circ = -3\sqrt{2}$

3. **(a)** $\mathbf{u} \cdot \mathbf{v} = -34 < 0$, obtuse **(b)** $\mathbf{u} \cdot \mathbf{v} = 6 > 0$, acute

(c) $\mathbf{u} \cdot \mathbf{v} = -1 < 0$, obtuse **(d)** $\mathbf{u} \cdot \mathbf{v} = 0$, orthogonal

4. Let the points be P, Q, R in order, then $\overrightarrow{PQ} = \langle 2 - (-1), -2 - 2, 0 - 3\rangle = \langle 3, -4, -3\rangle$, $\overrightarrow{QR} = \langle 3 - 2, 1 - (-2), -4 - 0\rangle = \langle 1, 3, -4\rangle$, $\overrightarrow{RP} = \langle -1 - 3, 2 - 1, 3 - (-4)\rangle = \langle -4, 1, 7\rangle$;
since $\overrightarrow{QP} \cdot \overrightarrow{QR} = -3(1) + 4(3) + 3(-4) = -3 < 0$, $\angle PQR$ is obtuse;
since $\overrightarrow{RP} \cdot \overrightarrow{RQ} = -4(-1) + (-3) + 7(4) = 29 > 0$, $\angle PRQ$ is acute;
since $\overrightarrow{PR} \cdot \overrightarrow{PQ} = 4(3) - 1(-4) - 7(-3) = 37 > 0$, $\angle RPQ$ is acute

5. Since $\mathbf{v}_1 \cdot \mathbf{v}_i = \cos\phi_i$, the answers are, in order, $\sqrt{2}/2, 0, -\sqrt{2}/2, -1, -\sqrt{2}/2, 0, \sqrt{2}/2$

6. Proceed as in Exercise 5; $25/2, -25/2, -25, -25/2, 25/2$

7. **(a)** $\overrightarrow{AB} = \langle 1, 3, -2\rangle, \overrightarrow{BC} = \langle 4, -2, -1\rangle, \overrightarrow{AB} \cdot \overrightarrow{BC} = 0$ so $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are orthogonal; it is a right triangle with the right angle at vertex B.

(b) Let A, B, and C be the vertices $(-1, 0)$, $(2, -1)$, and (1,4) with corresponding interior angles α, β, and γ, then

$$\cos\alpha = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\|\overrightarrow{AB}\|\,\|\overrightarrow{AC}\|} = \frac{\langle 3,-1\rangle \cdot \langle 2,4\rangle}{\sqrt{10}\sqrt{20}} = 1/(5\sqrt{2}),\ \alpha \approx 82^\circ$$

$$\cos\beta = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{\|\overrightarrow{BA}\|\,\|\overrightarrow{BC}\|} = \frac{\langle -3,1\rangle \cdot \langle -1,5\rangle}{\sqrt{10}\sqrt{26}} = 4/\sqrt{65},\ \beta \approx 60^\circ$$

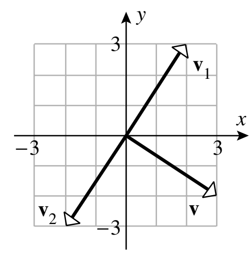
$$\cos\gamma = \frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{\|\overrightarrow{CA}\|\,\|\overrightarrow{CB}\|} = \frac{\langle -2,-4\rangle \cdot \langle 1,-5\rangle}{\sqrt{20}\sqrt{26}} = 9/\sqrt{130},\ \gamma \approx 38^\circ$$

8. $\overrightarrow{AB} \cdot \overrightarrow{AP} = [2\mathbf{i}+\mathbf{j}+2\mathbf{k}] \cdot [(k-1)\mathbf{i}+(k+1)\mathbf{j}+(k-3)\mathbf{k}]$
$= 2(k-1)+(k+1)+2(k-3) = 5k-7 = 0, k = 7/5.$

9. **(a)** $\mathbf{v}\cdot\mathbf{v}_1 = -ab+ba = 0$; $\mathbf{v}\cdot\mathbf{v}_2 = ab+b(-a) = 0$

(b) Let $\mathbf{v}_1 = 2\mathbf{i}+3\mathbf{j}$, $\mathbf{v}_2 = -2\mathbf{i}-3\mathbf{j}$;

take $\mathbf{u}_1 = \dfrac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \dfrac{2}{\sqrt{13}}\mathbf{i} + \dfrac{3}{\sqrt{13}}\mathbf{j}$, $\mathbf{u}_2 = -\mathbf{u}_1$.



10. By inspection, $3\mathbf{i}-4\mathbf{j}$ is orthogonal to and has the same length as $4\mathbf{i}+3\mathbf{j}$ so $\mathbf{u}_1 = (4\mathbf{i}+3\mathbf{j})+(3\mathbf{i}-4\mathbf{j}) = 7\mathbf{i}-\mathbf{j}$ and $\mathbf{u}_2 = (4\mathbf{i}+3\mathbf{j})+(-1)(3\mathbf{i}-4\mathbf{j}) = \mathbf{i}+7\mathbf{j}$ each make an angle of 45° with $4\mathbf{i}+3\mathbf{j}$; unit vectors in the directions of $\mathbf{u}_1$ and $\mathbf{u}_2$ are $(7\mathbf{i}-\mathbf{j})/\sqrt{50}$ and $(\mathbf{i}+7\mathbf{j})/\sqrt{50}$.

11. **(a)** The dot product of a vector $\mathbf{u}$ and a scalar $\mathbf{v}\cdot\mathbf{w}$ is not defined.

(b) The sum of a scalar $\mathbf{u}\cdot\mathbf{v}$ and a vector $\mathbf{w}$ is not defined.

(c) $\mathbf{u}\cdot\mathbf{v}$ is not a vector.

(d) The dot product of a scalar k and a vector $\mathbf{u}+\mathbf{v}$ is not defined.

12. (b): $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = (6\mathbf{i}-\mathbf{j}+2\mathbf{k})\cdot((2\mathbf{i}+7\mathbf{j}+4\mathbf{k})+(\mathbf{i}+\mathbf{j}-3\mathbf{k})) = (6\mathbf{i}-\mathbf{j}+2\mathbf{k})\cdot(3\mathbf{i}+8\mathbf{j}+\mathbf{k}) = 12$;
$\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w} = (6\mathbf{i}-\mathbf{j}+2\mathbf{k})\cdot(2\mathbf{i}+7\mathbf{j}+4\mathbf{k}) + (6\mathbf{i}-\mathbf{j}+2\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}-3\mathbf{k}) = 13-1 = 12$
(c): $k(\mathbf{u}\cdot\mathbf{v}) = -5(13) = -65$; $(k\mathbf{u})\cdot\mathbf{v} = (-30\mathbf{i}+5\mathbf{j}-10\mathbf{k})\cdot(2\mathbf{i}+7\mathbf{j}+4\mathbf{k}) = -65$;
$\mathbf{u}\cdot(k\mathbf{v}) = (6\mathbf{i}-\mathbf{j}+2\mathbf{k})\cdot(-10\mathbf{i}-35\mathbf{j}-20\mathbf{k}) = -65$

13. **(a)** $\langle 1,2\rangle\cdot(\langle 28,-14\rangle+\langle 6,0\rangle) = \langle 1,2\rangle\cdot\langle 34,-14\rangle = 6$
(b) $\|6\mathbf{w}\| = 6\|\mathbf{w}\| = 36$ **(c)** $24\sqrt{5}$ **(d)** $24\sqrt{5}$

14. false, for example $\mathbf{a} = \langle 1,2\rangle, \mathbf{b} = \langle -1,0\rangle, \mathbf{c} = \langle 5,-3\rangle$

15. **(a)** $\|\mathbf{v}\| = \sqrt{3}$ so $\cos\alpha = \cos\beta = 1/\sqrt{3}$, $\cos\gamma = -1/\sqrt{3}$, $\alpha = \beta \approx 55^\circ$, $\gamma \approx 125^\circ$

(b) $\|\mathbf{v}\| = 3$ so $\cos\alpha = 2/3$, $\cos\beta = -2/3$, $\cos\gamma = 1/3$, $\alpha \approx 48^\circ$, $\beta \approx 132^\circ$, $\gamma \approx 71^\circ$

16. **(a)** $\|\mathbf{v}\| = 7$ so $\cos\alpha = 3/7$, $\cos\beta = -2/7$, $\cos\gamma = -6/7$, $\alpha \approx 65^\circ$, $\beta \approx 107^\circ$, $\gamma \approx 149^\circ$

(b) $\|\mathbf{v}\| = 5$, $\cos\alpha = 3/5$, $\cos\beta = 0$, $\cos\gamma = -4/5$, $\alpha \approx 53^\circ$, $\beta = 90^\circ$, $\gamma \approx 143^\circ$

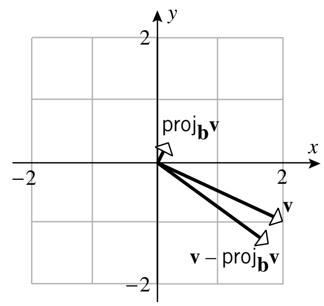
17. $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \dfrac{v_1^2}{\|\mathbf{v}\|^2} + \dfrac{v_2^2}{\|\mathbf{v}\|^2} + \dfrac{v_3^2}{\|\mathbf{v}\|^2} = \left(v_1^2 + v_2^2 + v_3^2\right)/\|\mathbf{v}\|^2 = \|\mathbf{v}\|^2/\|\mathbf{v}\|^2 = 1$

18. Let $\mathbf{v} = \langle x, y, z\rangle$, then $x = \sqrt{x^2+y^2}\cos\theta,\ y = \sqrt{x^2+y^2}\sin\theta,\ \sqrt{x^2+y^2} = \|\mathbf{v}\|\cos\lambda$, and $z = \|\mathbf{v}\|\sin\lambda$, so $x/\|\mathbf{v}\| = \cos\theta\cos\lambda,\ y/\|\mathbf{v}\| = \sin\theta\cos\lambda,$ and $z/\|\mathbf{v}\| = \sin\lambda$.

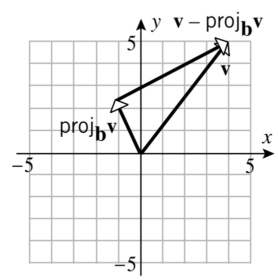
19. $\cos\alpha = \dfrac{\sqrt{3}}{2}\dfrac{1}{2} = \dfrac{\sqrt{3}}{4},\ \cos\beta = \dfrac{\sqrt{3}}{2}\dfrac{\sqrt{3}}{2} = \dfrac{3}{4},\ \cos\gamma = \dfrac{1}{2};\ \alpha \approx 64^\circ, \beta \approx 41^\circ, \gamma = 60^\circ$

20. Let $\mathbf{u}_1 = \|\mathbf{u}_1\|\langle\cos\alpha_1, \cos\beta_1, \cos\gamma_1\rangle, \mathbf{u}_2 = \|\mathbf{u}_2\|\langle\cos\alpha_2, \cos\beta_2, \cos\gamma_2\rangle$, $\mathbf{u}_1$ and $\mathbf{u}_2$ are perpendicular if and only if $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$ so $\|\mathbf{u}_1\|\ \|\mathbf{u}_2\|(\cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2) = 0$, $\cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2 = 0$.

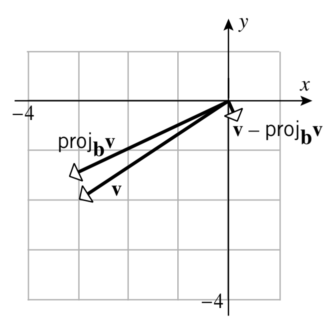
21. **(a)** $\dfrac{\mathbf{b}}{\|\mathbf{b}\|} = \langle 3/5, 4/5\rangle$, so $\text{proj}_{\mathbf{b}}\mathbf{v} = \langle 6/25, 8/25\rangle$
and $\mathbf{v} - \text{proj}_{\mathbf{b}}\mathbf{v} = \langle 44/25, -33/25\rangle$



(b) $\dfrac{\mathbf{b}}{\|\mathbf{b}\|} = \langle 1/\sqrt{5}, -2/\sqrt{5}\rangle$, so $\text{proj}_{\mathbf{b}}\mathbf{v} = \langle -6/5, 12/5\rangle$
and $\mathbf{v} - \text{proj}_{\mathbf{b}}\mathbf{v} = \langle 26/5, 13/5\rangle$



(c) $\dfrac{\mathbf{b}}{\|\mathbf{b}\|} = \langle 2/\sqrt{5}, 1/\sqrt{5}\rangle$, so $\text{proj}_{\mathbf{b}}\mathbf{v} = \langle -16/5, -8/5\rangle$
and $\mathbf{v} - \text{proj}_{\mathbf{b}}\mathbf{v} = \langle 1/5, -2/5\rangle$



22. **(a)** $\dfrac{\mathbf{b}}{\|\mathbf{b}\|} = \langle 1/3, 2/3, 2/3\rangle$, so $\text{proj}_{\mathbf{b}}\mathbf{v} = \langle 2/3, 4/3, 4/3\rangle$ and $\mathbf{v} - \text{proj}_{\mathbf{b}}\mathbf{v} = \langle 4/3, -7/3, 5/3\rangle$

(b) $\dfrac{\mathbf{b}}{\|\mathbf{b}\|} = \langle 2/7, 3/7, -6/7\rangle$, so $\text{proj}_{\mathbf{b}}\mathbf{v} = \langle -74/49, -111/49, 222/49\rangle$
and $\mathbf{v} - \text{proj}_{\mathbf{b}}\mathbf{v} = \langle 270/49, 62/49, 121/49\rangle$

23. **(a)** $\text{proj}_{\mathbf{b}}\mathbf{v} = \langle -1, -1\rangle$, so $\mathbf{v} = \langle -1, -1\rangle + \langle 3, -3\rangle$

(b) $\text{proj}_{\mathbf{b}}\mathbf{v} = \langle 16/5, 0, -8/5\rangle$, so $\mathbf{v} = \langle 16/5, 0, -8/5\rangle + \langle -1/5, 1, -2/5\rangle$

24. **(a)** $\text{proj}_{\mathbf{b}}\mathbf{v} = \langle 1,1\rangle$, so $\mathbf{v} = \langle 1,1\rangle + \langle -4,4\rangle$

(b) $\text{proj}_{\mathbf{b}}\mathbf{v} = \langle 0,-8/5,4/5\rangle$, so $\mathbf{v} = \langle 0,-8/5,4/5\rangle + \langle -2,13/5,26/5\rangle$

25. $\overrightarrow{AP} = -\mathbf{i}+3\mathbf{j}$, $\overrightarrow{AB} = 3\mathbf{i}+4\mathbf{j}$, $\|\text{proj}_{\overrightarrow{AB}}\overrightarrow{AP}\| = |\overrightarrow{AP}\cdot\overrightarrow{AB}|/\|\overrightarrow{AB}\| = 9/5$
$\|\overrightarrow{AP}\| = \sqrt{10}$, $\sqrt{10-81/25} = 13/5$

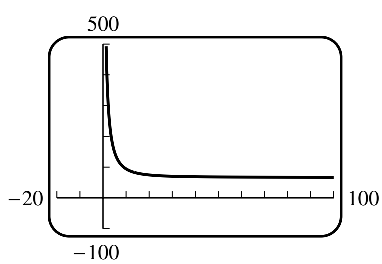
26. $\overrightarrow{AP} = -4\mathbf{i}+2\mathbf{k}$, $\overrightarrow{AB} = -3\mathbf{i}+2\mathbf{j}-4\mathbf{k}$, $\|\text{proj}_{\overrightarrow{AB}}\overrightarrow{AP}\| = |\overrightarrow{AP}\cdot\overrightarrow{AB}|/\|\overrightarrow{AB}\| = 4/\sqrt{29}$.
$\|\overrightarrow{AP}\| = \sqrt{20}$, $\sqrt{20-16/29} = \sqrt{564/29}$

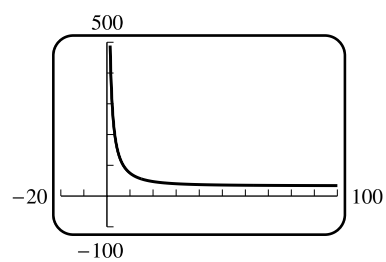
27. Let $\mathbf{F}$ be the downward force of gravity on the block, then $\|\mathbf{F}\| = 10(9.8) = 98$ N, and if $\mathbf{F}_1$ and $\mathbf{F}_2$ are the forces parallel to and perpendicular to the ramp, then $\|\mathbf{F}_1\| = \|\mathbf{F}_2\| = 49\sqrt{2}$ N. Thus the block exerts a force of $49\sqrt{2}$ N against the ramp and it requires a force of $49\sqrt{2}$ N to prevent the block from sliding down the ramp.

28. Let x denote the magnitude of the force in the direction of $\mathbf{Q}$. Then the force $\mathbf{F}$ acting on the block is $\mathbf{F} = x\mathbf{i} - 98\mathbf{j}$. Let $\mathbf{u} = -\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})$ and $\mathbf{v} = \frac{1}{\sqrt{2}}(\mathbf{i}-\mathbf{j})$ be the unit vectors in the directions along and against the ramp. Then $\mathbf{F}$ decomposes as $\mathbf{F} = -\frac{x-98}{\sqrt{2}}\mathbf{u} + \frac{x+98}{\sqrt{2}}\mathbf{v}$, and thus the block will not slide down the ramp provided $x \geq 98$ N.

29. Three forces act on the block: its weight $-300\mathbf{j}$; the tension in cable A, which has the form $a(-\mathbf{i}+\mathbf{j})$; and the tension in cable B, which has the form $b(\sqrt{3}\mathbf{i}-\mathbf{j})$, where a, b are positive constants. The sum of these forces is zero, which yields $a = 450+150\sqrt{3}, b = 150+150\sqrt{3}$. Thus the forces along cables A and B are, respectively,
$\|150(3+\sqrt{3})(\mathbf{i}-\mathbf{j})\| = 450\sqrt{2}+150\sqrt{6}$ lb, and $\|150(\sqrt{3}+1)(\sqrt{3}\mathbf{i}-\mathbf{j})\| = 300+300\sqrt{3}$ lb.

30. **(a)** Let $\mathbf{T}_A$ and $\mathbf{T}_B$ be the forces exerted on the block by cables A and B. Then $\mathbf{T}_A = a(-10\mathbf{i}+d\mathbf{j})$ and $\mathbf{T}_B = b(20\mathbf{i}+d\mathbf{j})$ for some positive a, b. Since $\mathbf{T}_A+\mathbf{T}_B-100\mathbf{j} = \mathbf{0}$, we find $a = \frac{200}{3d}, b = \frac{100}{3d}, \mathbf{T}_A = -\frac{2000}{3d}\mathbf{i} + \frac{200}{3}\mathbf{j}$, and $\mathbf{T}_B = \frac{2000}{3d}\mathbf{i} + \frac{100}{3}\mathbf{j}$. Thus
$\mathbf{T}_A = \frac{200}{3}\sqrt{1+\frac{100}{d^2}}, \mathbf{T}_B = \frac{100}{3}\sqrt{1+\frac{400}{d^2}}$, and the graphs are:





(b) An increase in d will decrease both forces.

(c) The inequality $\|\mathbf{T}_A\| \leq 150$ is equivalent to $d \geq \frac{40}{\sqrt{65}}$, and $\|\mathbf{T}_B\| \leq 150$ is equivalent to $d \geq \frac{40}{\sqrt{77}}$. Hence we must have $d \geq \frac{40}{65}$.

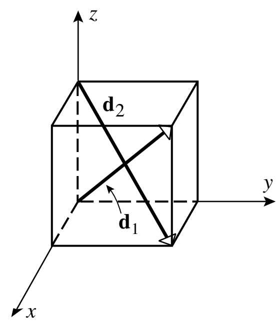
31. Let P and Q be the points (1,3) and (4,7) then $\overrightarrow{PQ} = 3\mathbf{i}+4\mathbf{j}$ so $W = \mathbf{F}\cdot\overrightarrow{PQ} = -12$ ft · lb.

32. $W = \mathbf{F}\cdot\overrightarrow{PQ} = \|\mathbf{F}\|\,\|\overrightarrow{PQ}\|\cos 45^\circ = (500)(100)\left(\sqrt{2}/2\right) = 25{,}000\sqrt{2}$ N · m $= 25{,}000\sqrt{2}$ J

33. $W = \mathbf{F} \cdot 15\mathbf{i} = 15 \cdot 50 \cos 60° = 375$ ft $\cdot$ lb.

34. $W = \mathbf{F} \cdot (15/\sqrt{3})(\mathbf{i} + \mathbf{j} + \mathbf{k}) = -15/\sqrt{3}$ N $\cdot$ m $= -5\sqrt{3}$ J

35. With the cube as shown in the diagram,
and a the length of each edge,
$\mathbf{d}_1 = a\mathbf{i} + a\mathbf{j} + a\mathbf{k}, \mathbf{d}_2 = a\mathbf{i} + a\mathbf{j} - a\mathbf{k}$,
$\cos\theta = (\mathbf{d}_1 \cdot \mathbf{d}_2) / (\|\mathbf{d}_1\| \, \|\mathbf{d}_2\|) = 1/3, \theta \approx 71°$



36. Take $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ along adjacent edges of the box, then $10\mathbf{i} + 15\mathbf{j} + 25\mathbf{k}$ is along a diagonal, and a unit vector in this direction is $\dfrac{2}{\sqrt{38}}\mathbf{i} + \dfrac{3}{\sqrt{38}}\mathbf{j} + \dfrac{5}{\sqrt{38}}\mathbf{k}$. The direction cosines are $\cos\alpha = 2/\sqrt{38}$, $\cos\beta = 3/\sqrt{38}$, and $\cos\gamma = 5/\sqrt{38}$ so $\alpha \approx 71°$, $\beta \approx 61°$, and $\gamma \approx 36°$.

37. $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are vectors along the diagonals,
$(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{v} = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$ so $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = 0$
if and only if $\|\mathbf{u}\| = \|\mathbf{v}\|$.

38. The diagonals have lengths $\|\mathbf{u} + \mathbf{v}\|$ and $\|\mathbf{u} - \mathbf{v}\|$ but
$\|\mathbf{u} + \mathbf{v}\|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \|\mathbf{u}\|^2 + 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2$, and
$\|\mathbf{u} - \mathbf{v}\|^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \|\mathbf{u}\|^2 - 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2$. If the parallelogram is a rectangle then $\mathbf{u} \cdot \mathbf{v} = 0$ so $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u} - \mathbf{v}\|^2$; the diagonals are equal. If the diagonals are equal, then $4\mathbf{u} \cdot \mathbf{v} = 0$, $\mathbf{u} \cdot \mathbf{v} = 0$ so $\mathbf{u}$ is perpendicular to $\mathbf{v}$ and hence the parallelogram is a rectangle.

39. $\|\mathbf{u} + \mathbf{v}\|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \|\mathbf{u}\|^2 + 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2$ and
$\|\mathbf{u} - \mathbf{v}\|^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \|\mathbf{u}\|^2 - 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2$, add to get
$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$

The sum of the squares of the lengths of the diagonals of a parallelogram is equal to twice the sum of the squares of the lengths of the sides.

40. $\|\mathbf{u} + \mathbf{v}\|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \|\mathbf{u}\|^2 + 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2$ and
$\|\mathbf{u} - \mathbf{v}\|^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \|\mathbf{u}\|^2 - 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2$, subtract to get
$\|\mathbf{u} + \mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2 = 4\mathbf{u} \cdot \mathbf{v}$, the result follows by dividing both sides by 4.

41. $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ so $\mathbf{v} \cdot \mathbf{v}_i = c_i\mathbf{v}_i \cdot \mathbf{v}_i$ because $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ if $i \neq j$,
thus $\mathbf{v} \cdot \mathbf{v}_i = c_i\|\mathbf{v}_i\|^2, c_i = \mathbf{v} \cdot \mathbf{v}_i/\|\mathbf{v}_i\|^2$ for $i = 1, 2, 3$.

42. $\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_1 \cdot \mathbf{v}_3 = \mathbf{v}_2 \cdot \mathbf{v}_3 = 0$ so they are mutually perpendicular. Let $\mathbf{v} = \mathbf{i} - \mathbf{j} + \mathbf{k}$, then
$c_1 = \dfrac{\mathbf{v} \cdot \mathbf{v}_1}{\|\mathbf{v}_1\|^2} = \dfrac{3}{7}$, $c_2 = \dfrac{\mathbf{v} \cdot \mathbf{v}_2}{\|\mathbf{v}_2\|^2} = -\dfrac{1}{3}$, and $c_3 = \dfrac{\mathbf{v} \cdot \mathbf{v}_3}{\|\mathbf{v}_3\|^2} = \dfrac{1}{21}$.

43. **(a)** $\mathbf{u} = x\mathbf{i} + (x^2+1)\mathbf{j}, \mathbf{v} = x\mathbf{i} - (x+1)\mathbf{j}, \theta = \cos^{-1}[(\mathbf{u}\cdot\mathbf{v})/(\|\mathbf{u}\|\|\mathbf{v}\|)]$.

Use a CAS to solve $d\theta/dx = 0$ to find that the minimum value of θ occurs when $x \approx -3.136742$ so the minimum angle is about $40°$. NB: Since $\cos^{-1} u$ is a decreasing function of u, it suffices to maximize $(\mathbf{u}\cdot\mathbf{v})/(\|\mathbf{u}\|\|\mathbf{v}\|)$, or, what is easier, its square.

(b) Solve $\mathbf{u}\cdot\mathbf{v} = 0$ for x to get $x \approx -0.682328$.

44. **(a)** $\mathbf{u} = \cos\theta_1\mathbf{i} \pm \sin\theta_1\mathbf{j}, \mathbf{v} = \pm\sin\theta_2\mathbf{j} + \cos\theta_2\mathbf{k}$, $\cos\theta = \mathbf{u}\cdot\mathbf{v} = \pm\sin\theta_1\sin\theta_2$

(b) $\cos\theta = \pm\sin^2 45° = \pm 1/2,\ \theta = 60°$

(c) Let $\theta(t) = \cos^{-1}(\sin t \sin 2t)$; solve $\theta'(t) = 0$ for t to find that $\theta_{\max} \approx 140°$ (reject, since θ is acute) when $t \approx 2.186276$ and that $\theta_{\min} \approx 40°$ when $t \approx 0.955317$; for $\theta_{\max}$ check the endpoints $t = 0, \pi/2$ to obtain $\theta_{\max} = \cos^{-1}(0) = \pi/2$.

45. Let $\mathbf{u} = \langle u_1, u_2, u_3\rangle, \mathbf{v} = \langle v_1, v_2, v_3\rangle, \mathbf{w} = \langle w_1, w_2, w_3\rangle$. Then

$$\begin{aligned}\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) &= \langle u_1(v_1+w_1), u_2(v_2+w_2), u_3(v_3+w_3)\rangle = \langle u_1v_1 + u_1w_1, u_2v_2+u_2w_2, u_3v_3+u_3w_3\rangle \\ &= \langle u_1v_1, u_2v_2, u_3v_3\rangle + \langle u_1w_1, u_2w_2, u_3w_3\rangle = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}\end{aligned}$$

$\mathbf{0}\cdot\mathbf{v} = 0\cdot v_1 + 0\cdot v_2 + 0\cdot v_3 = 0$

EXERCISE SET 12.4

1. **(a)** $$\mathbf{i}\times(\mathbf{i}+\mathbf{j}+\mathbf{k}) = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & 0\\ 1 & 1 & 1\end{vmatrix} = -\mathbf{j}+\mathbf{k}$$

(b) $\mathbf{i}\times(\mathbf{i}+\mathbf{j}+\mathbf{k}) = (\mathbf{i}\times\mathbf{i}) + (\mathbf{i}\times\mathbf{j}) + (\mathbf{i}\times\mathbf{k}) = -\mathbf{j}+\mathbf{k}$

2. **(a)** $$\mathbf{j}\times(\mathbf{i}+\mathbf{j}+\mathbf{k}) = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 1 & 0\\ 1 & 1 & 1\end{vmatrix} = \mathbf{i}-\mathbf{k}$$

$\mathbf{j}\times(\mathbf{i}+\mathbf{j}+\mathbf{k}) = (\mathbf{j}\times\mathbf{i}) + (\mathbf{j}\times\mathbf{j}) + (\mathbf{j}\times\mathbf{k}) = \mathbf{i}-\mathbf{k}$

(b) $$\mathbf{k}\times(\mathbf{i}+\mathbf{j}+\mathbf{k}) = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 0 & 1\\ 1 & 1 & 1\end{vmatrix} = -\mathbf{i}+\mathbf{j}$$

$\mathbf{k}\times(\mathbf{i}+\mathbf{j}+\mathbf{k}) = (\mathbf{k}\times\mathbf{i}) + (\mathbf{k}\times\mathbf{j}) + (\mathbf{k}\times\mathbf{k}) = \mathbf{j}-\mathbf{i}+\mathbf{0} = -\mathbf{i}+\mathbf{j}$

3. $\langle 7, 10, 9\rangle$ **4.** $-\mathbf{i} - 2\mathbf{j} - 7\mathbf{k}$ **5.** $\langle -4, -6, -3\rangle$ **6.** $\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$

7. **(a)** $\mathbf{v}\times\mathbf{w} = \langle -23, 7, -1\rangle, \mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = \langle -20, -67, -9\rangle$

(b) $\mathbf{u}\times\mathbf{v} = \langle -10, -14, 2\rangle, (\mathbf{u}\times\mathbf{v})\times\mathbf{w} = \langle -78, 52, -26\rangle$

(c) $(\mathbf{u}\times\mathbf{v})\times(\mathbf{v}\times\mathbf{w}) = \langle -10, -14, 2\rangle \times \langle -23, 7, -1\rangle = \langle 0, -56, -392\rangle$

(d) $(\mathbf{v}\times\mathbf{w})\times(\mathbf{u}\times\mathbf{v}) = \langle 0, 56, 392\rangle$

9. $\mathbf{u}\times\mathbf{v} = (\mathbf{i}+\mathbf{j})\times(\mathbf{i}+\mathbf{j}+\mathbf{k}) = \mathbf{k}-\mathbf{j}-\mathbf{k}+\mathbf{i} = \mathbf{i}-\mathbf{j}$, the direction cosines are $\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}, 0$

10. $\mathbf{u}\times\mathbf{v} = 12\mathbf{i} + 30\mathbf{j} - 6\mathbf{k}$, so $\pm\left(\dfrac{2}{\sqrt{30}}\mathbf{i} + \dfrac{\sqrt{5}}{\sqrt{6}}\mathbf{j} - \dfrac{1}{\sqrt{30}}\mathbf{k}\right)$

11. $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \langle 1,1,-3\rangle \times \langle -1,3,-1\rangle = \langle 8,4,4\rangle$, unit vectors are $\pm\dfrac{1}{\sqrt{6}}\langle 2,1,1\rangle$

12. A vector parallel to the yz-plane must be perpendicular to $\mathbf{i}$;
$\mathbf{i} \times (\mathbf{3i} - \mathbf{j} + \mathbf{2k}) = -2\mathbf{j} - \mathbf{k}$, $\|-2\mathbf{j}-\mathbf{k}\| = \sqrt{5}$, the unit vectors are $\pm(2\mathbf{j}+\mathbf{k})/\sqrt{5}$.

13. $A = \|\mathbf{u}\times\mathbf{v}\| = \|-7\mathbf{i}-\mathbf{j}+3\mathbf{k}\| = \sqrt{59}$

14. $A = \|\mathbf{u}\times\mathbf{v}\| = \|-6\mathbf{i}+4\mathbf{j}+7\mathbf{k}\| = \sqrt{101}$

15. $A = \dfrac{1}{2}\|\overrightarrow{PQ}\times\overrightarrow{PR}\| = \dfrac{1}{2}\|\langle -1,-5,2\rangle \times \langle 2,0,3\rangle\| = \dfrac{1}{2}\|\langle -15,7,10\rangle\| = \sqrt{374}/2$

16. $A = \dfrac{1}{2}\|\overrightarrow{PQ}\times\overrightarrow{PR}\| = \dfrac{1}{2}\|\langle -1,4,8\rangle \times \langle 5,2,12\rangle\| = \dfrac{1}{2}\|\langle 32,52,-22\rangle\| = 9\sqrt{13}$

17. 80 **18.** 29 **19.** -3 **20.** 1

21. $V = |\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})| = |-16| = 16$ **22.** $V = |\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})| = |45| = 45$

23. **(a)** $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = 0$, yes **(b)** $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = 0$, yes **(c)** $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = 245$, no

24. **(a)** $\mathbf{u}\cdot(\mathbf{w}\times\mathbf{v}) = -\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = -3$ **(b)** $(\mathbf{v}\times\mathbf{w})\cdot\mathbf{u} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = 3$
(c) $\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = 3$ **(d)** $\mathbf{v}\cdot(\mathbf{u}\times\mathbf{w}) = \mathbf{u}\cdot(\mathbf{w}\times\mathbf{v}) = -3$
(e) $(\mathbf{u}\times\mathbf{w})\cdot\mathbf{v} = \mathbf{u}\cdot(\mathbf{w}\times\mathbf{v}) = -3$ **(f)** $\mathbf{v}\cdot(\mathbf{w}\times\mathbf{w}) = 0$ because $\mathbf{w}\times\mathbf{w} = \mathbf{0}$

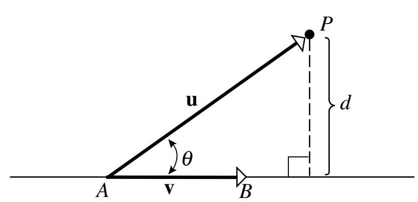
25. **(a)** $V = |\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})| = |-9| = 9$ **(b)** $A = \|\mathbf{u}\times\mathbf{w}\| = \|3\mathbf{i}-8\mathbf{j}+7\mathbf{k}\| = \sqrt{122}$
(c) $\mathbf{v}\times\mathbf{w} = -3\mathbf{i}-\mathbf{j}+2\mathbf{k}$ is perpendicular to the plane determined by $\mathbf{v}$ and $\mathbf{w}$; let θ be the angle between $\mathbf{u}$ and $\mathbf{v}\times\mathbf{w}$ then

$$\cos\theta = \frac{\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})}{\|\mathbf{u}\|\,\|\mathbf{v}\times\mathbf{w}\|} = \frac{-9}{\sqrt{14}\sqrt{14}} = -9/14$$

so the acute angle ϕ that $\mathbf{u}$ makes with the plane determined by $\mathbf{v}$ and $\mathbf{w}$ is
$\phi = \theta - \pi/2 = \sin^{-1}(9/14)$.

26. From the diagram,

$$d = \|\mathbf{u}\|\sin\theta = \frac{\|\mathbf{u}\|\|\mathbf{v}\|\sin\theta}{\|\mathbf{v}\|} = \frac{\|\mathbf{u}\times\mathbf{v}\|}{\|\mathbf{v}\|}$$



27. **(a)** $\mathbf{u} = \overrightarrow{AP} = -4\mathbf{i}+2\mathbf{k}$, $\mathbf{v} = \overrightarrow{AB} = -3\mathbf{i}+2\mathbf{j}-4\mathbf{k}$, $\mathbf{u}\times\mathbf{v} = -4\mathbf{i}-22\mathbf{j}-8\mathbf{k}$;
distance $= \|\mathbf{u}\times\mathbf{v}\|/\|\mathbf{v}\| = 2\sqrt{141/29}$
(b) $\mathbf{u} = \overrightarrow{AP} = 2\mathbf{i}+2\mathbf{j}$, $\mathbf{v} = \overrightarrow{AB} = -2\mathbf{i}+\mathbf{j}$, $\mathbf{u}\times\mathbf{v} = 6\mathbf{k}$; distance $= \|\mathbf{u}\times\mathbf{v}\|/\|\mathbf{v}\| = 6/\sqrt{5}$

28. Take $\mathbf{v}$ and $\mathbf{w}$ as sides of the (triangular) base, then area of base $= \dfrac{1}{2}\|\mathbf{v}\times\mathbf{w}\|$ and
height $= \|\text{proj}_{\mathbf{v}\times\mathbf{w}}\mathbf{u}\| = \dfrac{|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|}{\|\mathbf{v}\times\mathbf{w}\|}$ so $V = \dfrac{1}{3}$ (area of base) (height) $= \dfrac{1}{6}|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|$

29. $\overrightarrow{PQ} = \langle 3, -1, -3\rangle, \overrightarrow{PR} = \langle 2, -2, 1\rangle, \overrightarrow{PS} = \langle 4, -4, 3\rangle,$

$V = \dfrac{1}{6}|\ \overrightarrow{PQ} \cdot (\overrightarrow{PR} \times \overrightarrow{PS})| = \dfrac{1}{6}\,|-4| = 2/3$

30. **(a)** $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = -\dfrac{23}{49}$ **(b)** $\sin\theta = \dfrac{\|\mathbf{u}\times\mathbf{v}\|}{\|\mathbf{u}\|\|\mathbf{v}\|} = \dfrac{\|36\mathbf{i} - 24\mathbf{j}\|}{49} = \dfrac{12\sqrt{13}}{49}$

(c) $\dfrac{23^2}{49^2} + \dfrac{144\cdot 13}{49^2} = \dfrac{2401}{49^2} = 1$

31. From Theorems 12.3.3 and 12.4.5a it follows that $\sin\theta = \cos\theta$, so $\theta = \pi/4$.

32. $\|\mathbf{u}\times\mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2\sin^2\theta = \|\mathbf{u}\|^2\|\mathbf{v}\|^2(1-\cos^2\theta) = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2$

33. **(a)** $\mathbf{F} = 10\mathbf{j}$ and $\overrightarrow{PQ} = \mathbf{i}+\mathbf{j}+\mathbf{k}$, so the vector moment of $\mathbf{F}$ about P is

$\overrightarrow{PQ}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 0 & 10 & 0 \end{vmatrix} = -10\mathbf{i} + 10\mathbf{k}$, and the scalar moment is $10\sqrt{2}$ lb·ft.

The direction of rotation of the cube about P is counterclockwise looking along $\overrightarrow{PQ}\times\mathbf{F} = -10\mathbf{i}+10\mathbf{k}$ toward its initial point.

(b) $\mathbf{F} = 10\mathbf{j}$ and $\overrightarrow{PQ} = \mathbf{j}+\mathbf{k}$, so the vector moment of $\mathbf{F}$ about P is

$\overrightarrow{PQ}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ 0 & 10 & 0 \end{vmatrix} = -10\mathbf{i}$, and the scalar moment is 10 lb·ft. The direction of rotation of the cube about P is counterclockwise looking along $-10\mathbf{i}$ toward its initial point.

(c) $\mathbf{F} = 10\mathbf{j}$ and $\overrightarrow{PQ} = \mathbf{j}$, so the vector moment of $\mathbf{F}$ about P is

$\overrightarrow{PQ}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 10 & 0 \end{vmatrix} = \mathbf{0}$, and the scalar moment is 0 lb·ft. Since the force is parallel to the direction of motion, there is no rotation about P.

34. **(a)** $\mathbf{F} = \dfrac{1000}{\sqrt{2}}(-\mathbf{i}+\mathbf{k})$ and $\overrightarrow{PQ} = 2\mathbf{j} - \mathbf{k}$, so the vector moment of $\mathbf{F}$ about P is

$\overrightarrow{PQ}\times\mathbf{F} = 500\sqrt{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & -1 \\ -1 & 0 & 1 \end{vmatrix} = 500\sqrt{2}(2\mathbf{i}+\mathbf{j}+2\mathbf{k})$, and the scalar moment is $1500\sqrt{2}$ N·m.

(b) The direction angles of the vector moment of $\mathbf{F}$ about the point P are $\cos^{-1}(2/3) \approx 48^\circ, \cos^{-1}(1/3) \approx 71^\circ$, and $\cos^{-1}(2/3) \approx 48^\circ$.

35. Take the center of the bolt as the origin of the plane. Then $\mathbf{F}$ makes an angle 72° with the positive x-axis, so $\mathbf{F} = 200\cos 72^\circ\mathbf{i} + 200\sin 72^\circ\mathbf{j}$ and $\vec{PQ} = 0.2\,\mathbf{i} + 0.03\,\mathbf{j}$. The scalar moment is given by

$$\left\|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.2 & 0.03 & 0 \\ 200\cos 72^\circ & 200\sin 72^\circ & 0 \end{vmatrix}\right\| = \left|40\frac{1}{4}(\sqrt{5}-1) - 6\frac{1}{4}\sqrt{10+2\sqrt{5}}\right| \approx 36.1882 \text{ N·m}.$$

36. Part (b) : let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$; show that $\mathbf{u} \times (\mathbf{v} + \mathbf{w})$ and $(\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$ are the same.

Part (c) :
$$\begin{aligned}(\mathbf{u}+\mathbf{v}) \times \mathbf{w} &= -[\mathbf{w} \times (\mathbf{u}+\mathbf{v})] \text{ from Part (a)}\\ &= -[(\mathbf{w} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{v})] \text{ from Part (b)}\\ &= (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w}) \text{ from Part (a)}\end{aligned}$$

37. Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$; show that $k(\mathbf{u} \times \mathbf{v})$, $(k\mathbf{u}) \times \mathbf{v}$, and $\mathbf{u} \times (k\mathbf{v})$ are all the same; Part (e) is proved in a similar fashion.

38. Suppose the first two rows are interchanged. Then by definition,

$$\begin{aligned}\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} &= b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} - b_2 \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} + b_3 \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}\\ &= b_1(a_2c_3 - a_3c_2) - b_2(a_1c_3 - a_3c_1) + b_3(a_1c_2 - a_2c_1),\end{aligned}$$

which is the negative of the right hand side of (2) after expansion. If two other rows were to be exchanged, a similar proof would hold. Finally, suppose Δ were a determinant with two identical rows. Then the value is unchanged if we interchange those two rows, yet $\Delta = -\Delta$ by Part (b) of Theorem 12.4.1. Hence $\Delta = -\Delta, \Delta = 0$.

39. $-8\mathbf{i} - 8\mathbf{k}, -8\mathbf{i} - 20\mathbf{j} + 2\mathbf{k}$

40. **(a)** From the first formula in Exercise 39 it follows that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ is a linear combination of $\mathbf{v}$ and $\mathbf{w}$ and hence lies in the plane determined by them, and from the second formula it follows that $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$ and hence lies in their plane.

(b) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ is orthogonal to $\mathbf{v} \times \mathbf{w}$ and hence lies in the plane of $\mathbf{v}$ and $\mathbf{w}$; similarly for $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

41. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ lie in the same plane then $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c} \times \mathbf{d}$ are parallel so $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \mathbf{0}$

42. Let $\mathbf{u}$ and $\mathbf{v}$ be the vectors from a point on the curve to the points $(2, -1, 0)$ and $(3, 2, 2)$, respectively. Then $\mathbf{u} = (2-x)\mathbf{i} + (-1 - \ln x)\mathbf{j}$ and $\mathbf{v} = (3-x)\mathbf{i} + (2 - \ln x)\mathbf{j} + 2\mathbf{k}$. The area of the triangle is given by $A = (1/2)\|\mathbf{u} \times \mathbf{v}\|$; solve $dA/dx = 0$ for x to get $x = 2.091581$. The minimum area is 1.887850.

43. $\overrightarrow{PQ'} \times \mathbf{F} = \overrightarrow{PQ} \times \mathbf{F} + \overrightarrow{QQ'} \times \mathbf{F} = \overrightarrow{PQ} \times \mathbf{F}$, since $\mathbf{F}$ and $\overrightarrow{QQ'}$ are parallel.

EXERCISE SET 12.5

In many of the Exercises in this section other answers are also possible.

1. **(a)** L_1: $P(1,0), \mathbf{v} = \mathbf{j}, x = 1, y = t$
L_2: $P(0,1), \mathbf{v} = \mathbf{i}, x = t, y = 1$
L_3: $P(0,0), \mathbf{v} = \mathbf{i} + \mathbf{j}, x = t, y = t$

(b) L_1: $P(1,1,0), \mathbf{v} = \mathbf{k}, x = 1, y = 1, z = t$
L_2: $P(0,1,1), \mathbf{v} = \mathbf{i}, x = t, y = 1, z = 1$
L_3: $P(1,0,1), \mathbf{v} = \mathbf{j}, x = 1, y = t, z = 1$
L_4: $P(0,0,0), \mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}, x = t, y = t, z = t$

2. **(a)** L_1: $x = t, y = 1, 0 \le t \le 1$
L_2: $x = 1, y = t, 0 \le t \le 1$
L_3: $x = t, y = t, 0 \le t \le 1$

(b) L_1: $x = 1, y = 1, z = t, 0 \le t \le 1$
L_2: $x = t, y = 1, z = 1, 0 \le t \le 1$
L_3: $x = 1, y = t, z = 1, 0 \le t \le 1$
L_4: $x = t, y = t, z = t, 0 \le t \le 1$

3. **(a)** $\overrightarrow{P_1P_2} = \langle 2, 3\rangle$ so $x = 3 + 2t$, $y = -2 + 3t$ for the line; for the line segment add the condition $0 \le t \le 1$.

(b) $\overrightarrow{P_1P_2} = \langle -3, 6, 1\rangle$ so $x = 5 - 3t$, $y = -2 + 6t$, $z = 1 + t$ for the line; for the line segment add the condition $0 \le t \le 1$.

4. **(a)** $\overrightarrow{P_1P_2} = \langle -3, -5\rangle$ so $x = -3t$, $y = 1 - 5t$ for the line; for the line segment add the condition $0 \le t \le 1$.

(b) $\overrightarrow{P_1P_2} = \langle 0, 0, -3\rangle$ so $x = -1$, $y = 3$,$z = 5 - 3t$ for the line; for the line segment add the condition $0 \le t \le 1$.

5. **(a)** $x = 2 + t, y = -3 - 4t$ **(b)** $x = t, y = -t, z = 1 + t$

6. **(a)** $x = 3 + 2t, y = -4 + t$ **(b)** $x = -1 - t, y = 3t, z = 2$

7. **(a)** $\mathbf{r}_0 = 2\mathbf{i} - \mathbf{j}$ so $P(2, -1)$ is on the line, and $\mathbf{v} = 4\mathbf{i} - \mathbf{j}$ is parallel to the line.
(b) At $t = 0, P(-1, 2, 4)$ is on the line, and $\mathbf{v} = 5\mathbf{i} + 7\mathbf{j} - 8\mathbf{k}$ is parallel to the line.

8. **(a)** At $t = 0, P(-1, 5)$ is on the line, and $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$ is parallel to the line.
(b) $\mathbf{r}_0 = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$ so $P(1, 1, -2)$ is on the line, and $\mathbf{v} = \mathbf{j}$ is parallel to the line.

9. **(a)** $\langle x, y\rangle = \langle -3, 4\rangle + t\langle 1, 5\rangle; \mathbf{r} = -3\mathbf{i} + 4\mathbf{j} + t(\mathbf{i} + 5\mathbf{j})$
(b) $\langle x, y, z\rangle = \langle 2, -3, 0\rangle + t\langle -1, 5, 1\rangle; \mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + t(-\mathbf{i} + 5\mathbf{j} + \mathbf{k})$

10. **(a)** $\langle x, y\rangle = \langle 0, -2\rangle + t\langle 1, 1\rangle; \mathbf{r} = -2\mathbf{j} + t(\mathbf{i} + \mathbf{j})$
(b) $\langle x, y, z\rangle = \langle 1, -7, 4\rangle + t\langle 1, 3, -5\rangle; \mathbf{r} = \mathbf{i} - 7\mathbf{j} + 4\mathbf{k} + t(\mathbf{i} + 3\mathbf{j} - 5\mathbf{k})$

11. $x = -5 + 2t$, $y = 2 - 3t$ **12.** $x = t$, $y = 3 - 2t$

13. $2x + 2yy' = 0$, $y' = -x/y = -(3)/(-4) = 3/4$, $\mathbf{v} = 4\mathbf{i} + 3\mathbf{j}$; $x = 3 + 4t$, $y = -4 + 3t$

14. $y' = 2x = 2(-2) = -4$, $\mathbf{v} = \mathbf{i} - 4\mathbf{j}$; $x = -2 + t$, $y = 4 - 4t$

15. $x = -1 + 3t$, $y = 2 - 4t$, $z = 4 + t$ **16.** $x = 2 - t$, $y = -1 + 2t$, $z = 5 + 7t$

17. The line is parallel to the vector $\langle 2, -1, 2\rangle$ so $x = -2 + 2t$, $y = -t$, $z = 5 + 2t$.

18. The line is parallel to the vector $\langle 1, 1, 0\rangle$ so $x = t$, $y = t$, $z = 0$.

19. **(a)** $y = 0, 2 - t = 0, t = 2, x = 7$ **(b)** $x = 0, 1 + 3t = 0, t = -1/3, y = 7/3$

(c) $y = x^2, 2 - t = (1 + 3t)^2, 9t^2 + 7t - 1 = 0, t = \dfrac{-7 \pm \sqrt{85}}{18}, x = \dfrac{-1 \pm \sqrt{85}}{6}, y = \dfrac{43 \mp \sqrt{85}}{18}$

20. $(4t)^2 + (3t)^2 = 25, 25t^2 = 25, t = \pm 1$, the line intersects the circle at $\pm\langle 4, 3\rangle$

21. **(a)** $z = 0$ when $t = 3$ so the point is $(-2, 10, 0)$
(b) $y = 0$ when $t = -2$ so the point is $(-2, 0, -5)$
(c) x is always -2 so the line does not intersect the yz-plane

22. **(a)** $z = 0$ when $t = 4$ so the point is (7,7,0)
(b) $y = 0$ when $t = -3$ so the point is $(-7, 0, 7)$
(c) $x = 0$ when $t = 1/2$ so the point is $(0, 7/2, 7/2)$

23. $(1+t)^2+(3-t)^2=16$, $t^2-2t-3=0$, $(t+1)(t-3)=0$; $t=-1, 3$. The points of intersection are $(0,4,-2)$ and (4,0,6).

24. $2(3t)+3(-1+2t)=6$, $12t=9$; $t=3/4$. The point of intersection is $(5/4, 9/4, 1/2)$.

25. The lines intersect if we can find values of t_1 and t_2 that satisfy the equations $2+t_1=2+t_2$, $2+3t_1=3+4t_2$, and $3+t_1=4+2t_2$. Solutions of the first two of these equations are $t_1=-1$, $t_2=-1$ which also satisfy the third equation so the lines intersect at $(1,-1,2)$.

26. Solve the equations $-1+4t_1=-13+12t_2$, $3+t_1=1+6t_2$, and $1=2+3t_2$. The third equation yields $t_2=-1/3$ which when substituted into the first and second equations gives $t_1=-4$ in both cases; the lines intersect at $(-17,-1,1)$.

27. The lines are parallel, respectively, to the vectors $\langle 7,1,-3\rangle$ and $\langle -1,0,2\rangle$. These vectors are not parallel so the lines are not parallel. The system of equations $1+7t_1=4-t_2$, $3+t_1=6$, and $5-3t_1=7+2t_2$ has no solution so the lines do not intersect.

28. The vectors $\langle 8,-8,10\rangle$ and $\langle 8,-3,1\rangle$ are not parallel so the lines are not parallel. The lines do not intersect because the system of equations $2+8t_1=3+8t_2$, $6-8t_1=5-3t_2$, $10t_1=6+t_2$ has no solution.

29. The lines are parallel, respectively, to the vectors $\mathbf{v}_1=\langle -2,1,-1\rangle$ and $\mathbf{v}_2=\langle -4,2,-2\rangle$; $\mathbf{v}_2=2\mathbf{v}_1$, $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel so the lines are parallel.

30. The lines are not parallel because the vectors $\langle 3,-2,3\rangle$ and $\langle 9,-6,8\rangle$ are not parallel.

31. $\overrightarrow{P_1P_2}=\langle 3,-7,-7\rangle$, $\overrightarrow{P_2P_3}=\langle -9,-7,-3\rangle$; these vectors are not parallel so the points do not lie on the same line.

32. $\overrightarrow{P_1P_2}=\langle 2,-4,-4\rangle$, $\overrightarrow{P_2P_3}=\langle 1,-2,-2\rangle$; $\overrightarrow{P_1P_2}=2\,\overrightarrow{P_2P_3}$ so the vectors are parallel and the points lie on the same line.

33. If t_2 gives the point $\langle -1+3t_2, 9-6t_2\rangle$ on the second line, then $t_1=4-3t_2$ yields the point $\langle 3-(4-3t_2), 1+2(4-3t_2)\rangle=\langle -1+3t_2, 9-6t_2\rangle$ on the first line, so each point of L_2 is a point of L_1; the converse is shown with $t_2=(4-t_1)/3$.

34. If t_1 gives the point $\langle 1+3t_1, -2+t_1, 2t_1\rangle$ on L_1, then $t_2=(1-t_1)/2$ gives the point $\langle 4-6(1-t_1)/2, -1-2(1-t_1)/2, 2-4(1-t_1)/2\rangle=\langle 1+3t_1, -2+t_1, 2t_1\rangle$ on L_2, so each point of L_1 is a point of L_2; the converse is shown with $t_1=1-2t_2$.

35. The line segment joining the points (1,0) and $(-3,6)$.

36. The line segment joining the points $(-2,1,4)$ and (7,1,1).

37. $A(3,0,1)$ and $B(2,1,3)$ are on the line, and (method of Exercise 25) $\overrightarrow{AP}=-5\mathbf{i}+\mathbf{j}$, $\overrightarrow{AB}=-\mathbf{i}+\mathbf{j}+2\mathbf{k}$, $\|\text{proj}_{\overrightarrow{AB}}\overrightarrow{AP}\|=|\overrightarrow{AP}\cdot\overrightarrow{AB}|/\|\overrightarrow{AB}\|=\sqrt{6}$ and $\|\overrightarrow{AP}\|=\sqrt{26}$, so distance $=\sqrt{26-6}=2\sqrt{5}$. Using the method of Exercise 26, distance $=\dfrac{\|\overrightarrow{AP}\times\overrightarrow{AB}\|}{\|\overrightarrow{AB}\|}=2\sqrt{5}$.

38. $A(2,-1,0)$ and $B(3,-2,3)$ are on the line, and (method of Exercise 25)
$\overrightarrow{AP} = -\mathbf{i}+5\mathbf{j}-3\mathbf{k}$, $\overrightarrow{AB} = \mathbf{i}-\mathbf{j}+3\mathbf{k}$, $\|\text{proj}_{\overrightarrow{AB}}\overrightarrow{AP}\| = |\overrightarrow{AP}\cdot\overrightarrow{AB}|/\|\overrightarrow{AB}\| = \dfrac{15}{\sqrt{11}}$ and $\|\overrightarrow{AP}\| = \sqrt{35}$, so distance $= \sqrt{35-225/11} = 4\sqrt{10/11}$. Using the method of Exercise 26, distance $= \dfrac{\|\overrightarrow{AP}\times\overrightarrow{AB}\|}{\|\overrightarrow{AB}\|} = 4\sqrt{10/11}$.

39. The vectors $\mathbf{v}_1 = -\mathbf{i}+2\mathbf{j}+\mathbf{k}$ and $\mathbf{v}_2 = 2\mathbf{i}-4\mathbf{j}-2\mathbf{k}$ are parallel to the lines, $\mathbf{v}_2 = -2\mathbf{v}_1$ so $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel. Let $t=0$ to get the points $P(2,0,1)$ and $Q(1,3,5)$ on the first and second lines, respectively. Let $\mathbf{u} = \overrightarrow{PQ} = -\mathbf{i}+3\mathbf{j}+4\mathbf{k}$, $\mathbf{v} = \frac{1}{2}\mathbf{v}_2 = \mathbf{i}-2\mathbf{j}-\mathbf{k}$; $\mathbf{u}\times\mathbf{v} = 5\mathbf{i}+3\mathbf{j}-\mathbf{k}$; by the method of Exercise 26 of Section 12.4, distance $= \|\mathbf{u}\times\mathbf{v}\|/\|\mathbf{v}\| = \sqrt{35/6}$.

40. The vectors $\mathbf{v}_1 = 2\mathbf{i}+4\mathbf{j}-6\mathbf{k}$ and $\mathbf{v}_2 = 3\mathbf{i}+6\mathbf{j}-9\mathbf{k}$ are parallel to the lines, $\mathbf{v}_2 = (3/2)\mathbf{v}_1$ so $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel. Let $t=0$ to get the points $P(0,3,2)$ and $Q(1,0,0)$ on the first and second lines, respectively. Let $\mathbf{u} = \overrightarrow{PQ} = \mathbf{i}-3\mathbf{j}-2\mathbf{k}$, $\mathbf{v} = \frac{1}{2}\mathbf{v}_1 = \mathbf{i}+2\mathbf{j}-3\mathbf{k}$; $\mathbf{u}\times\mathbf{v} = 13\mathbf{i}+\mathbf{j}+5\mathbf{k}$, distance $= \|\mathbf{u}\times\mathbf{v}\|/\|\mathbf{v}\| = \sqrt{195/14}$ (Exer. 26, Section 12.4).

41. **(a)** The line is parallel to the vector $\langle x_1-x_0, y_1-y_0, z_1-z_0\rangle$ so
$x = x_0+(x_1-x_0)\,t$, $y = y_0+(y_1-y_0)\,t$, $z = z_0+(z_1-z_0)\,t$

(b) The line is parallel to the vector $\langle a,b,c\rangle$ so $x = x_1+at$, $y = y_1+bt$, $z = z_1+ct$

42. Solve each of the given parametric equations (2) for t to get $t = (x-x_0)/a$, $t = (y-y_0)/b$, $t = (z-z_0)/c$, so (x,y,z) is on the line if and only if $(x-x_0)/a = (y-y_0)/b = (z-z_0)/c$.

43. **(a)** It passes through the point $(1,-3,5)$ and is parallel to $\mathbf{v} = 2\mathbf{i}+4\mathbf{j}+\mathbf{k}$

(b) $\langle x,y,z\rangle = \langle 1+2t, -3+4t, 5+t\rangle$

44. Let the desired point be $P(x_0,y_0,z_0)$, then $\overrightarrow{P_1P} = (2/3)\,\overrightarrow{P_1P_2}$,
$\langle x_0-1, y_0-4, z_0+3\rangle = (2/3)\langle 0,1,2\rangle = \langle 0, 2/3, 4/3\rangle$; equate corresponding components to get $x_0 = 1$, $y_0 = 14/3$, $z_0 = -5/3$.

45. **(a)** Let $t=3$ and $t=-2$, respectively, in the equations for L_1 and L_2.

(b) $\mathbf{u} = 2\mathbf{i}-\mathbf{j}-2\mathbf{k}$ and $\mathbf{v} = \mathbf{i}+3\mathbf{j}-\mathbf{k}$ are parallel to L_1 and L_2,
$\cos\theta = \mathbf{u}\cdot\mathbf{v}/(\|\mathbf{u}\|\,\|\mathbf{v}\|) = 1/(3\sqrt{11})$, $\theta \approx 84°$.

(c) $\mathbf{u}\times\mathbf{v} = 7\mathbf{i}+7\mathbf{k}$ is perpendicular to both L_1 and L_2, and hence so is $\mathbf{i}+\mathbf{k}$, thus $x = 7+t$, $y=-1$, $z=-2+t$.

46. **(a)** Let $t = 1/2$ and $t=1$, respectively, in the equations for L_1 and L_2.

(b) $\mathbf{u} = 4\mathbf{i}-2\mathbf{j}+2\mathbf{k}$ and $\mathbf{v} = \mathbf{i}-\mathbf{j}+4\mathbf{k}$ are parallel to L_1 and L_2,
$\cos\theta = \mathbf{u}\cdot\mathbf{v}/(\|\mathbf{u}\|\,\|\mathbf{v}\|) = 14/\sqrt{432}$, $\theta \approx 48°$.

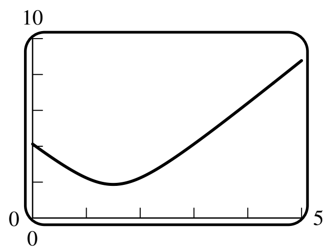
(c) $\mathbf{u}\times\mathbf{v} = -6\mathbf{i}-14\mathbf{j}-2\mathbf{k}$ is perpendicular to both L_1 and L_2, and hence so is $3\mathbf{i}+7\mathbf{j}+\mathbf{k}$, thus $x = 2+3t$, $y = 7t$, $z = 3+t$.

47. (0,1,2) is on the given line ($t=0$) so $\mathbf{u} = \mathbf{j}-\mathbf{k}$ is a vector from this point to the point (0,2,1), $\mathbf{v} = 2\mathbf{i}-\mathbf{j}+\mathbf{k}$ is parallel to the given line. $\mathbf{u}\times\mathbf{v} = -2\mathbf{j}-2\mathbf{k}$, and hence $\mathbf{w} = \mathbf{j}+\mathbf{k}$, is perpendicular to both lines so $\mathbf{v}\times\mathbf{w} = -2\mathbf{i}-2\mathbf{j}+2\mathbf{k}$, and hence $\mathbf{i}+\mathbf{j}-\mathbf{k}$, is parallel to the line we seek. Thus $x = t$, $y = 2+t$, $z = 1-t$ are parametric equations of the line.

48. $(-2,4,2)$ is on the given line $(t=0)$ so $\mathbf{u}=5\mathbf{i}-3\mathbf{j}-4\mathbf{k}$ is a vector from this point to the point $(3,1,-2)$, $\mathbf{v}=2\mathbf{i}+2\mathbf{j}+\mathbf{k}$ is parallel to the given line. $\mathbf{u}\times\mathbf{v}=5\mathbf{i}-13\mathbf{j}+16\mathbf{k}$ is perpendicular to both lines so $\mathbf{v}\times(\mathbf{u}\times\mathbf{v})=45\mathbf{i}-27\mathbf{j}-36\mathbf{k}$, and hence $5\mathbf{i}-3\mathbf{j}-4\mathbf{k}$ is parallel to the line we seek. Thus $x=3+5t$, $y=1-3t$, $z=-2-4t$ are parametric equations of the line.

49. **(a)** When $t=0$ the bugs are at $(4,1,2)$ and (0,1,1) so the distance between them is $\sqrt{4^2+0^2+1^2}=\sqrt{17}$ cm.

(b)



(c) The distance has a minimum value.

(d) Minimize D^2 instead of D (the distance between the bugs).
$D^2=[t-(4-t)]^2+[(1+t)-(1+2t)]^2+[(1+2t)-(2+t)]^2=6t^2-18t+17$,
$d(D^2)/dt=12t-18=0$ when $t=3/2$; the minimum distance is $\sqrt{6(3/2)^2-18(3/2)+17}=\sqrt{14}/2$ cm.

50. The line intersects the xz-plane when $t=-1$, the xy-plane when $t=3/2$. Along the line, $T=25t^2(1+t)(3-2t)$ for $-1\le t\le 3/2$. Solve $dT/dt=0$ for t to find that the maximum value of T is about 50.96 when $t\approx 1.073590$.

EXERCISE SET 12.6

1. $x=3, y=4, z=5$

2. $x=x_0, y=y_0, z=z_0$

3. $(x-2)+4(y-6)+2(z-1)=0$, $x+4y+2z=28$

4. $-(x+1)+7(y+1)+6(z-2)=0$, $-x+7y+6z=6$

5. $z=0$

6. $2x-3y-4z=0$

7. $\mathbf{n}=\mathbf{i}-\mathbf{j}, x-y=0$

8. $\mathbf{n}=\mathbf{i}+\mathbf{j}, P(1,0,0), (x-1)+y=0, x+y=1$

9. $\mathbf{n}=\mathbf{j}+\mathbf{k}, P(0,1,0), (y-1)+z=0, y+z=1$

10. $\mathbf{n}=\mathbf{j}-\mathbf{k}, y-z=0$

11. $\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}=\langle 2,1,2\rangle\times\langle 3,-1,-2\rangle=\langle 0,10,-5\rangle$,for convenience choose $\langle 0,2,-1\rangle$ which is also normal to the plane. Use any of the given points to get $2y-z=1$

12. $\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}=\langle -1,-1,-2\rangle\times\langle -4,1,1\rangle=\langle 1,9,-5\rangle, x+9y-5z=16$

13. **(a)** parallel, because $\langle 2,-8,-6\rangle$ and $\langle -1,4,3\rangle$ are parallel

(b) perpendicular, because $\langle 3,-2,1\rangle$ and $\langle 4,5,-2\rangle$ are orthogonal

(c) neither, because $\langle 1,-1,3\rangle$ and $\langle 2,0,1\rangle$ are neither parallel nor orthogonal

14. **(a)** neither, because $\langle 3,-2,1\rangle$ and $\langle 6,-4,3\rangle$ are neither parallel nor orthogonal

(b) parallel, because $\langle 4,-1,-2\rangle$ and $\langle 1,-1/4,-1/2\rangle$ are parallel

(c) perpendicular, because $\langle 1,4,7\rangle$ and $\langle 5,-3,1\rangle$ are orthogonal

15. **(a)** parallel, because $\langle 2,-1,-4\rangle$ and $\langle 3,2,1\rangle$ are orthogonal

(b) neither, because $\langle 1,2,3\rangle$ and $\langle 1,-1,2\rangle$ are neither parallel nor orthogonal

(c) perpendicular, because $\langle 2,1,-1\rangle$ and $\langle 4,2,-2\rangle$ are parallel

16. **(a)** parallel, because $\langle -1,1,-3\rangle$ and $\langle 2,2,0\rangle$ are orthogonal

(b) perpendicular, because $\langle -2,1,-1\rangle$ and $\langle 6,-3,3\rangle$ are parallel

(c) neither, because $\langle 1,-1,1\rangle$ and $\langle 1,1,1\rangle$ are neither parallel nor orthogonal

17. **(a)** $3t-2t+t-5=0$, $t=5/2$ so $x=y=z=5/2$, the point of intersection is $(5/2,5/2,5/2)$

(b) $2(2-t)+(3+t)+t=1$ has no solution so the line and plane do not intersect

18. **(a)** $2(3t)-5t+(-t)+1=0,\ 1=0$ has no solution so the line and the plane do not intersect.

(b) $(1+t)-(-1+3t)+4(2+4t)=7$, $t=-3/14$ so $x=1-3/14=11/14$,
$y=-1-9/14=-23/14$, $z=2-12/14=8/7$, the point is $(11/14,-23/14,8/7)$

19. $\mathbf{n}_1=\langle 1,0,0\rangle, \mathbf{n}_2=\langle 2,-1,1\rangle, \mathbf{n}_1\cdot\mathbf{n}_2=2$ so

$$\cos\theta=\frac{\mathbf{n}_1\cdot\mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|}=\frac{2}{\sqrt{1}\sqrt{6}}=2/\sqrt{6}, \theta=\cos^{-1}(2/\sqrt{6})\approx 35^\circ$$

20. $\mathbf{n}_1=\langle 1,2,-2\rangle, \mathbf{n}_2=\langle 6,-3,2\rangle, \mathbf{n}_1\cdot\mathbf{n}_2=-4$ so

$$\cos\theta=\frac{(-\mathbf{n}_1)\cdot\mathbf{n}_2}{\|-\mathbf{n}_1\|\,\|\mathbf{n}_2\|}=\frac{4}{(3)(7)}=4/21, \theta=\cos^{-1}(4/21)\approx 79^\circ$$

(Note: $-\mathbf{n}_1$ is used instead of $\mathbf{n}_1$ to get a value of θ in the range $[0,\pi/2]$)

21. $\langle 4,-2,7\rangle$ is normal to the desired plane and (0,0,0) is a point on it; $4x-2y+7z=0$

22. $\mathbf{v}=\langle 3,2,-1\rangle$ is parallel to the line and $\mathbf{n}=\langle 1,-2,1\rangle$ is normal to the given plane so $\mathbf{v}\times\mathbf{n}=\langle 0,-4,-8\rangle$ is normal to the desired plane. Let $t=0$ in the line to get $(-2,4,3)$ which is also a point on the desired plane, use this point and (for convenience) the normal $\langle 0,1,2\rangle$ to find that $y+2z=10$.

23. Find two points P_1 and P_2 on the line of intersection of the given planes and then find an equation of the plane that contains P_1, P_2, and the given point $P_0(-1,4,2)$. Let (x_0,y_0,z_0) be on the line of intersection of the given planes; then $4x_0-y_0+z_0-2=0$ and $2x_0+y_0-2z_0-3=0$, eliminate y_0 by addition of the equations to get $6x_0-z_0-5=0$; if $x_0=0$ then $z_0=-5$, if $x_0=1$ then $z_0=1$. Substitution of these values of x_0 and z_0 into either of the equations of the planes gives the corresponding values $y_0=-7$ and $y_0=3$ so $P_1(0,-7,-5)$ and $P_2(1,3,1)$ are on the line of intersection of the planes. $\overrightarrow{P_0P_1}\times\overrightarrow{P_0P_2}=\langle 4,-13,21\rangle$ is normal to the desired plane whose equation is $4x-13y+21z=-14$.

24. $\langle 1,2,-1\rangle$ is parallel to the line and hence normal to the plane $x+2y-z=10$

25. $\mathbf{n}_1=\langle 2,1,1\rangle$ and $\mathbf{n}_2=\langle 1,2,1\rangle$ are normals to the given planes, $\mathbf{n}_1\times\mathbf{n}_2=\langle -1,-1,3\rangle$ so $\langle 1,1,-3\rangle$ is normal to the desired plane whose equation is $x+y-3z=6$.

26. $\mathbf{n} = \langle 4, -1, 3\rangle$ is normal to the given plane, $\overrightarrow{P_1P_2} = \langle 3, -1, -1\rangle$ is parallel to the line through the given points, $\mathbf{n} \times \overrightarrow{P_1P_2} = \langle 4, 13, -1\rangle$ is normal to the desired plane whose equation is $4x + 13y - z = 1$.

27. $\mathbf{n}_1 = \langle 2, -1, 1\rangle$ and $\mathbf{n}_2 = \langle 1, 1, -2\rangle$ are normals to the given planes, $\mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, 5, 3\rangle$ is normal to the desired plane whose equation is $x + 5y + 3z = -6$.

28. Let $t = 0$ and $t = 1$ to get the points $P_1(-1, 0, -4)$ and $P_2(0, 1, -2)$ that lie on the line. Denote the given point by P_0, then $\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2} = \langle 7, -1, -3\rangle$ is normal to the desired plane whose equation is $7x - y - 3z = 5$.

29. The plane is the perpendicular bisector of the line segment that joins $P_1(2, -1, 1)$ and $P_2(3, 1, 5)$. The midpoint of the line segment is $(5/2, 0, 3)$ and $\overrightarrow{P_1P_2} = \langle 1, 2, 4\rangle$ is normal to the plane so an equation is $x + 2y + 4z = 29/2$.

30. $\mathbf{n}_1 = \langle 2, -1, 1\rangle$ and $\mathbf{n}_2 = \langle 0, 1, 1\rangle$ are normals to the given planes, $\mathbf{n}_1 \times \mathbf{n}_2 = \langle -2, -2, 2\rangle$ so $\mathbf{n} = \langle 1, 1, -1\rangle$ is parallel to the line of intersection of the planes. $\mathbf{v} = \langle 3, 1, 2\rangle$ is parallel to the given line, $\mathbf{v} \times \mathbf{n} = \langle -3, 5, 2\rangle$ so $\langle 3, -5, -2\rangle$ is normal to the desired plane. Let $t = 0$ to find the point (0,1,0) that lies on the given line and hence on the desired plane. An equation of the plane is $3x - 5y - 2z = -5$.

31. The line is parallel to the line of intersection of the planes if it is parallel to both planes. Normals to the given planes are $\mathbf{n}_1 = \langle 1, -4, 2\rangle$ and $\mathbf{n}_2 = \langle 2, 3, -1\rangle$ so $\mathbf{n}_1 \times \mathbf{n}_2 = \langle -2, 5, 11\rangle$ is parallel to the line of intersection of the planes and hence parallel to the desired line whose equations are $x = 5 - 2t$, $y = 5t$, $z = -2 + 11t$.

32. Denote the points by A, B, C, and D, respectively. The points lie in the same plane if $\overrightarrow{AB} \times \overrightarrow{AC}$ and $\overrightarrow{AB} \times \overrightarrow{AD}$ are parallel (method 1). $\overrightarrow{AB} \times \overrightarrow{AC} = \langle 0, -10, 5\rangle$, $\overrightarrow{AB} \times \overrightarrow{AD} = \langle 0, 16, -8\rangle$, these vectors are parallel because $\langle 0, -10, 5\rangle = (-10/16)\langle 0, 16, -8\rangle$. The points lie in the same plane if D lies in the plane determined by A, B, C (method 2), and since $\overrightarrow{AB} \times \overrightarrow{AC} = \langle 0, -10, 5\rangle$, an equation of the plane is $-2y + z + 1 = 0$, $2y - z = 1$ which is satisfied by the coordinates of D.

33. $\mathbf{v} = \langle 0, 1, 1\rangle$ is parallel to the line.

(a) For any t, $6 \cdot 0 + 4t - 4t = 0$, so $(0, t, t)$ is in the plane.

(b) $\mathbf{n} = \langle 5, -3, 3\rangle$ is normal to the plane, $\mathbf{v} \cdot \mathbf{n} = 0$ so the line is parallel to the plane. (0,0,0) is on the line, $(0, 0, 1/3)$ is on the plane. The line is below the plane because (0,0,0) is below $(0, 0, 1/3)$.

(c) $\mathbf{n} = \langle 6, 2, -2\rangle$, $\mathbf{v} \cdot \mathbf{n} = 0$ so the line is parallel to the plane. (0,0,0) is on the line, $(0, 0, -3/2)$ is on the plane. The line is above the plane because (0,0,0) is above $(0, 0, -3/2)$.

34. The intercepts correspond to the points $A(a, 0, 0)$, $B(0, b, 0)$, and $C(0, 0, c)$. $\overrightarrow{AB} \times \overrightarrow{AC} = \langle bc, ac, ab\rangle$ is normal to the plane so $bcx + acy + abz = abc$ or $x/a + y/b + z/c = 1$.

35. $\mathbf{v}_1 = \langle 1, 2, -1\rangle$ and $\mathbf{v}_2 = \langle -1, -2, 1\rangle$ are parallel, respectively, to the given lines and to each other so the lines are parallel. Let $t = 0$ to find the points $P_1(-2, 3, 4)$ and $P_2(3, 4, 0)$ that lie, respectively, on the given lines. $\mathbf{v}_1 \times \overrightarrow{P_1P_2} = \langle -7, -1, -9\rangle$ so $\langle 7, 1, 9\rangle$ is normal to the desired plane whose equation is $7x + y + 9z = 25$.

36. The system $4t_1 - 1 = 12t_2 - 13$, $t_1 + 3 = 6t_2 + 1$, $1 = 3t_2 + 2$ has the solution (Exercise 26, Section 12.5) $t_1 = -4$, $t_2 = -1/3$ so $(-17, -1, 1)$ is the point of intersection. $\mathbf{v}_1 = \langle 4, 1, 0\rangle$ and $\mathbf{v}_2 = \langle 12, 6, 3\rangle$ are (respectively) parallel to the lines, $\mathbf{v}_1 \times \mathbf{v}_2 = \langle 3, -12, 12\rangle$ so $\langle 1, -4, 4\rangle$ is normal to the desired plane whose equation is $x - 4y + 4z = -9$.

37. $\mathbf{n}_1 = \langle -2, 3, 7\rangle$ and $\mathbf{n}_2 = \langle 1, 2, -3\rangle$ are normals to the planes, $\mathbf{n}_1 \times \mathbf{n}_2 = \langle -23, 1, -7\rangle$ is parallel to the line of intersection. Let $z = 0$ in both equations and solve for x and y to get $x = -11/7$, $y = -12/7$ so $(-11/7, -12/7, 0)$ is on the line, a parametrization of which is
$x = -11/7 - 23t$, $y = -12/7 + t$, $z = -7t$.

38. Similar to Exercise 37 with $\mathbf{n}_1 = \langle 3, -5, 2\rangle$, $\mathbf{n}_2 = \langle 0, 0, 1\rangle$, $\mathbf{n}_1 \times \mathbf{n}_2 = \langle -5, -3, 0\rangle$. $z = 0$ so $3x - 5y = 0$, let $x = 0$ then $y = 0$ and (0,0,0) is on the line, a parametrization of which is
$x = -5t$, $y = -3t$, $z = 0$.

39. $D = |2(1) - 2(-2) + (3) - 4|/\sqrt{4 + 4 + 1} = 5/3$

40. $D = |3(0) + 6(1) - 2(5) - 5|/\sqrt{9 + 36 + 4} = 9/7$

41. (0,0,0) is on the first plane so $D = |6(0) - 3(0) - 3(0) - 5|/\sqrt{36 + 9 + 9} = 5/\sqrt{54}$.

42. (0,0,1) is on the first plane so $D = |(0) + (0) + (1) + 1|/\sqrt{1 + 1 + 1} = 2/\sqrt{3}$.

43. (1,3,5) and (4,6,7) are on L_1 and L_2, respectively. $\mathbf{v}_1 = \langle 7, 1, -3\rangle$ and $\mathbf{v}_2 = \langle -1, 0, 2\rangle$ are, respectively, parallel to L_1 and L_2, $\mathbf{v}_1 \times \mathbf{v}_2 = \langle 2, -11, 1\rangle$ so the plane $2x - 11y + z + 51 = 0$ contains L_2 and is parallel to L_1, $D = |2(1) - 11(3) + (5) + 51|/\sqrt{4 + 121 + 1} = 25/\sqrt{126}$.

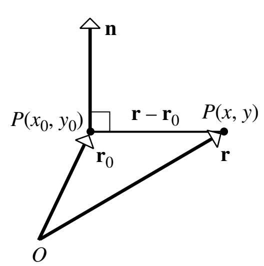
44. (3,4,1) and (0,3,0) are on L_1 and L_2, respectively. $\mathbf{v}_1 = \langle -1, 4, 2\rangle$ and $\mathbf{v}_2 = \langle 1, 0, 2\rangle$ are parallel to L_1 and L_2, $\mathbf{v}_1 \times \mathbf{v}_2 = \langle 8, 4, -4\rangle = 4\langle 2, 1, -1\rangle$ so $2x + y - z - 3 = 0$ contains L_2 and is parallel to L_1, $D = |2(3) + (4) - (1) - 3|/\sqrt{4 + 1 + 1} = \sqrt{6}$.

45. The distance between $(2, 1, -3)$ and the plane is $|2 - 3(1) + 2(-3) - 4|/\sqrt{1 + 9 + 4} = 11/\sqrt{14}$ which is the radius of the sphere; an equation is $(x - 2)^2 + (y - 1)^2 + (z + 3)^2 = 121/14$.

46. The vector $2\mathbf{i} + \mathbf{j} - \mathbf{k}$ is normal to the plane and hence parallel to the line so parametric equations of the line are $x = 3 + 2t$, $y = 1 + t$, $z = -t$. Substitution into the equation of the plane yields $2(3 + 2t) + (1 + t) - (-t) = 0$, $t = -7/6$; the point of intersection is $(2/3, -1/6, 7/6)$.

47. $\mathbf{v} = \langle 1, 2, -1\rangle$ is parallel to the line, $\mathbf{n} = \langle 2, -2, -2\rangle$ is normal to the plane, $\mathbf{v} \cdot \mathbf{n} = 0$ so $\mathbf{v}$ is parallel to the plane because $\mathbf{v}$ and $\mathbf{n}$ are perpendicular. $(-1, 3, 0)$ is on the line so
$D = |2(-1) - 2(3) - 2(0) + 3|/\sqrt{4 + 4 + 4} = 5/\sqrt{12}$

48. **(a)**



(b) $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = a(x - x_0) + b(y - y_0) = 0$

(c) See the proof of Theorem 12.6.1. Since a and b are not both zero, there is at least one point (x_0, y_0) that satisfies $ax + by + d = 0$, so $ax_0 + by_0 + d = 0$. If (x, y) also satisfies $ax + by + d = 0$ then, subtracting, $a(x - x_0) + b(y - y_0) = 0$, which is the equation of a line with $\mathbf{n} = \langle a, b\rangle$ as normal.

(d) Let $Q(x_1, y_1)$ be a point on the line, and position the normal $\mathbf{n} = \langle a, b\rangle$, with length $\sqrt{a^2+b^2}$, so that its initial point is at Q. The distance is the orthogonal projection of $\overrightarrow{QP_0} = \langle x_0 - x_1, y_0 - y_1\rangle$ onto $\mathbf{n}$. Then

$$D = \|\text{proj}_{\mathbf{n}}\, \overrightarrow{QP_0}\| = \left\| \frac{\overrightarrow{QP_0} \cdot \mathbf{n}}{\|\mathbf{n}\|^2}\mathbf{n} \right\| = \frac{|ax_0 + by_0 + d|}{\sqrt{a^2+b^2}}.$$

49. $D = |2(-3) + (5) - 1|/\sqrt{4+1} = 2/\sqrt{5}$

50. **(a)** If $\langle x_0, y_0, z_0\rangle$ lies on the second plane, so that $ax_0 + by_0 + cz_0 + d_2 = 0$, then by Theorem 12.6.2, the distance between the planes is $D = \dfrac{|ax_0 + by_0 + cz_0 + d_1|}{\sqrt{a^2+b^2+c^2}} = \dfrac{|-d_2 + d_1|}{\sqrt{a^2+b^2+c^2}}$

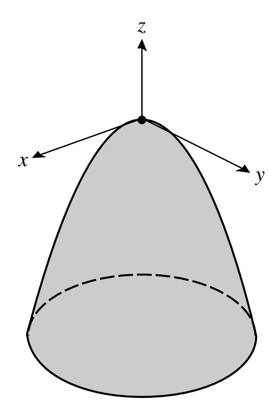
(b) The distance between the planes $-2x + y + z = 0$ and $-2x + y + z + \dfrac{5}{3} = 0$ is $D = \dfrac{|0 - 5/3|}{\sqrt{4+1+1}} = \dfrac{5}{3\sqrt{6}}$.

EXERCISE SET 12.7

1. **(a)** elliptic paraboloid, $a = 2, b = 3$
(b) hyperbolic paraboloid, $a = 1, b = 5$
(c) hyperboloid of one sheet, $a = b = c = 4$
(d) circular cone, $a = b = 1$
(e) elliptic paraboloid, $a = 2, b = 1$
(f) hyperboloid of two sheets, $a = b = c = 1$

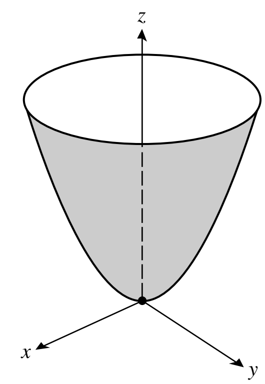
2. **(a)** ellipsoid, $a = \sqrt{2}, b = 2, e = \sqrt{3}$
(b) hyperbolic paraboloid, $a = b = 1$
(c) hyperboloid of one sheet, $a = 1, b = 3, c = 1$
(d) hyperboloid of two sheets, $a = 1, b = 2, c = 1$
(e) elliptic paraboloid, $a = \sqrt{2}, b = \sqrt{2}/2$
(f) elliptic cone, $a = 2, b = \sqrt{3}$

3. **(a)** $-z = x^2 + y^2$, circular paraboloid opening down the negative z-axis

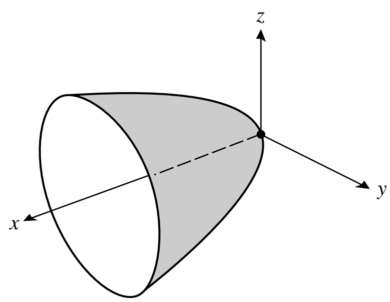


(b) $z = x^2 + y^2$, circular paraboloid, no change
(c) $z = x^2 + y^2$, circular paraboloid, no change

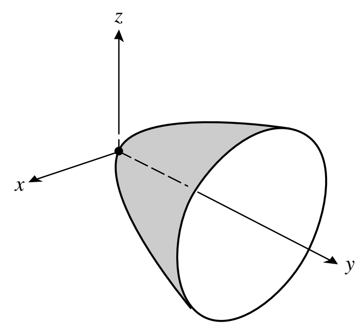
(d) $z = x^2 + y^2$, circular paraboloid, no change



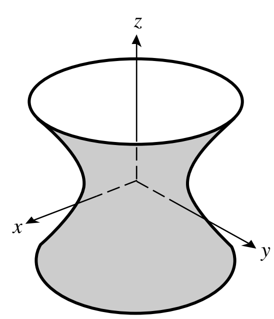
(e) $x = y^2 + z^2$, circular paraboloid opening along the positive x-axis



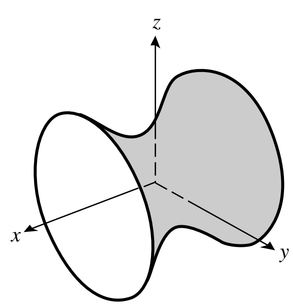
(f) $y = x^2 + z^2$, circular paraboloid opening along the positive y-axis



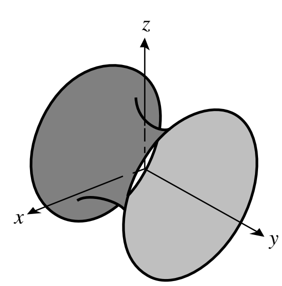
4. **(a)** $x^2 + y^2 - z^2 = 1$, no change
(b) $x^2 + y^2 - z^2 = 1$, no change
(c) $x^2 + y^2 - z^2 = 1$, no change
(d) $x^2 + y^2 - z^2 = 1$, no change



(e) $-x^2 + y^2 + z^2 = 1$, hyperboloid of one sheet with x-axis as axis



(f) $x^2 - y^2 + z^2 = 1$, hyperboloid of one sheet with y-axis as axis



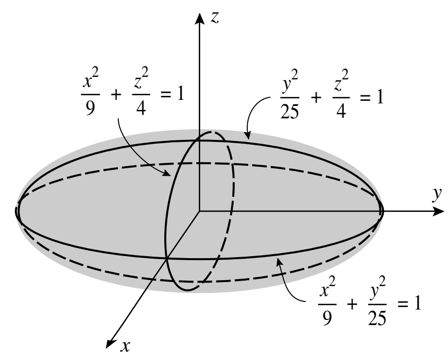
5. **(a)** hyperboloid of one sheet, axis is y-axis
(b) hyperboloid of two sheets separated by yz-plane

(c) elliptic paraboloid opening along the positive x-axis
(d) elliptic cone with x-axis as axis
(e) hyperbolic paraboloid straddling the z-axis
(f) paraboloid opening along the negative y-axis

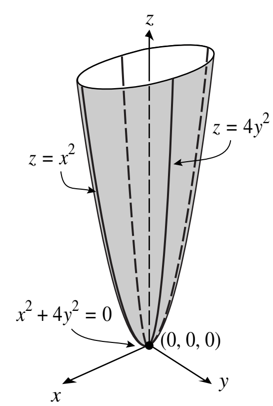
6. **(a)** same **(b)** same **(c)** same
(d) same **(e)** $y = \frac{x^2}{a^2} - \frac{z^2}{c^2}$ **(f)** $y = \frac{x^2}{a^2} + \frac{z^2}{c^2}$

7. **(a)** $x = 0 : \frac{y^2}{25} + \frac{z^2}{4} = 1; y = 0 : \frac{x^2}{9} + \frac{z^2}{4} = 1;$

$z = 0 : \frac{x^2}{9} + \frac{y^2}{25} = 1$

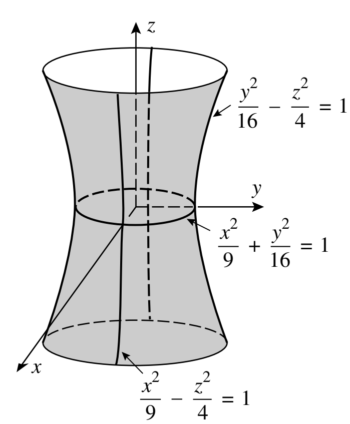


(b) $x = 0 : z = 4y^2; y = 0 : z = x^2;$
$z = 0 : x = y = 0$

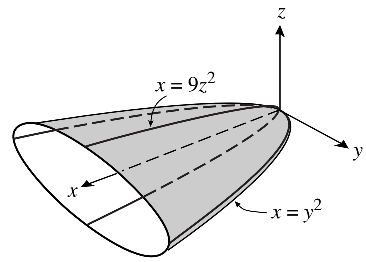


(c) $x = 0 : \frac{y^2}{16} - \frac{z^2}{4} = 1; y = 0 : \frac{x^2}{9} - \frac{z^2}{4} = 1;$

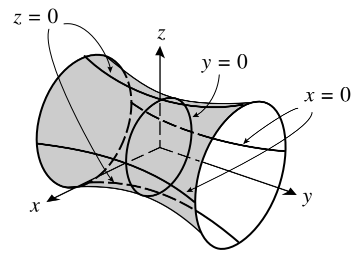
$z = 0 : \frac{x^2}{9} + \frac{y^2}{16} = 1$



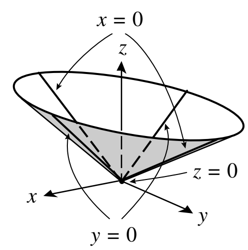
8. **(a)** $x = 0 : y = z = 0; y = 0 : x = 9z^2; z = 0 : x = y^2$



(b) $x = 0 : -y^2 + 4z^2 = 4; y = 0 : x^2 + z^2 = 1;$
$z = 0 : 4x^2 - y^2 = 4$



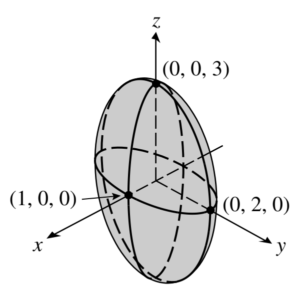
(c) $x = 0 : z = \pm\dfrac{y}{2}; y = 0 : z = \pm x; z = 0 : x = y = 0$



9. **(a)** $4x^2 + z^2 = 3$; ellipse **(b)** $y^2 + z^2 = 3$; circle **(c)** $y^2 + z^2 = 20$; circle
(d) $9x^2 - y^2 = 20$; hyperbola **(e)** $z = 9x^2 + 16$; parabola **(f)** $9x^2 + 4y^2 = 4$; ellipse

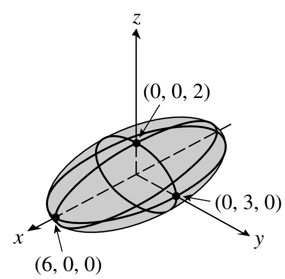
10. **(a)** $y^2 - 4z^2 = 27$; hyperbola **(b)** $9x^2 + 4z^2 = 25$; ellipse **(c)** $9z^2 - x^2 = 4$; hyperbola
(d) $x^2 + 4y^2 = 9$; ellipse **(e)** $z = 1 - 4y^2$; parabola **(f)** $x^2 - 4y^2 = 4$; hyperbola

11.



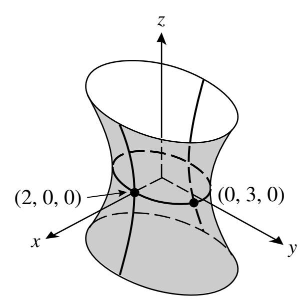
Ellipsoid

12.



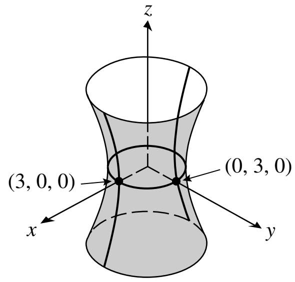
Ellipsoid

13.



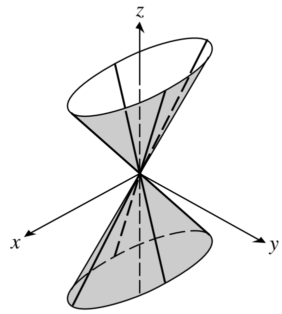
Hyperboloid of one sheet

14.



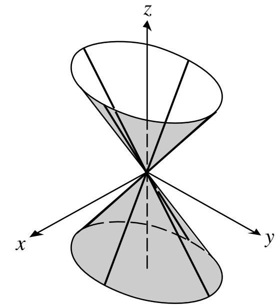
Hyperboloid
of one sheet

15.



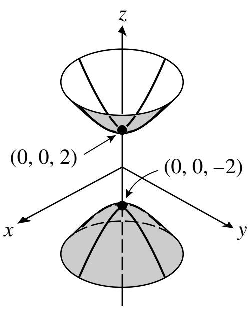
Elliptic cone

16.



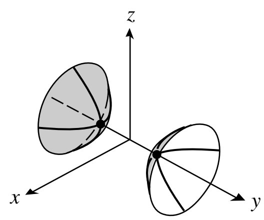
Elliptic cone

17.



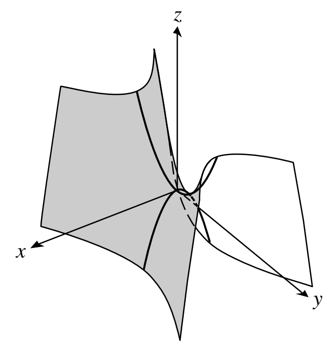
Hyperboloid
of two sheets

18.



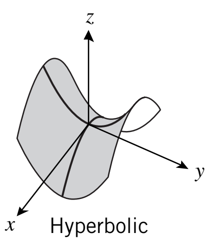
Hyperboloid
of two sheets

19.



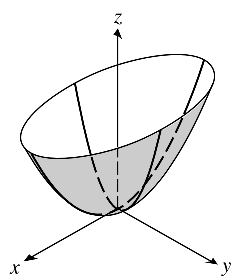
Hyperbolic paraboloid

20.



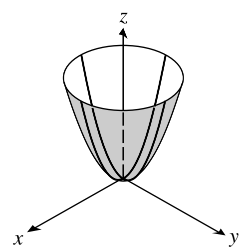
Hyperbolic
paraboloid

21.



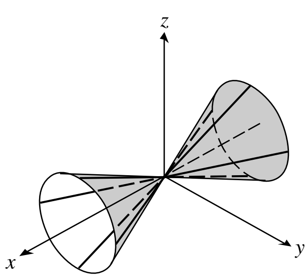
Elliptic paraboloid

22.



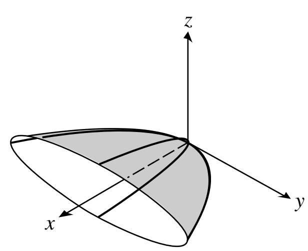
Circular paraboloid

23.



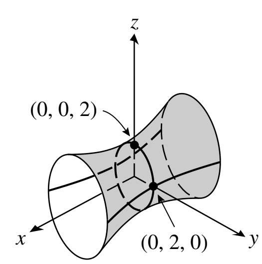
Circular cone

24.



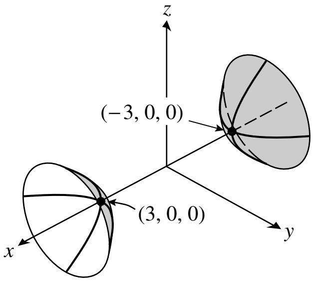
Elliptic paraboloid

25.



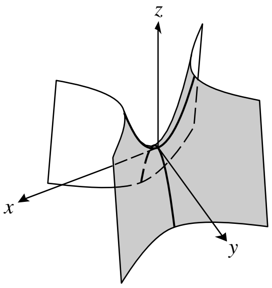
Hyperboloid
of one sheet

26.



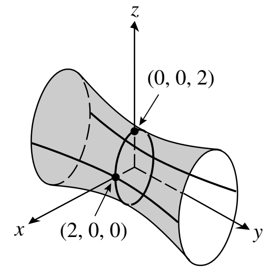
Hyperboloid
of two sheets

27.



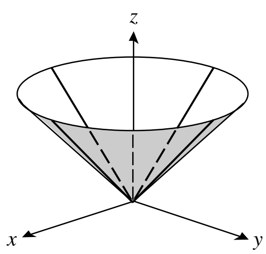
Hyperbolic
paraboloid

28.



Hyperboloid
of one sheet

29.



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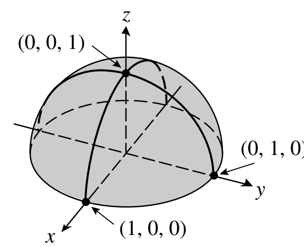

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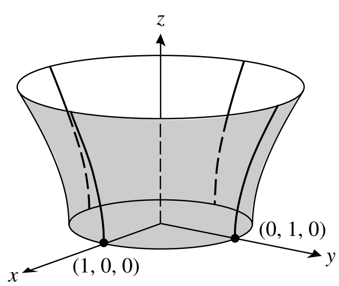

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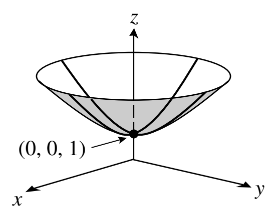

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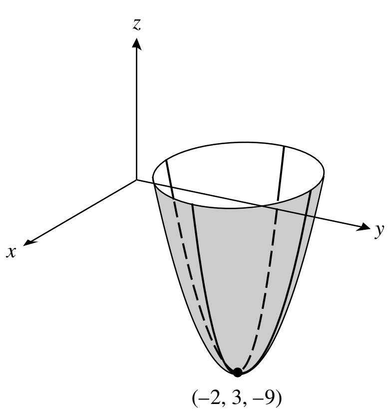
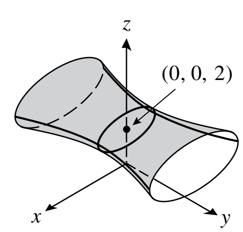


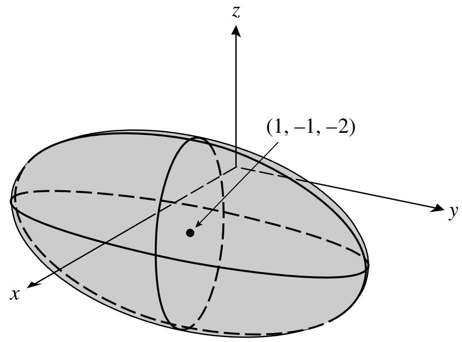
Circular paraboloid

34.



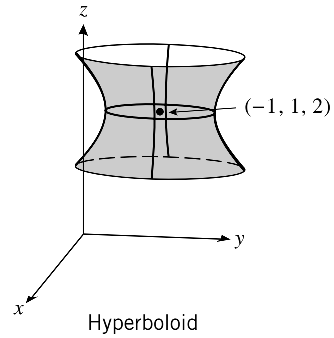
Hyperboloid
of one sheet

35.



Ellipsoid

36.



Hyperboloid
of one sheet

37. **(a)** $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$ **(b)** $6, 4$ **(c)** $(\pm\sqrt{5}, 0, \sqrt{2})$

(d) The focal axis is parallel to the x-axis.

38. **(a)** $\dfrac{y^2}{4}+\dfrac{z^2}{2}=1$ **(b)** $4, 2\sqrt{2}$ **(c)** $(3, \pm\sqrt{2}, 0)$

(d) The focal axis is parallel to the y-axis.

39. **(a)** $\dfrac{y^2}{4}-\dfrac{x^2}{4}=1$ **(b)** $(0, \pm 2, 4)$ **(c)** $(0, \pm 2\sqrt{2}, 4)$

(d) The focal axis is parallel to the y-axis.

40. **(a)** $\dfrac{x^2}{4}-\dfrac{y^2}{4}=1$ **(b)** $(\pm 2, 0, -4)$ **(c)** $(\pm 2\sqrt{2}, 0, -4)$

(e) The focal axis is parallel to the x-axis.

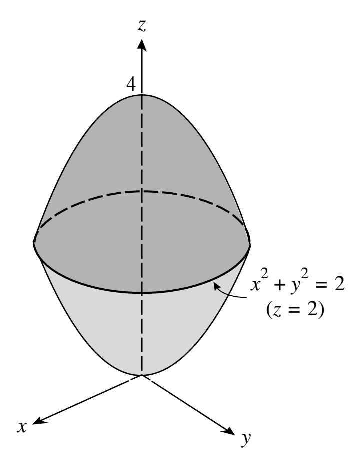
41. **(a)** $z+4=y^2$ **(b)** $(2, 0, -4)$ **(c)** $(2, 0, -15/4)$

(d) The focal axis is parallel to the z-axis.

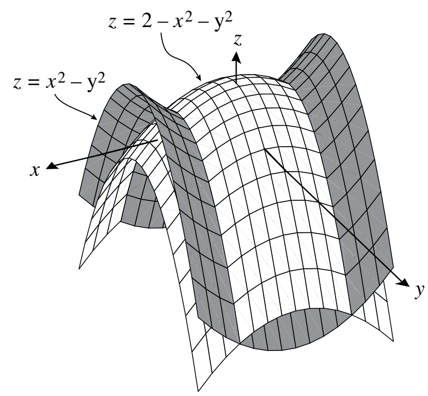
42. **(a)** $z-4=-x^2$ **(b)** $(0, 2, 4)$ **(c)** $(0, 2, 15/4)$

(d) The focal axis is parallel to the z-axis.

43. $x^2+y^2=4-x^2-y^2, x^2+y^2=2$; circle of radius $\sqrt{2}$ in the plane $z=2$, centered at $(0,0,2)$



44. $y^2 + z = 4 - 2(y^2 + z), y^2 + z = 4/3$;
parabolas in the planes $x = \pm 2/\sqrt{3}$
which open in direction of the negative z-axis



45. $y = 4(x^2 + z^2)$

46. $y^2 = 4(x^2 + z^2)$

47. $|z - (-1)| = \sqrt{x^2 + y^2 + (z-1)^2}$, $z^2 + 2z + 1 = x^2 + y^2 + z^2 - 2z + 1$, $z = (x^2 + y^2)/4$; circular paraboloid

48. $|z + 1| = 2\sqrt{x^2 + y^2 + (z-1)^2}$, $z^2 + 2z + 1 = 4\left(x^2 + y^2 + z^2 - 2z + 1\right)$,
$4x^2 + 4y^2 + 3z^2 - 10z + 3 = 0$, $\dfrac{x^2}{4/3} + \dfrac{y^2}{4/3} + \dfrac{(z-5/3)^2}{16/9} = 1$; ellipsoid, center at $(0, 0, 5/3)$.

49. If $z = 0$, $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2} = 1$; if $y = 0$ then $\dfrac{x^2}{a^2} + \dfrac{z^2}{c^2} = 1$; since $c < a$ the major axis has length $2a$, the minor axis length $2c$.

50. $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2} + \dfrac{z^2}{b^2} = 1$, where $a = 6378.1370, b = 6356.5231$.

51. Each slice perpendicular to the z-axis for $|z| < c$ is an ellipse whose equation is
$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{c^2 - z^2}{c^2}$, or $\dfrac{x^2}{(a^2/c^2)(c^2 - z^2)} + \dfrac{y^2}{(b^2/c^2)(c^2 - z^2)} = 1$, the area of which is
$\pi\left(\dfrac{a}{c}\sqrt{c^2 - z^2}\right)\left(\dfrac{b}{c}\sqrt{c^2 - z^2}\right) = \pi\dfrac{ab}{c^2}\left(c^2 - z^2\right)$ so $V = 2\displaystyle\int_0^c \pi\dfrac{ab}{c^2}\left(c^2 - z^2\right) dz = \dfrac{4}{3}\pi abc$.

EXERCISE SET 12.8

1. **(a)** $(8, \pi/6, -4)$ **(b)** $\left(5\sqrt{2}, 3\pi/4, 6\right)$ **(c)** $(2, \pi/2, 0)$ **(d)** $(8, 5\pi/3, 6)$

2. **(a)** $(2, 7\pi/4, 1)$ **(b)** $(1, \pi/2, 1)$ **(c)** $(4\sqrt{2}, 3\pi/4, -7)$ **(d)** $(2\sqrt{2}, 7\pi/4, -2)$

3. **(a)** $\left(2\sqrt{3}, 2, 3\right)$ **(b)** $\left(-4\sqrt{2}, 4\sqrt{2}, -2\right)$ **(c)** $(5, 0, 4)$ **(d)** $(-7, 0, -9)$

4. **(a)** $\left(3, -3\sqrt{3}, 7\right)$ **(b)** $(0, 1, 0)$ **(c)** $(0, 3, 5)$ **(d)** $(0, 4, -1)$

5. **(a)** $\left(2\sqrt{2}, \pi/3, 3\pi/4\right)$ **(b)** $(2, 7\pi/4, \pi/4)$ **(c)** $(6,\ \pi/2,\ \pi/3)$ **(d)** $(10, 5\pi/6,\ \pi/2)$

6. **(a)** $\left(8\sqrt{2}, \pi/4, \pi/6\right)$ **(b)** $\left(2\sqrt{2}, 5\pi/3, 3\pi/4\right)$ **(c)** $(2, 0, \pi/2)$ **(d)** $(4, \pi/6, \pi/6)$

7. **(a)** $(5\sqrt{6}/4, 5\sqrt{2}/4, 5\sqrt{2}/2)$ **(b)** $(7,0,0)$
(c) $(0,0,1)$ **(d)** $(0, -2, 0)$

8. **(a)** $\left(-\sqrt{2}/4, \sqrt{6}/4, -\sqrt{2}/2\right)$ **(b)** $\left(3\sqrt{2}/4, -3\sqrt{2}/4, -3\sqrt{3}/2\right)$
(c) $(2\sqrt{6}, 2\sqrt{2}, 4\sqrt{2})$ **(d)** $(0, 2\sqrt{3}, 2)$

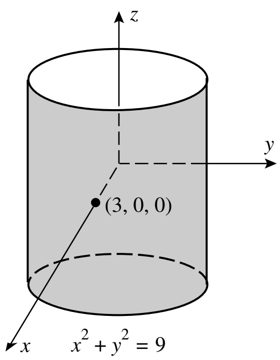
9. **(a)** $\left(2\sqrt{3}, \pi/6, \pi/6\right)$ **(b)** $\left(\sqrt{2}, \pi/4, 3\pi/4\right)$
(c) $(2, 3\pi/4, \pi/2)$ **(d)** $\left(4\sqrt{3}, 1, 2\pi/3\right)$

10. **(a)** $\left(4\sqrt{2}, 5\pi/6, \pi/4\right)$ **(b)** $\left(2\sqrt{2}, 0, 3\pi/4\right)$
(c) $(5, \pi/2, \tan^{-1}(4/3))$ **(d)** $(2\sqrt{10}, \pi, \tan^{-1} 3)$

11. **(a)** $\left(5\sqrt{3}/2, \pi/4, -5/2\right)$ **(b)** $(0, 7\pi/6, -1)$
(c) $(0, 0, 3)$ **(d)** $(4, \pi/6, 0)$

12. **(a)** $(0, \pi/2, 5)$ **(b)** $(3\sqrt{2}, 0, -3\sqrt{2})$
(c) $(0, 3\pi/4, -\sqrt{2})$ **(d)** $(5/2, 2\pi/3, -5\sqrt{3}/2)$

15.



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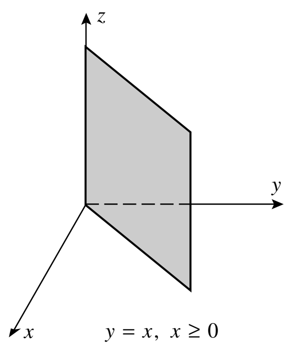

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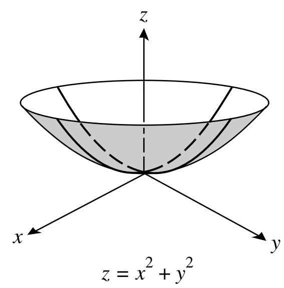

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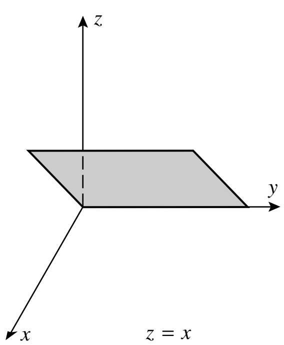

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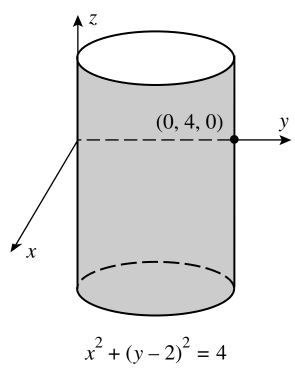

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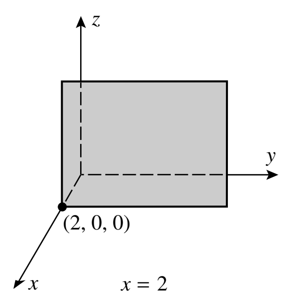

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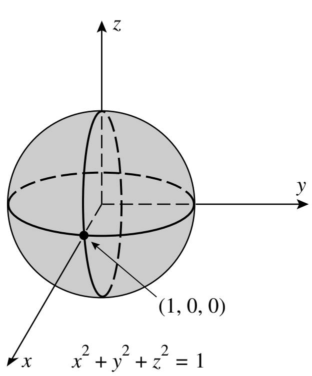
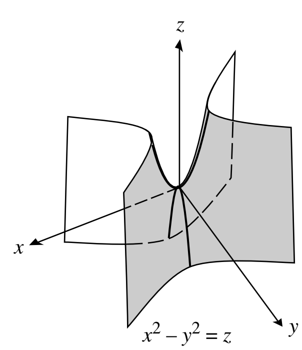
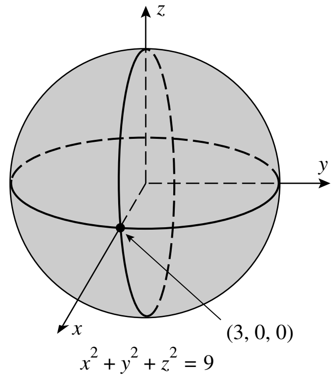


22.



23.



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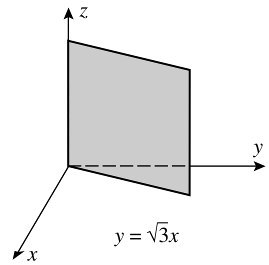

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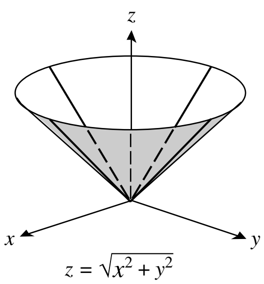

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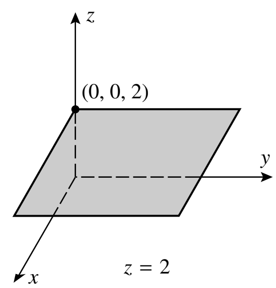

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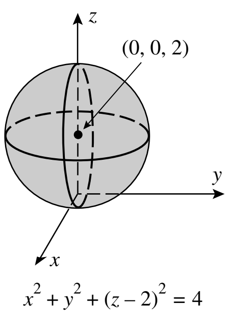
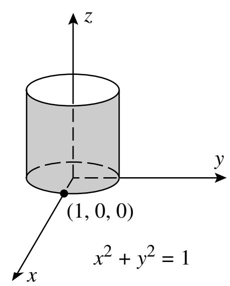


28.



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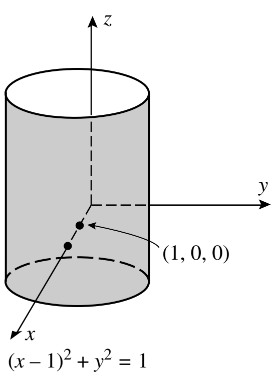

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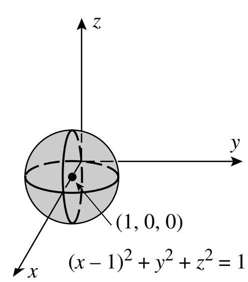


31. **(a)** $z = 3$ **(b)** $\rho\cos\phi = 3, \rho = 3\sec\phi$

32. **(a)** $r\sin\theta = 2, r = 2\csc\theta$ **(b)** $\rho\sin\phi\sin\theta = 2, \rho = 2\csc\phi\csc\theta$

33. **(a)** $z = 3r^2$ **(b)** $\rho\cos\phi = 3\rho^2\sin^2\phi, \rho = \dfrac{1}{3}\csc\phi\cot\phi$

34. **(a)** $z = \sqrt{3}r$ **(b)** $\rho\cos\phi = \sqrt{3}\rho\sin\phi, \tan\phi = \dfrac{1}{\sqrt{3}}, \phi = \dfrac{\pi}{6}$

35. **(a)** $r = 2$ **(b)** $\rho\sin\phi = 2, \rho = 2\csc\phi$

36. **(a)** $r^2 - 6r\sin\theta = 0,\ r = 6\sin\theta$ **(b)** $\rho\sin\phi = 6\sin\theta, \rho = 6\sin\theta\csc\phi$

37. **(a)** $r^2 + z^2 = 9$ **(b)** $\rho = 3$

38. **(a)** $z^2 = r^2\cos^2\theta - r^2\sin^2\theta = r^2(\cos^2\theta - \sin^2\theta),\ z^2 = r^2\cos 2\theta$
(b) Use the result in Part (a) with $r = \rho\sin\phi$, $z = \rho\cos\phi$ to get $\rho^2\cos^2\phi = \rho^2\sin^2\phi\cos 2\theta$, $\cot^2\phi = \cos 2\theta$

39. **(a)** $2r\cos\theta + 3r\sin\theta + 4z = 1$
(b) $2\rho\sin\phi\cos\theta + 3\rho\sin\phi\sin\theta + 4\rho\cos\phi = 1$

40. **(a)** $r^2 - z^2 = 1$
(b) Use the result of Part (a) with $r = \rho\sin\phi$, $z = \rho\cos\phi$ to get $\rho^2\sin^2\phi - \rho^2\cos^2\phi = 1$, $\rho^2\cos 2\phi = -1$

41. **(a)** $r^2\cos^2\theta = 16 - z^2$
(b) $x^2 = 16 - z^2,\ x^2 + y^2 + z^2 = 16 + y^2,\ \rho^2 = 16 + \rho^2\sin^2\phi\sin^2\theta,\ \rho^2\left(1 - \sin^2\phi\sin^2\theta\right) = 16$

42. **(a)** $r^2 + z^2 = 2z$ **(b)** $\rho^2 = 2\rho\cos\phi,\ \rho = 2\cos\phi$

43. all points on or above the paraboloid $z = x^2 + y^2$, that are also on or below the plane $z = 4$

44. a right circular cylindrical solid of height 3 and radius 1 whose axis is the line $x = 0, y = 1$

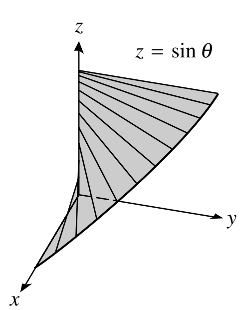
45. all points on or between concentric spheres of radii 1 and 3 centered at the origin

46. all points on or above the cone $\phi = \pi/6$, that are also on or below the sphere $\rho = 2$

47. $\theta = \pi/6,\ \phi = \pi/6$, spherical $(4000, \pi/6,\ \pi/6)$, rectangular $\left(1000\sqrt{3}, 1000, 2000\sqrt{3}\right)$

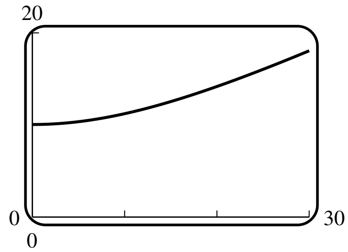
48. **(a)** $y = r\sin\theta = a\sin\theta$ but $az = a\sin\theta$ so $y = az$, which is a plane that contains the curve of intersection of $z = \sin\theta$ and the circular cylinder $r = a$. From Exercise 60, Section 11.4, the curve of intersection of a plane and a circular cylinder is an ellipse.

(b)



49. **(a)** $(10, \pi/2, 1)$ **(b)** $(0, 10, 1)$ **(c)** $(\sqrt{101}, \pi/2, \tan^{-1} 10)$

50.



51. Using spherical coordinates: for point A, $\theta_A = 360° - 60° = 300°, \phi_A = 90° - 40° = 50°$; for point B, $\theta_B = 360° - 40° = 320°, \phi_B = 90° - 20° = 70°$. Unit vectors directed from the origin to the points A and B, respectively, are

$$\mathbf{u}_A = \sin 50° \cos 300° \mathbf{i} + \sin 50° \sin 300° \mathbf{j} + \cos 50° \mathbf{k},$$
$$\mathbf{u}_B = \sin 70° \cos 320° \mathbf{i} + \sin 70° \sin 320° \mathbf{j} + \cos 70° \mathbf{k}$$

The angle α between $\mathbf{u}_A$ and $\mathbf{u}_B$ is $\alpha = \cos^{-1}(\mathbf{u}_A \cdot \mathbf{u}_B) \approx 0.459486$ so the shortest distance is $6370\alpha \approx 2927$ km.

CHAPTER 12 SUPPLEMENTARY EXERCISES

2. **(c)** $\mathbf{F} = -\mathbf{i} - \mathbf{j}$

(d) $\|\langle 1, -2, 2\rangle\| = 3$, so $\|\mathbf{r} - \langle 1, -2, 2\rangle\| = 3$, or $(x-1)^2 + (y+2)^2 + (z-2)^2 = 9$

3. **(b)** $x = \cos 120° = -1/2, y = \pm \sin 120° = \pm\sqrt{3}/2$

(d) true: $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\||\sin(\theta)| = 1$

4. **(d)** $x + 2y - z = 0$

5. **(b)** $(y, x, z), (x, z, y), (z, y, x)$

(c) the set of points $\{(5, \theta, 1)\}, 0 \le \theta \le 2\pi$

(d) the set of points $\{(\rho, \pi/4, 0)\}, 0 \le \rho < +\infty$

6. $(x+3)^2 + (y-5)^2 + (z+4)^2 = r^2$,

(a) $r^2 = 4^2 = 16$ **(b)** $r^2 = 5^2 = 25$ **(c)** $r^2 = 3^2 = 9$

7. **(a)** $\overrightarrow{AB} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $\overrightarrow{AC} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\overrightarrow{AB} \times \overrightarrow{AC} = -4\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, area $= \dfrac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \sqrt{26}/2$

(b) area $= \dfrac{1}{2}h\|\overrightarrow{AB}\| = \dfrac{3}{2}h = \dfrac{1}{2}\sqrt{26}$, $h = \sqrt{26}/3$

8. The sphere $x^2 + (y-1)^2 + (z+3)^2 = 16$ has center $Q(0, 1, -3)$ and radius 4, and $\|\overrightarrow{PQ}\| = \sqrt{1^2 + 4^2} = \sqrt{17}$, so minimum distance is $\sqrt{17} - 4$, maximum distance is $\sqrt{17} + 4$.

9. **(a)** $\mathbf{a} \cdot \mathbf{b} = 0$, $4c + 3 = 0$, $c = -3/4$

(b) Use $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\| \cos\theta$ to get $4c + 3 = \sqrt{c^2 + 1}(5)\cos(\pi/4)$, $4c + 3 = 5\sqrt{c^2 + 1}/\sqrt{2}$ Square both sides and rearrange to get $7c^2 + 48c - 7 = 0$, $(7c - 1)(c + 7) = 0$ so $c = -7$ (invalid) or $c = 1/7$.

(c) Proceed as in (b) with $\theta = \pi/6$ to get $11c^2 - 96c + 39 = 0$ and use the quadratic formula to get $c = \left(48 \pm 25\sqrt{3}\right)/11$.

(d) $\mathbf{a}$ must be a scalar multiple of $\mathbf{b}$, so $c\mathbf{i} + \mathbf{j} = k(4\mathbf{i} + 3\mathbf{j}), k = 1/3, c = 4/3$.

10. $\overrightarrow{OS} = \overrightarrow{OP} + \overrightarrow{PS} = 3\mathbf{i} + 4\mathbf{j} + \overrightarrow{QR} = 3\mathbf{i} + 4\mathbf{j} + (4\mathbf{i} + \mathbf{j}) = 7\mathbf{i} + 5\mathbf{j}$

11. **(a)** the plane through the origin which is perpendicular to $\mathbf{r}_0$

(b) the plane through the tip of $\mathbf{r}_0$ which is perpendicular to $\mathbf{r}_0$

12. The normals to the planes are given by $\langle a_1, b_1, c_1\rangle$ and $\langle a_2, b_2, c_2\rangle$, so the condition is $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

13. Since $\overrightarrow{AC} \cdot (\overrightarrow{AB} \times \overrightarrow{AD}) = \overrightarrow{AC} \cdot (\overrightarrow{AB} \times \overrightarrow{CD}) + \overrightarrow{AC} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = \mathbf{0} + \mathbf{0} = \mathbf{0}$, the volume of the parallelopiped determined by $\overrightarrow{AB}, \overrightarrow{AC}$, and $\overrightarrow{AD}$ is zero, thus A, B, C, and D are coplanar (lie in the same plane). Since $\overrightarrow{AB} \times \overrightarrow{CD} \neq \mathbf{0}$, the lines are not parallel. Hence they must intersect.

14. The points P lie on the plane determined by A, B and C.

15. **(a)** false, for example $\mathbf{i} \cdot \mathbf{j} = 0$

(b) false, for example $\mathbf{i} \times \mathbf{i} = \mathbf{0}$

(c) true; $0 = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \cos\theta = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \sin\theta$, so either $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$ since $\cos\theta = \sin\theta = 0$ is impossible.

16. **(a)** Replace $\mathbf{u}$ with $\mathbf{a} \times \mathbf{b}$, $\mathbf{v}$ with $\mathbf{c}$, and $\mathbf{w}$ with $\mathbf{d}$ in the first formula of Exercise 39.

(b) From the second formula of Exercise 39,
$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} + (\mathbf{b} \times \mathbf{c}) \times \mathbf{a} + (\mathbf{c} \times \mathbf{a}) \times \mathbf{b}$
$= (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{c} \cdot \mathbf{b})\mathbf{a} + (\mathbf{a} \cdot \mathbf{b})\mathbf{c} - (\mathbf{a} \cdot \mathbf{c})\mathbf{b} + (\mathbf{b} \cdot \mathbf{c})\mathbf{a} - (\mathbf{b} \cdot \mathbf{a})\mathbf{c} = \mathbf{0}$

17. $\|\mathbf{u} - \mathbf{v}\|^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\| \cos\theta = 2(1 - \cos\theta) = 4\sin^2(\theta/2)$, so $\|\mathbf{u} - \mathbf{v}\| = 2\sin(\theta/2)$

18. $\overrightarrow{AB} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$, $\overrightarrow{AC} = -2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$, $\overrightarrow{AD} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$

(a) From Theorem 12.4.6 and formula (9) of Section 12.4, $\begin{vmatrix} 1 & -2 & -2 \\ -2 & -1 & -2 \\ 1 & 2 & -3 \end{vmatrix} = 29$, so $V = 29$.

(b) The plane containing A, B, and C has normal $\overrightarrow{AB} \times \overrightarrow{AC} = 2\mathbf{i} + 6\mathbf{j} - 5\mathbf{k}$, so the equation of the plane is $2(x - 1) + 6(y + 1) - 5(z - 2) = 0, 2x + 6y - 5z = -14$. From Theorem 12.6.2, $D = \dfrac{|2(2) + 6(1) - 5(-1) + 14|}{\sqrt{65}} = \dfrac{29}{\sqrt{65}}$.

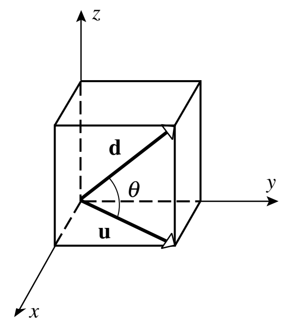
19. **(a)** $\langle 2, 1, -1\rangle \times \langle 1, 2, 1\rangle = \langle 3, -3, 3\rangle$, so the line is parallel to $\mathbf{i} - \mathbf{j} + \mathbf{k}$. By inspection, $(0, 2, -1)$ lies on both planes, so the line has an equation $\mathbf{r} = 2\mathbf{j} - \mathbf{k} + t(\mathbf{i} - \mathbf{j} + \mathbf{k})$, that is, $x = t, y = 2 - t, z = -1 + t$.

(b) $\cos\theta = \dfrac{\langle 2, 1, -1\rangle \cdot \langle 1, 2, 1\rangle}{\|\langle 2, 1, -1\rangle\|\|\langle 1, 2, 1\rangle\|} = 1/2$, so $\theta = \pi/3$

20. Let $\alpha = 50°, \beta = 70°$, then $\gamma = \cos^{-1}\sqrt{1 - \cos^2\alpha - \cos^2\beta} \approx 47°$.

21. $5\langle\cos 60°, \cos 120°, \cos 135°\rangle = \langle 5/2, -5/2, -5\sqrt{2}/2\rangle$

22. **(a)** Let k be the length of an edge and introduce a coordinate system as shown in the figure, then $\mathbf{d} = \langle k, k, k\rangle, \mathbf{u} = \langle k, k, 0\rangle$, $\cos\theta = \dfrac{\mathbf{d}\cdot\mathbf{u}}{\|\mathbf{d}\|\ \|\mathbf{u}\|} = \dfrac{2k^2}{\left(k\sqrt{3}\right)\left(k\sqrt{2}\right)} = 2/\sqrt{6}$
so $\theta = \cos^{-1}(2/\sqrt{6}) \approx 35°$



(b) $\mathbf{v} = \langle -k, 0, k\rangle, \cos\theta = \dfrac{\mathbf{d}\cdot\mathbf{v}}{\|\mathbf{d}\|\ \|\mathbf{v}\|} = 0$ so $\theta = \pi/2$ radians.

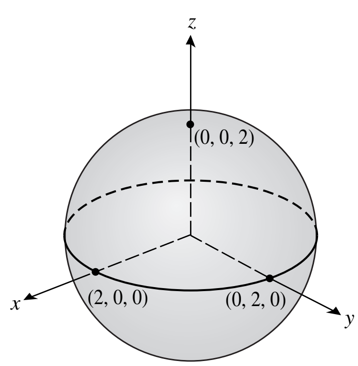
23. **(a)** $(x-3)^2 + 4(y+1)^2 - (z-2)^2 = 9$, hyperboloid of one sheet
(b) $(x+3)^2 + (y-2)^2 + (z+6)^2 = 49$, sphere
(c) $(x-1)^2 + (y+2)^2 - z^2 = 0$, circular cone

24. **(a)** perpendicular, since $\langle 2,1,2\rangle \cdot \langle -1,-2,2\rangle = 0$
(b) L_1: $\langle x,y,z\rangle = \langle 1+2t, -\dfrac{3}{2}+t, -1+2t\rangle$; L_2: $\langle x,y,z\rangle = \langle 4-t, 3-2t, -4+2t\rangle$
(c) Solve simultaneously $1+2t_1 = 4-t_2, -\dfrac{3}{2}+t_1 = 3-2t_2, -1+2t_1 = -4+2t_2$, solution $t_1 = \dfrac{1}{2}, t_2 = 2, x = 2, y = -1, z = 0$

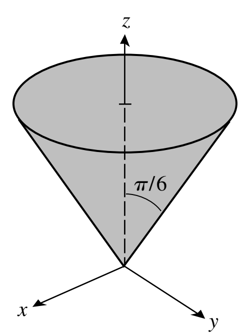
25. **(a)** $r^2 = z; \rho^2\sin^2\phi = \rho\cos\phi, \rho = \cot\phi\csc\phi$
(b) $r^2(\cos^2\theta - \sin^2\theta) - z^2 = 0, z^2 = r^2\cos 2\theta;$
$\rho^2\sin^2\phi\cos^2\theta - \rho^2\sin^2\phi\sin^2\theta - \rho^2\cos^2\phi = 0, \cos 2\theta = \cot^2\phi$
$\sin^2\phi(\cos^2\theta - \sin^2\theta) - \cos^2\phi = 0$

26. **(a)** $z = r^2\cos^2\theta - r^2\sin^2\theta = x^2 - y^2$ **(b)** $(\rho\sin\phi\cos\theta)(\rho\cos\phi) = 1, xz = 1$

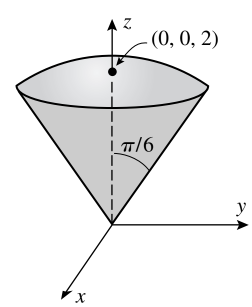
27. **(a)**



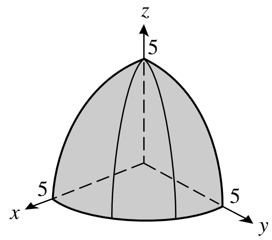
(b)



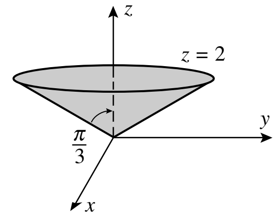
(c)



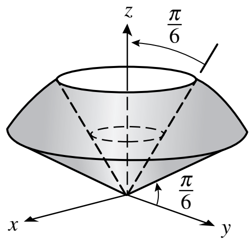
28. (a)



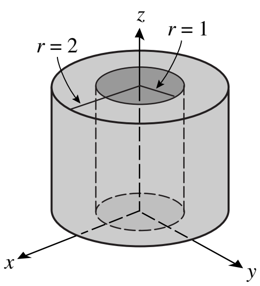
(b)



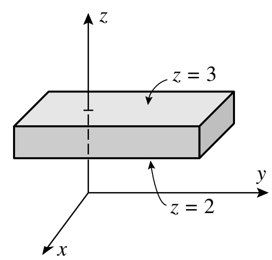
(c)



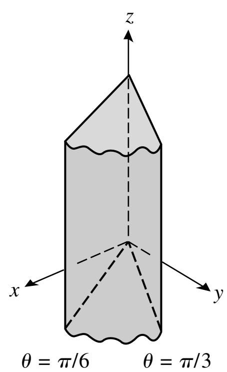
29. (a)



(b)



(c)



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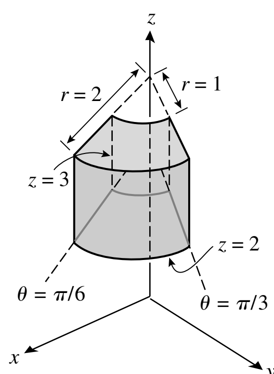
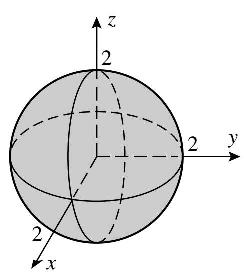
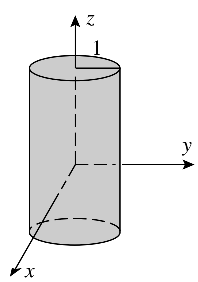


30. (a)



(b)



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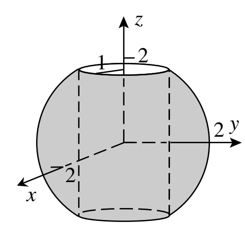


31. **(a)** At $x = c$ the trace of the surface is the circle $y^2 + z^2 = [f(c)]^2$, so the surface is given by $y^2 + z^2 = [f(x)]^2$

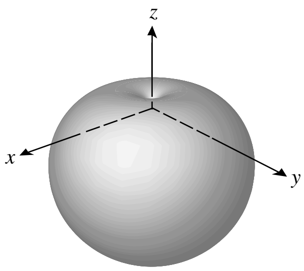
(b) $y^2 + z^2 = e^{2x}$ **(c)** $y^2 + z^2 = 4 - \frac{3}{4}x^2$, so let $f(x) = \sqrt{4 - \frac{3}{4}x^2}$

32. **(a)** Permute x and y in Exercise 31a: $x^2 + z^2 = [f(y)]^2$

(b) Permute x and z in Exercise 31a: $x^2 + y^2 = [f(z)]^2$

(c) Permute y and z in Exercise 31a: $y^2 + z^2 = [f(x)]^2$

33.



34. $\overrightarrow{PQ} = \langle 1, -1, 6 \rangle$, and $W = \mathbf{F} \cdot \overrightarrow{\mathbf{PQ}} = 13$ lb·ft

35. $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}, \overrightarrow{PQ} = \mathbf{i} + 4\mathbf{j} - 3\mathbf{k}, W = \mathbf{F} \cdot \overrightarrow{PQ} = -11$ N·m $= -11$ J

36. $\mathbf{F}_1 = 250 \cos 38°\mathbf{i} + 250 \sin 38°\mathbf{j}, \mathbf{F} = 1000\mathbf{i}, \mathbf{F}_2 = \mathbf{F} - \mathbf{F}_1 = (1000 - 250 \cos 38°)\mathbf{i} - 250 \sin 38°\mathbf{j}$;

$\|F_2\| = 1000\sqrt{\frac{17}{16} - \frac{1}{2}\cos 38°} \approx 817.62$ N, $\theta = \tan^{-1} \frac{250 \sin 38°}{250 \cos 38° - 1000} \approx -11°$

37. **(a)** $\mathbf{F} = -6\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$

(b) $\overrightarrow{OA} = \langle 5, 0, 2 \rangle$, so the vector moment is $\overrightarrow{OA} \times \mathbf{F} = -6\mathbf{i} + 18\mathbf{j} + 15\mathbf{k}$