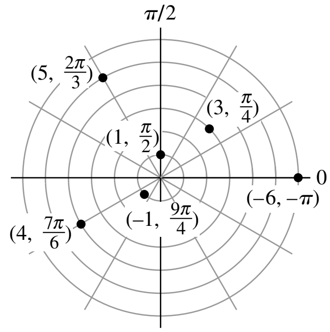
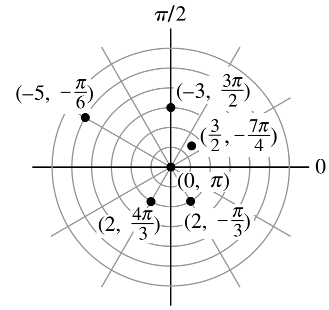
CHAPTER 11
Analytic Geometry in Calculus

EXERCISE SET 11.1

1.



2.



3. **(a)** $(3\sqrt{3}, 3)$ **(b)** $(-7/2, 7\sqrt{3}/2)$ **(c)** $(3\sqrt{3}, 3)$
(d) $(0, 0)$ **(e)** $(-7\sqrt{3}/2, 7/2)$ **(f)** $(-5, 0)$

4. **(a)** $(-4\sqrt{2}, -4\sqrt{2})$ **(b)** $(7\sqrt{2}/2, -7\sqrt{2}/2)$ **(c)** $(4\sqrt{2}, 4\sqrt{2})$
(d) $(5, 0)$ **(e)** $(0, -2)$ **(f)** $(0, 0)$

5. **(a)** both $(5, \pi)$ **(b)** $(4, 11\pi/6), (4, -\pi/6)$ **(c)** $(2, 3\pi/2), (2, -\pi/2)$
(d) $(8\sqrt{2}, 5\pi/4), (8\sqrt{2}, -3\pi/4)$ **(e)** both $(6, 2\pi/3)$ **(f)** both $(\sqrt{2}, \pi/4)$

6. **(a)** $(2, 5\pi/6)$ **(b)** $(-2, 11\pi/6)$ **(c)** $(2, -7\pi/6)$ **(d)** $(-2, -\pi/6)$

7. **(a)** $(5, 0.6435)$ **(b)** $(\sqrt{29}, 5.0929)$ **(c)** $(1.2716, 0.6658)$

8. **(a)** $(5, 2.2143)$ **(b)** $(3.4482, 2.6260)$ **(c)** $(\sqrt{4+\pi^2/36}, 0.2561)$

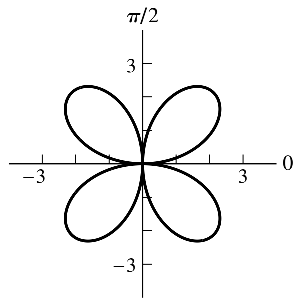
9. **(a)** $r^2 = x^2 + y^2 = 4$; circle **(b)** $y = 4$; horizontal line
(c) $r^2 = 3r\cos\theta$, $x^2 + y^2 = 3x$, $(x - 3/2)^2 + y^2 = 9/4$; circle
(d) $3r\cos\theta + 2r\sin\theta = 6$, $3x + 2y = 6$; line

10. **(a)** $r\cos\theta = 5$, $x = 5$; vertical line
(b) $r^2 = 2r\sin\theta$, $x^2 + y^2 = 2y$, $x^2 + (y-1)^2 = 1$; circle
(c) $r^2 = 4r\cos\theta + 4r\sin\theta$, $x^2 + y^2 = 4x + 4y$, $(x-2)^2 + (y-2)^2 = 8$; circle
(d) $r = \dfrac{1}{\cos\theta}\dfrac{\sin\theta}{\cos\theta}$, $r\cos^2\theta = \sin\theta$, $r^2\cos^2\theta = r\sin\theta$, $x^2 = y$; parabola

11. **(a)** $r\cos\theta = 7$ **(b)** $r = 3$
(c) $r^2 - 6r\sin\theta = 0$, $r = 6\sin\theta$
(d) $4(r\cos\theta)(r\sin\theta) = 9$, $4r^2\sin\theta\cos\theta = 9$, $r^2\sin 2\theta = 9/2$

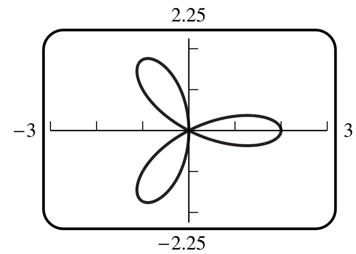
12. **(a)** $r\sin\theta = -3$ **(b)** $r = \sqrt{5}$
(c) $r^2 + 4r\cos\theta = 0$, $r = -4\cos\theta$
(d) $r^4\cos^2\theta = r^2\sin^2\theta$, $r^2 = \tan^2\theta$, $r = \tan\theta$

13.



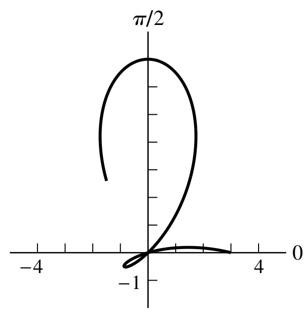
$r = 3\sin 2\theta$

14.



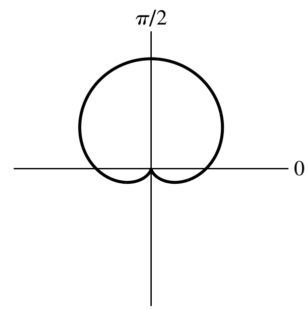
$r = 2\cos 3\theta$

15.



$r = 3 - 4\sin 3\theta$

16.



$r = 2 + 2\sin\theta$

17. **(a)** $r = 5$

(b) $(x-3)^2 + y^2 = 9,\ r = 6\cos\theta$

(c) Example 6, $r = 1 - \cos\theta$

18. **(a)** From (8-9), $r = a \pm b\sin\theta$ or $r = a \pm b\cos\theta$. The curve is not symmetric about the y-axis, so Theorem 11.2.1(a) eliminates the sine function, thus $r = a \pm b\cos\theta$. The cartesian point $(-3, 0)$ is either the polar point $(3, \pi)$ or $(-3, 0)$, and the cartesian point $(-1, 0)$ is either the polar point $(1, \pi)$ or $(-1, 0)$. A solution is $a = 1, b = -2$; we may take the equation as $r = 1 - 2\cos\theta$.

(b) $x^2 + (y + 3/2)^2 = 9/4, r = -3\sin\theta$

(c) Figure 11.1.18, $a = 1, n = 3, r = \sin 3\theta$

19. **(a)** Figure 11.1.18, $a = 3, n = 2, r = 3\sin 2\theta$

(b) From (8-9), symmetry about the y-axis and Theorem 11.1.1(b), the equation is of the form $r = a \pm b\sin\theta$. The cartesian points $(3, 0)$ and $(0, 5)$ give $a = 3$ and $5 = a + b$, so $b = 2$ and $r = 3 + 2\sin\theta$.

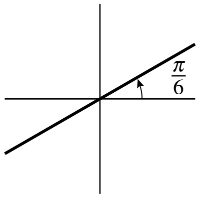
(c) Example 8, $r^2 = 9\cos 2\theta$

20. **(a)** Example 6 rotated through $\pi/2$ radian: $a = 3, r = 3 - 3\sin\theta$

(b) Figure 11.1.18, $a = 1, r = \cos 5\theta$

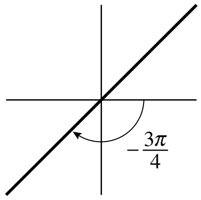
(c) $x^2 + (y-2)^2 = 4,\ r = 4\sin\theta$

21.



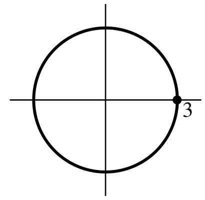
Line

22.



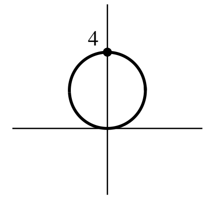
Line

23.



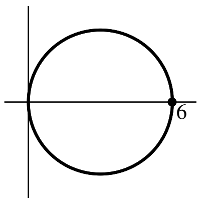
Circle

24.



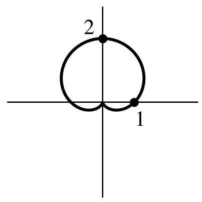
Circle

25.



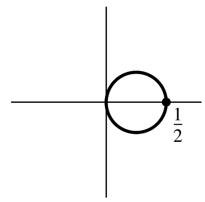
Circle

26.



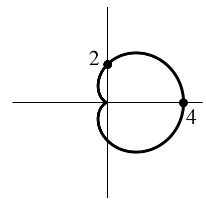
Cardioid

27.



Circle

28.



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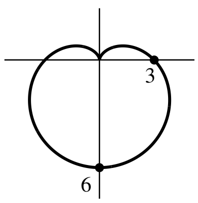

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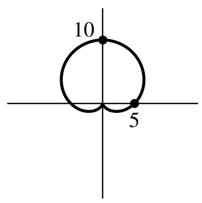

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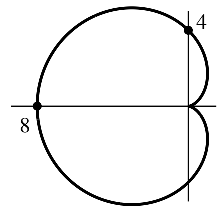

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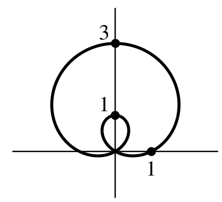
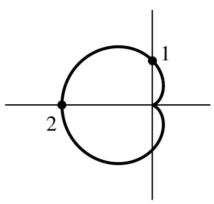


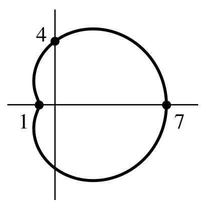
Limaçon

33.



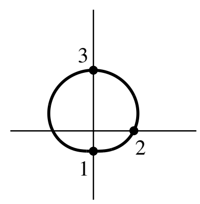
Cardioid

34.



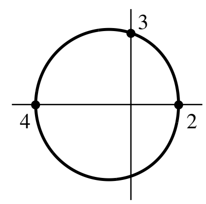
Limaçon

35.



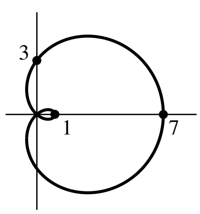
Limaçon

36.



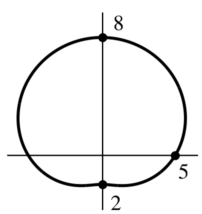
Limaçon

37.



Limaçon

38.



Limaçon

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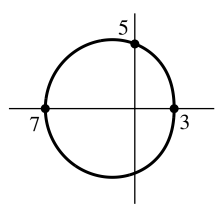

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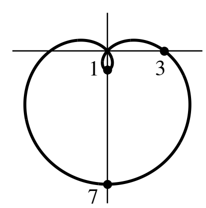

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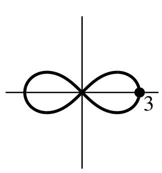
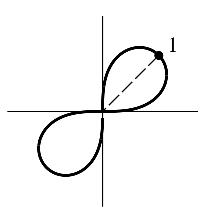
41.



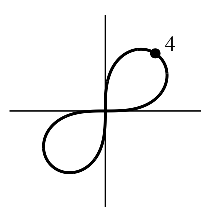
Lemniscate

42.



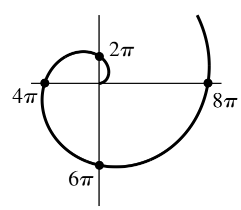
Lemniscate

43.



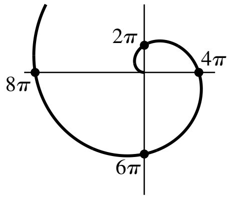
Lemniscate

44.



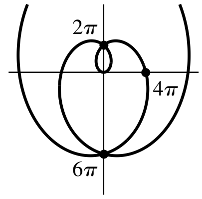
Spiral

45.



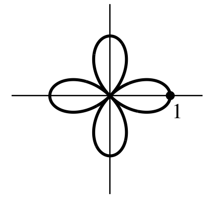
Spiral

46.



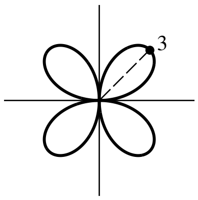
Spiral

47.



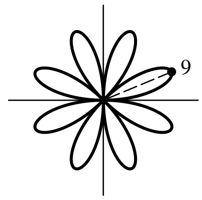
Four-petal rose

48.



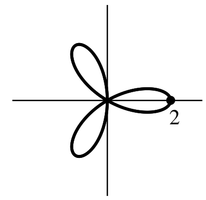
Four-petal rose

49.



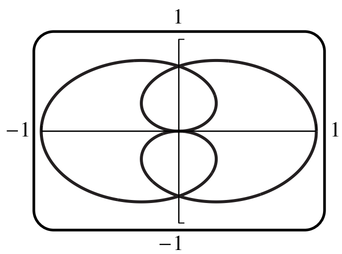
Eight-petal rose

50.

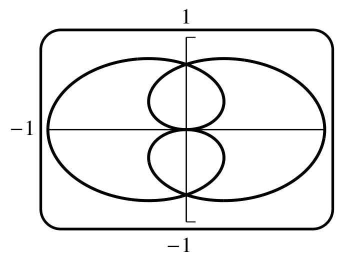


Three-petal rose

51.



52.



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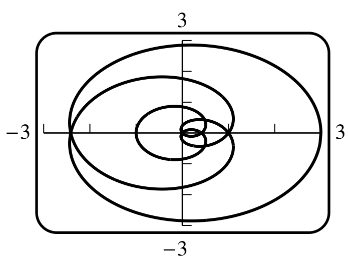
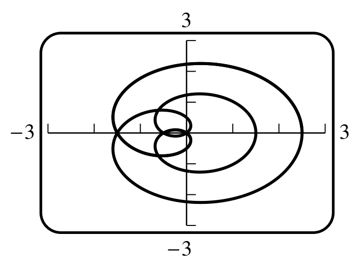
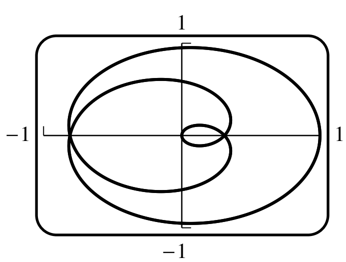


54.



55.



56. $0 \le \theta \le 8\pi$

57. **(a)** $-4\pi < \theta < 4\pi$

58. In I, along the x-axis, $x = r$ grows ever slower with θ. In II $x = r$ grows linearly with θ. Hence I: $r = \sqrt{\theta}$; II: $r = \theta$.

59. **(a)** $r = a/\cos\theta, x = r\cos\theta = a$, a family of vertical lines

(b) $r = b/\sin\theta, y = r\sin\theta = b$, a family of horizontal lines

60. The image of (r_0, θ_0) under a rotation through an angle α is $(r_0, \theta_0 + \alpha)$. Hence $(f(\theta), \theta)$ lies on the original curve if and only if $(f(\theta), \theta + \alpha)$ lies on the rotated curve, i.e. (r, θ) lies on the rotated curve if and only if $r = f(\theta - \alpha)$.

61. **(a)** $r = 1 + \cos(\theta - \pi/4) = 1 + \dfrac{\sqrt{2}}{2}(\cos\theta + \sin\theta)$

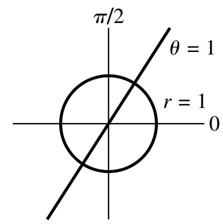
(b) $r = 1 + \cos(\theta - \pi/2) = 1 + \sin\theta$

(c) $r = 1 + \cos(\theta - \pi) = 1 - \cos\theta$

(d) $r = 1 + \cos(\theta - 5\pi/4) = 1 - \dfrac{\sqrt{2}}{2}(\cos\theta + \sin\theta)$

62. $r^2 = 4\cos 2(\theta - \pi/2) = -4\cos 2\theta$

63. Either $r - 1 = 0$ or $\theta - 1 = 0$, so the graph consists of the circle $r = 1$ and the line $\theta = 1$.



64. **(a)** $r^2 = Ar\sin\theta + Br\cos\theta$, $x^2 + y^2 = Ay + Bx$, $(x - B/2)^2 + (y - A/2)^2 = (A^2 + B^2)/4$, which is a circle of radius $\dfrac{1}{2}\sqrt{A^2 + B^2}$.

(b) Formula (4) follows by setting $A = 0, B = 2a$, $(x - a)^2 + y^2 = a^2$, the circle of radius a about $(a, 0)$. Formula (5) is derived in a similar fashion.

65. $y = r\sin\theta = (1 + \cos\theta)\sin\theta = \sin\theta + \sin\theta\cos\theta$,
$dy/d\theta = \cos\theta - \sin^2\theta + \cos^2\theta = 2\cos^2\theta + \cos\theta - 1 = (2\cos\theta - 1)(\cos\theta + 1)$;
$dy/d\theta = 0$ if $\cos\theta = 1/2$ or if $\cos\theta = -1$; $\theta = \pi/3$ or π (or $\theta = -\pi/3$, which leads to the minimum point).
If $\theta = \pi/3, \pi$, then $y = 3\sqrt{3}/4, 0$ so the maximum value of y is $3\sqrt{3}/4$ and the polar coordinates of the highest point are $(3/2, \pi/3)$.

66. $x = r\cos\theta = (1 + \cos\theta)\cos\theta = \cos\theta + \cos^2\theta$, $dx/d\theta = -\sin\theta - 2\sin\theta\cos\theta = -\sin\theta(1 + 2\cos\theta)$, $dx/d\theta = 0$ if $\sin\theta = 0$ or if $\cos\theta = -1/2$; $\theta = 0,\ 2\pi/3$, or π. If $\theta = 0,\ 2\pi/3,\ \pi$, then $x = 2, -1/4, 0$ so the minimum value of x is $-1/4$. The leftmost point has polar coordinates $(1/2, 2\pi/3)$.

67. **(a)** Let (x_1, y_1) and (x_2, y_2) be the rectangular coordinates of the points (r_1, θ_1) and (r_2, θ_2) then

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(r_2\cos\theta_2 - r_1\cos\theta_1)^2 + (r_2\sin\theta_2 - r_1\sin\theta_1)^2}$$
$$= \sqrt{r_1^2 + r_2^2 - 2r_1r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)} = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}.$$

An alternate proof follows directly from the Law of Cosines.

(b) Let P and Q have polar coordinates $(r_1, \theta_1), (r_2, \theta_2)$, respectively, then the perpendicular from OQ to OP has length $h = r_2\sin(\theta_2 - \theta_1)$ and $A = \frac{1}{2}hr_1 = \frac{1}{2}r_1r_2\sin(\theta_2 - \theta_1)$.

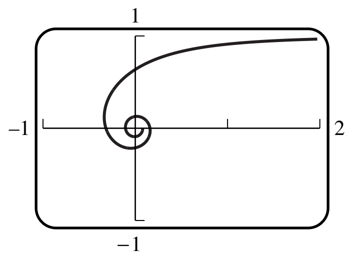
(c) From Part (a), $d = \sqrt{9 + 4 - 2\cdot 3\cdot 2\cos(\pi/6 - \pi/3)} = \sqrt{13 - 6\sqrt{3}} \approx 1.615$

(d) $A = \dfrac{1}{2}2\sin(5\pi/6 - \pi/3) = 1$

68. **(a)** $0 = (r^2 + a^2)^2 - a^4 - 4a^2r^2\cos^2\theta = r^4 + a^4 + 2r^2a^2 - a^4 - 4a^2r^2\cos^2\theta$
$= r^4 + 2r^2a^2 - 4a^2r^2\cos^2\theta$, so $r^2 = 2a^2(2\cos^2\theta - 1) = 2a^2\cos 2\theta$.

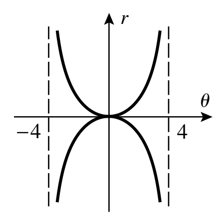
(b) The distance from the point (r, θ) to $(a, 0)$ is (from Exercise 67(a))
$\sqrt{r^2 + a^2 - 2ra\cos(\theta - 0)} = \sqrt{r^2 - 2ar\cos\theta + a^2}$, and to the point (a, π) is
$\sqrt{r^2 + a^2 - 2ra\cos(\theta - \pi)} = \sqrt{r^2 + 2ar\cos\theta + a^2}$, and their product is
$\sqrt{(r^2 + a^2)^2 - 4a^2r^2\cos^2\theta} = \sqrt{r^4 + a^4 + 2a^2r^2(1 - 2\cos^2\theta)}$
$= \sqrt{4a^4\cos^2 2\theta + a^4 + 2a^2(2a^2\cos 2\theta)(-\cos 2\theta)} = a^2$

69. $\lim_{\theta\to 0^+} y = \lim_{\theta\to 0^+} r\sin\theta = \lim_{\theta\to 0^+} \dfrac{\sin\theta}{\theta} = 1$, and $\lim_{\theta\to 0^+} x = \lim_{\theta\to 0^+} r\cos\theta = \lim_{\theta\to 0^+} \dfrac{\cos\theta}{\theta} = +\infty$.



70. $\lim_{\theta\to 0^\pm} y = \lim_{\theta\to 0^\pm} r\sin\theta = \lim_{\theta\to 0^\pm} \dfrac{\sin\theta}{\theta^2} = \lim_{\theta\to 0^\pm} \dfrac{\sin\theta}{\theta} \lim_{\theta\to 0^\pm} \dfrac{1}{\theta} = 1 \cdot \lim_{\theta\to 0^\pm} \dfrac{1}{\theta}$, so $\lim_{\theta\to 0^\pm} y$ does not exist.

71. Note that $r \to \pm\infty$ as θ approaches odd multiples of $\pi/2$;
$x = r\cos\theta = 4\tan\theta\cos\theta = 4\sin\theta$,
$y = r\sin\theta = 4\tan\theta\sin\theta$
so $x \to \pm 4$ and $y \to \pm\infty$ as θ approaches odd multiples of $\pi/2$.



72. $\lim_{\theta\to(\pi/2)^-} x = \lim_{\theta\to(\pi/2)^-} r\cos\theta = \lim_{\theta\to(\pi/2)^-} 2\sin^2\theta = 2$, and $\lim_{\theta\to(\pi/2)^-} y = +\infty$,
so $x = 2$ is a vertical asymptote.

73. Let $r = a\sin n\theta$ (the proof for $r = a\cos n\theta$ is similar). If θ starts at 0, then θ would have to increase by some positive integer multiple of π radians in order to reach the starting point and begin to retrace the curve. Let (r, θ) be the coordinates of a point P on the curve for $0 \le \theta < 2\pi$. Now $a\sin n(\theta + 2\pi) = a\sin(n\theta + 2\pi n) = a\sin n\theta = r$ so P is reached again with coordinates $(r, \theta + 2\pi)$ thus the curve is traced out either exactly once or exactly twice for $0 \le \theta < 2\pi$. If for $0 \le \theta < \pi$, $P(r, \theta)$ is reached again with coordinates $(-r, \theta + \pi)$ then the curve is traced out exactly once for $0 \le \theta < \pi$, otherwise exactly once for $0 \le \theta < 2\pi$. But

$$a\sin n(\theta + \pi) = a\sin(n\theta + n\pi) = \begin{cases} a\sin n\theta, & n \text{ even} \\ -a\sin n\theta, & n \text{ odd} \end{cases}$$

so the curve is traced out exactly once for $0 \le \theta < 2\pi$ if n is even, and exactly once for $0 \le \theta < \pi$ if n is odd.

EXERCISE SET 11.2

1. (a) $dy/dx = \dfrac{1/2}{2t} = 1/(4t)$; $dy/dx\big|_{t=-1} = -1/4; dy/dx\big|_{t=1} = 1/4$

(b) $x = (2y)^2 + 1, dx/dy = 8y, dy/dx\big|_{y=\pm(1/2)} = \pm 1/4$

2. **(a)** $dy/dx = (4\cos t)/(-3\sin t) = -(4/3)\cot t;\ dy/dx\big|_{t=\pi/4} = -4/3,\ dy/dx\big|_{t=7\pi/4} = 4/3$

(b) $(x/3)^2 + (y/4)^2 = 1,\ 2x/9 + (2y/16)(dy/dx) = 0,\ dy/dx = -16x/9y,$

$dy/dx\big|_{\substack{x=3/\sqrt{2}\\ y=4/\sqrt{2}}} = -4/3; dy/dx\big|_{\substack{x=3/\sqrt{2}\\ y=-4/\sqrt{2}}} = 4/3$

3. $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\dfrac{dy}{dx} = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)\dfrac{dt}{dx} = -\dfrac{1}{4t^2}(1/2t) = -1/(8t^3)$; positive when $t = -1$, negative when $t = 1$

4. $\dfrac{d^2y}{dx^2} = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)\dfrac{dt}{dx} = \dfrac{-(4/3)(-\csc^2 t)}{-3\sin t} = -\dfrac{4}{9}\csc^3 t$; negative at $t = \pi/4$, positive at $t = 7\pi/4$.

5. $dy/dx = \dfrac{2}{1/(2\sqrt{t})} = 4\sqrt{t},\ d^2y/dx^2 = \dfrac{2/\sqrt{t}}{1/(2\sqrt{t})} = 4,\ dy/dx\big|_{t=1} = 4,\ d^2y/dx^2\big|_{t=1} = 4$

6. $dy/dx = \dfrac{t^2}{t} = t,\ d^2y/dx^2 = \dfrac{1}{t},\ dy/dx\big|_{t=2} = 2,\ d^2y/dx^2\big|_{t=2} = 1/2$

7. $dy/dx = \dfrac{\sec^2 t}{\sec t\tan t} = \csc t,\ d^2y/dx^2 = \dfrac{-\csc t\cot t}{\sec t\tan t} = -\cot^3 t,$

$dy/dx\big|_{t=\pi/3} = 2/\sqrt{3},\ d^2y/dx^2\big|_{t=\pi/3} = -1/(3\sqrt{3})$

8. $dy/dx = \dfrac{\sinh t}{\cosh\ t} = \tanh t,\ \dfrac{d^2y}{dx^2} = \text{sech}^2 t/\cosh\ t = \text{sech}^3 t,\ dy/dx\big|_{t=0} = 0, d^2y/dx^2\big|_{t=0} = 1$

9. $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{-\cos\theta}{2-\sin\theta};\ \dfrac{d^2y}{dx^2} = \dfrac{d}{d\theta}\left(\dfrac{dy}{dx}\right)/\dfrac{dx}{d\theta} = \dfrac{1}{(2-\sin\theta)^2}\dfrac{1}{2-\sin\theta} = \dfrac{1}{(2-\sin\theta)^3};$

$\dfrac{dy}{dx}\Big|_{\theta=\pi/3} = \dfrac{-1/2}{2-\sqrt{3}/2} = \dfrac{-1}{4-\sqrt{3}};\ \dfrac{d^2y}{dx^2}\Big|_{\theta=\pi/3} = \dfrac{1}{(2-\sqrt{3}/2)^3} = \dfrac{8}{(4-\sqrt{3})^3}$

10. $\dfrac{dy}{dx} = \dfrac{3\cos\phi}{-\sin\phi} = -3\cot\phi;\ \dfrac{d^2y}{dx^2} = \dfrac{d}{d\phi}(-3\cot\phi)\dfrac{d\phi}{dx} = -3(-\csc^2\phi)(-\csc\phi) = -3\csc^3\phi;$

$\dfrac{dy}{dx}\Big|_{\phi=5\pi/6} = 3\sqrt{3};\ \dfrac{d^2y}{dx^2}\Big|_{\phi=5\pi/6} = -24$

11. **(a)** $dy/dx = \dfrac{-e^{-t}}{e^t} = -e^{-2t}$; for $t = 1$, $dy/dx = -e^{-2}$, $(x, y) = (e, e^{-1})$; $y - e^{-1} = -e^{-2}(x - e)$, $y = -e^{-2}x + 2e^{-1}$

(b) $y = 1/x, dy/dx = -1/x^2, m = -1/e^2, y - e^{-1} = -\dfrac{1}{e^2}(x - e), y = -\dfrac{1}{e^2}x + \dfrac{2}{e}$

12. $dy/dx = \dfrac{16t-2}{2} = 8t - 1$; for $t = 1$, $dy/dx = 7$, $(x, y) = (6, 10)$; $y - 10 = 7(x - 6)$, $y = 7x - 32$

13. $dy/dx = \dfrac{4\cos t}{-2\sin t} = -2\cot t$

(a) $dy/dx = 0$ if $\cot t = 0$, $t = \pi/2 + n\pi$ for $n = 0, \pm 1, \cdots$

(b) $dx/dy = -\dfrac{1}{2}\tan t = 0$ if $\tan t = 0$, $t = n\pi$ for $n = 0, \pm 1, \cdots$

14. $dy/dx = \dfrac{2t+1}{6t^2-30t+24} = \dfrac{2t+1}{6(t-1)(t-4)}$

(a) $dy/dx = 0$ if $t = -1/2$

(b) $dx/dy = \dfrac{6(t-1)(t-4)}{2t+1} = 0$ if $t = 1, 4$

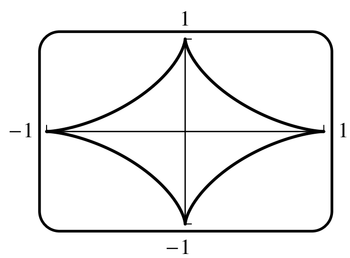
15. $x = y = 0$ when $t = 0, \pi$; $\dfrac{dy}{dx} = \dfrac{2\cos 2t}{\cos t}$; $\left.\dfrac{dy}{dx}\right|_{t=0} = 2$, $\left.\dfrac{dy}{dx}\right|_{t=\pi} = -2$, the equations of the tangent lines are $y = -2x, y = 2x$.

16. $y(t) = 0$ has three solutions, $t = 0, \pm\pi/2$; the last two correspond to the crossing point.
For $t = \pm\pi/2$, $m = \dfrac{dy}{dx} = \dfrac{2}{\pm\pi}$; the tangent lines are given by $y = \pm\dfrac{2}{\pi}(x-2)$.

17. If $y = 4$ then $t^2 = 4$, $t = \pm 2$, $x = 0$ for $t = \pm 2$ so $(0, 4)$ is reached when $t = \pm 2$.
$dy/dx = 2t/(3t^2 - 4)$. For $t = 2$, $dy/dx = 1/2$ and for $t = -2$, $dy/dx = -1/2$. The tangent lines are $y = \pm x/2 + 4$.

18. If $x = 3$ then $t^2 - 3t + 5 = 3$, $t^2 - 3t + 2 = 0$, $(t-1)(t-2) = 0$, $t = 1$ or 2. If $t = 1$ or 2 then $y = 1$ so $(3, 1)$ is reached when $t = 1$ or 2. $dy/dx = (3t^2 + 2t - 10)/(2t - 3)$. For $t = 1$, $dy/dx = 5$, the tangent line is $y - 1 = 5(x - 3)$, $y = 5x - 14$. For $t = 2$, $dy/dx = 6$, the tangent line is $y - 1 = 6(x - 3)$, $y = 6x - 17$.

19. **(a)**



(b) $\dfrac{dx}{dt} = -3\cos^2 t \sin t$ and $\dfrac{dy}{dt} = 3\sin^2 t \cos t$ are both zero when $t = 0, \pi/2, \pi, 3\pi/2, 2\pi$, so singular points occur at these values of t.

20. **(a)** when $y = 0$

(b) $\dfrac{dx}{dy} = \dfrac{a - a\cos\theta}{a\sin\theta} = 0$ when $\theta = 2n\pi, n = 0, 1, \ldots$ (which is when $y = 0$).

21. Substitute $\theta = \pi/3$, $r = 1$, and $dr/d\theta = -\sqrt{3}$ in equation (7) gives slope $m = 1/\sqrt{3}$.

22. As in Exercise 21, $\theta = \pi/4$, $dr/d\theta = \sqrt{2}/2$, $r = 1 + \sqrt{2}/2$, $m = -1 - \sqrt{2}$

23. As in Exercise 21, $\theta = 2$, $dr/d\theta = -1/4$, $r = 1/2$, $m = \dfrac{\tan 2 - 2}{2\tan 2 + 1}$

24. As in Exercise 21, $\theta = \pi/6$, $dr/d\theta = 4\sqrt{3}a$, $r = 2a$, $m = 3\sqrt{3}/5$

25. As in Exercise 21, $\theta = 3\pi/4$, $dr/d\theta = -3\sqrt{2}/2$, $r = \sqrt{2}/2$, $m = -2$

26. As in Exercise 21, $\theta = \pi$, $dr/d\theta = 3$, $r = 4$, $m = 4/3$

27. $m = \dfrac{dy}{dx} = \dfrac{r\cos\theta + (\sin\theta)(dr/d\theta)}{-r\sin\theta + (\cos\theta)(dr/d\theta)} = \dfrac{\cos\theta + 2\sin\theta\cos\theta}{-\sin\theta + \cos^2\theta - \sin^2\theta}$; if $\theta = 0, \pi/2, \pi$, then $m = 1, 0, -1$.

28. $m = \dfrac{dy}{dx} = \dfrac{\cos\theta(4\sin\theta - 1)}{4\cos^2\theta + \sin\theta - 2}$; if $\theta = 0, \pi/2, \pi$ then $m = -1/2, 0, 1/2$.

29. $dx/d\theta = -a\sin\theta(1 + 2\cos\theta)$, $dy/d\theta = a(2\cos\theta - 1)(\cos\theta + 1)$

(a) horizontal if $dy/d\theta = 0$ and $dx/d\theta \neq 0$. $dy/d\theta = 0$ when $\cos\theta = 1/2$ or $\cos\theta = -1$ so $\theta = \pi/3$, $5\pi/3$, or π; $dx/d\theta \neq 0$ for $\theta = \pi/3$ and $5\pi/3$. For the singular point $\theta = \pi$ we find that $\lim_{\theta\to\pi} dy/dx = 0$. There is a horizontal tangent line at $(3a/2, \pi/3)$, $(0, \pi)$, and $(3a/2, 5\pi/3)$.

(b) vertical if $dy/d\theta \neq 0$ and $dx/d\theta = 0$. $dx/d\theta = 0$ when $\sin\theta = 0$ or $\cos\theta = -1/2$ so $\theta = 0$, π, $2\pi/3$, or $4\pi/3$; $dy/d\theta \neq 0$ for $\theta = 0$, $2\pi/3$, and $4\pi/3$. The singular point $\theta = \pi$ was discussed in Part (a). There is a vertical tangent line at $(2a, 0)$, $(a/2, 2\pi/3)$, and $(a/2, 4\pi/3)$.

30. $dx/d\theta = a(\cos^2\theta - \sin^2\theta) = a\cos 2\theta$, $dy/d\theta = 2a\sin\theta\cos\theta = a\sin 2\theta$

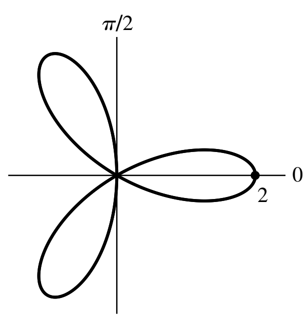
(a) horizontal if $dy/d\theta = 0$ and $dx/d\theta \neq 0$. $dy/d\theta = 0$ when $\theta = 0, \pi/2, \pi, 3\pi/2$;
$dx/d\theta \neq 0$ for $(0,0), (a, \pi/2), (0, \pi), (-a, 3\pi/2)$; in reality only two distinct points

(b) vertical if $dy/d\theta \neq 0$ and $dx/d\theta = 0$. $dx/d\theta = 0$ when $\theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$; $dy/d\theta \neq 0$ there, so vertical tangent line at $(a/\sqrt{2}, \pi/4), (a/\sqrt{2}, 3\pi/4), (-a/\sqrt{2}, 5\pi/4), (-a/\sqrt{2}, 7\pi/4)$, only two distinct points

31. $dy/d\theta = (d/d\theta)(\sin^2\theta\cos^2\theta) = (\sin 4\theta)/2 = 0$ at $\theta = 0, \pi/4, \pi/2, 3\pi/4, \pi$; at the same points, $dx/d\theta = (d/d\theta)(\sin\theta\cos^3\theta) = \cos^2\theta(4\cos^2\theta - 3)$. Next, $\dfrac{dx}{d\theta} = 0$ at $\theta = \pi/2$, a singular point; and $\theta = 0, \pi$ both give the same point, so there are just three points with a horizontal tangent.

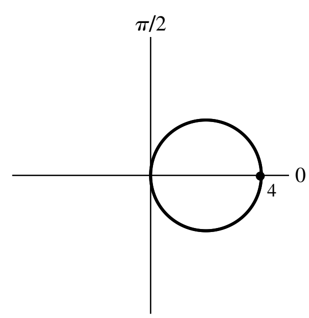
32. $dx/d\theta = 4\sin^2\theta - \sin\theta - 2$, $dy/d\theta = \cos\theta(1 - 4\sin\theta)$. $dy/d\theta = 0$ when $\cos\theta = 0$ or $\sin\theta = 1/4$ so $\theta = \pi/2$, $3\pi/2$, $\sin^{-1}(1/4)$, or $\pi - \sin^{-1}(1/4)$; $dx/d\theta \neq 0$ at these points, so there is a horizontal tangent at each one.

33.



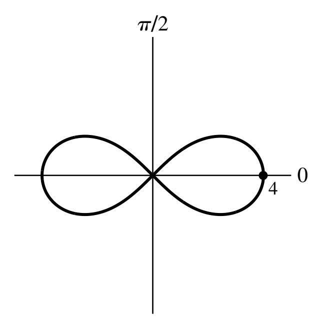
$\theta_0 = \pi/6, \pi/2, 5\pi/6$

34.



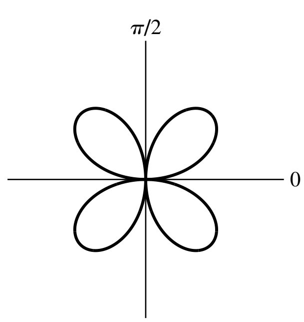
$\theta_0 = \pi/2$

35.



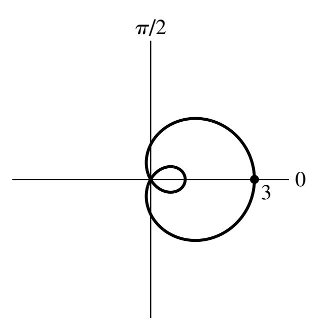
$\theta_0 = \pm\pi/4$

36.



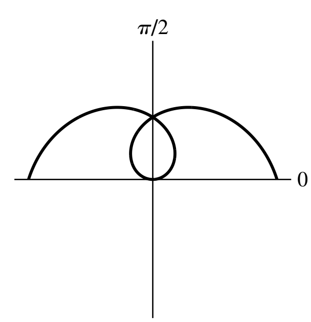
$\theta_0 = 0, \pi/2$

37.



$\theta_0 = 2\pi/3, 4\pi/3$

38.



$\theta_0 = 0$

39. $r^2 + (dr/d\theta)^2 = a^2 + 0^2 = a^2$, $L = \int_0^{2\pi} a d\theta = 2\pi a$

40. $r^2 + (dr/d\theta)^2 = (2a\cos\theta)^2 + (-2a\sin\theta)^2 = 4a^2$, $L = \int_{-\pi/2}^{\pi/2} 2a d\theta = 2\pi a$

41. $r^2 + (dr/d\theta)^2 = [a(1-\cos\theta)]^2 + [a\sin\theta]^2 = 4a^2\sin^2(\theta/2)$, $L = 2\int_0^{\pi} 2a\sin(\theta/2)d\theta = 8a$

42. $r^2 + (dr/d\theta)^2 = [\sin^2(\theta/2)]^2 + [\sin(\theta/2)\cos(\theta/2)]^2 = \sin^2(\theta/2)$, $L = \int_0^{\pi} \sin(\theta/2)d\theta = 2$

43. $r^2 + (dr/d\theta)^2 = (e^{3\theta})^2 + (3e^{3\theta})^2 = 10e^{6\theta}$, $L = \int_0^2 \sqrt{10}e^{3\theta}d\theta = \sqrt{10}(e^6-1)/3$

44. $r^2 + (dr/d\theta)^2 = [\sin^3(\theta/3)]^2 + [\sin^2(\theta/3)\cos(\theta/3)]^2 = \sin^4(\theta/3)$,

$L = \int_0^{\pi/2} \sin^2(\theta/3)d\theta = (2\pi - 3\sqrt{3})/8$

45. **(a)** From (3), $\dfrac{dy}{dx} = \dfrac{3\sin t}{1 - 3\cos t}$

(b) At $t = 10$, $\dfrac{dy}{dx} = \dfrac{3\sin 10}{1 - 3\cos 10} \approx -0.46402$, $\theta \approx \tan^{-1}(-0.46402) = -0.4345$

46. **(a)** $\dfrac{dy}{dx} = 0$ when $\dfrac{dy}{dt} = 2\sin t = 0, t = 0, \pi, 2\pi, 3\pi$

(b) $\dfrac{dx}{dt} = 0$ when $1 - 2\cos t = 0, \cos t = 1/2, t = \pi/3, 5\pi/3, 7\pi/3$

47. **(a)** $r^2 + (dr/d\theta)^2 = (\cos n\theta)^2 + (-n\sin n\theta)^2 = \cos^2 n\theta + n^2\sin^2 n\theta$
$= (1 - \sin^2 n\theta) + n^2\sin^2 n\theta = 1 + (n^2-1)\sin^2 n\theta$,

$L = 2\int_0^{\pi/(2n)} \sqrt{1 + (n^2-1)\sin^2 n\theta}d\theta$

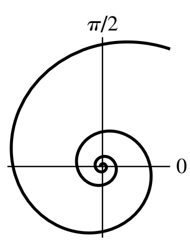
(b) $L = 2\int_0^{\pi/4} \sqrt{1 + 3\sin^2 2\theta}d\theta \approx 2.42$

(c)

n	2	3	4	5	6	7	8	9	10	11
L	2.42211	2.22748	2.14461	2.10100	2.07501	2.05816	2.04656	2.03821	2.03199	2.02721

n	12	13	14	15	16	17	18	19	20
L	2.02346	2.02046	2.01802	2.01600	2.01431	2.01288	2.01167	2.01062	2.00971

48. **(a)**



(b) $r^2 + (dr/d\theta)^2 = (e^{-\theta})^2 + (-e^{-\theta})^2 = 2e^{-2\theta}$, $L = 2\int_0^{+\infty} e^{-2\theta}d\theta$

(c) $L = \lim_{\theta_0 \to +\infty} 2\int_0^{\theta_0} e^{-2\theta}d\theta = \lim_{\theta_0 \to +\infty}(1 - e^{-2\theta_0}) = 1$

49. $x' = 2t,\ y' = 2,\ (x')^2 + (y')^2 = 4t^2 + 4$

$$S = 2\pi \int_0^4 (2t)\sqrt{4t^2 + 4}dt = 8\pi \int_0^4 t\sqrt{t^2 + 1}dt = \frac{8\pi}{3}(t^2 + 1)^{3/2}\Big]_0^4 = \frac{8\pi}{3}(17\sqrt{17} - 1)$$

50. $x' = e^t(\cos t - \sin t),\ y' = e^t(\cos t + \sin t),\ (x')^2 + (y')^2 = 2e^{2t}$

$$S = 2\pi \int_0^{\pi/2} (e^t \sin t)\sqrt{2e^{2t}}dt = 2\sqrt{2}\pi \int_0^{\pi/2} e^{2t}\sin t\, dt$$

$$= 2\sqrt{2}\pi \left[\frac{1}{5}e^{2t}(2\sin t - \cos t)\right]_0^{\pi/2} = \frac{2\sqrt{2}}{5}\pi(2e^{\pi} + 1)$$

51. $x' = -2\sin t\cos t,\ y' = 2\sin t\cos t,\ (x')^2 + (y')^2 = 8\sin^2 t\cos^2 t$

$$S = 2\pi \int_0^{\pi/2} \cos^2 t\sqrt{8\sin^2 t\cos^2 t}\, dt = 4\sqrt{2}\pi \int_0^{\pi/2} \cos^3 t\sin t\, dt = -\sqrt{2}\pi\cos^4 t\Big]_0^{\pi/2} = \sqrt{2}\pi$$

52. $x' = 1,\ y' = 4t,\ (x')^2 + (y')^2 = 1 + 16t^2,\ S = 2\pi \int_0^1 t\sqrt{1 + 16t^2}\, dt = \frac{\pi}{24}(17\sqrt{17} - 1)$

53. $x' = -r\sin t,\ y' = r\cos t,\ (x')^2 + (y')^2 = r^2,\ S = 2\pi \int_0^{\pi} r\sin t\sqrt{r^2}\, dt = 2\pi r^2 \int_0^{\pi} \sin t\, dt = 4\pi r^2$

54. $\frac{dx}{d\phi} = a(1 - \cos\phi),\ \frac{dy}{d\phi} = a\sin\phi,\ \left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2 = 2a^2(1 - \cos\phi)$

$$S = 2\pi \int_0^{2\pi} a(1 - \cos\phi)\sqrt{2a^2(1 - \cos\phi)}\, d\phi = 2\sqrt{2}\pi a^2 \int_0^{2\pi} (1 - \cos\phi)^{3/2}d\phi,$$

but $1 - \cos\phi = 2\sin^2\frac{\phi}{2}$ so $(1 - \cos\phi)^{3/2} = 2\sqrt{2}\sin^3\frac{\phi}{2}$ for $0 \le \phi \le \pi$ and, taking advantage of the symmetry of the cycloid, $S = 16\pi a^2 \int_0^{\pi} \sin^3\frac{\phi}{2}d\phi = 64\pi a^2/3$

55. (a) $\frac{dr}{dt} = 2$ and $\frac{d\theta}{dt} = 1$ so $\frac{dr}{d\theta} = \frac{dr/dt}{d\theta/dt} = \frac{2}{1} = 2$, $r = 2\theta + C$, $r = 10$ when $\theta = 0$ so $10 = C, r = 2\theta + 10$.

(b) $r^2 + (dr/d\theta)^2 = (2\theta + 10)^2 + 4$, during the first 5 seconds the rod rotates through an angle of $(1)(5) = 5$ radians so $L = \int_0^5 \sqrt{(2\theta + 10)^2 + 4}d\theta$, let $u = 2\theta + 10$ to get

$$L = \frac{1}{2}\int_{10}^{20} \sqrt{u^2 + 4}du = \frac{1}{2}\left[\frac{u}{2}\sqrt{u^2 + 4} + 2\ln|u + \sqrt{u^2 + 4}|\right]_{10}^{20}$$

$$= \frac{1}{2}\left[10\sqrt{404} - 5\sqrt{104} + 2\ln\frac{20 + \sqrt{404}}{10 + \sqrt{104}}\right] \approx 75.7 \text{ mm}$$

56. $x = r\cos\theta, y = r\sin\theta, \frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta, \frac{dy}{d\theta} = r\cos\theta + \frac{dr}{d\theta}\sin\theta,$

$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$, and Formula (6) of Section 8.4 becomes

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

EXERCISE SET 11.3

1. **(a)** $\displaystyle\int_{\pi/2}^{\pi} \frac{1}{2}(1-\cos\theta)^2\,d\theta$ **(b)** $\displaystyle\int_0^{\pi/2} \frac{1}{2}4\cos^2\theta\,d\theta$ **(c)** $\displaystyle\int_0^{\pi/2} \frac{1}{2}\sin^2 2\theta\,d\theta$

(d) $\displaystyle\int_0^{2\pi} \frac{1}{2}\theta^2\,d\theta$ **(e)** $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{1}{2}(1-\sin\theta)^2\,d\theta$ **(f)** $\displaystyle 2\int_0^{\pi/4} \frac{1}{2}\cos^2 2\theta\,d\theta$

2. **(a)** $3\pi/8+1$ **(b)** $\pi/2$ **(c)** $\pi/8$

(d) $4\pi^3/3$ **(e)** $3\pi/4$ **(f)** $\pi/8$

3. **(a)** $\displaystyle A=\int_0^{2\pi}\frac{1}{2}a^2\,d\theta=\pi a^2$ **(b)** $\displaystyle A=\int_0^{\pi}\frac{1}{2}4a^2\sin^2\theta\,d\theta=\pi a^2$

(c) $\displaystyle A=\int_{-\pi/2}^{\pi/2}\frac{1}{2}4a^2\cos^2\theta\,d\theta=\pi a^2$

4. **(a)** $\displaystyle r^2=r\sin\theta+r\cos\theta,\ x^2+y^2-y-x=0,\ \left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{2}$

(b) $\displaystyle A=\int_{-\pi/4}^{3\pi/4}\frac{1}{2}(\sin\theta+\cos\theta)^2\,d\theta=\pi/2$

5. $\displaystyle A=2\int_0^{\pi}\frac{1}{2}(2+2\cos\theta)^2d\theta=6\pi$

6. $\displaystyle A=\int_0^{\pi/2}\frac{1}{2}(1+\sin\theta)^2d\theta=3\pi/8+1$

7. $\displaystyle A=6\int_0^{\pi/6}\frac{1}{2}(16\cos^2 3\theta)d\theta=4\pi$

8. The petal in the first quadrant has area $\displaystyle\int_0^{\pi/2}\frac{1}{2}4\sin^2 2\theta\,d\theta=\pi/2$, so total area $=2\pi$.

9. $\displaystyle A=2\int_{2\pi/3}^{\pi}\frac{1}{2}(1+2\cos\theta)^2d\theta=\pi-3\sqrt{3}/2$

10. $\displaystyle A=\int_1^3\frac{2}{\theta^2}d\theta=4/3$

11. area $\displaystyle =A_1-A_2=\int_0^{\pi/2}\frac{1}{2}4\cos^2\theta\,d\theta-\int_0^{\pi/4}\frac{1}{2}\cos 2\theta\,d\theta=\pi/2-\frac{1}{4}$

12. area $\displaystyle =A_1-A_2=\int_0^{\pi}\frac{1}{2}(1+\cos\theta)^2\,d\theta-\int_0^{\pi/2}\frac{1}{2}\cos^2\theta\,d\theta=5\pi/8$

13. The circles intersect when $\cos t=\sqrt{3}\sin t, \tan t=1/\sqrt{3}, t=\pi/6$, so

$$A=A_1+A_2=\int_0^{\pi/6}\frac{1}{2}(4\sqrt{3}\sin t)^2\,dt+\int_{\pi/6}^{\pi/2}\frac{1}{2}(4\cos t)^2\,dt=2\pi-3\sqrt{3}+4\pi/3-\sqrt{3}=10\pi/3-4\sqrt{3}.$$

14. The curves intersect when $1+\cos t=3\cos t, \cos t=1/2, t=\pm\pi/3$, and hence

$$\text{total area}=2\int_0^{\pi/3}\frac{1}{2}(1+\cos t)^2\,dt+2\int_{\pi/3}^{\pi/2}\frac{1}{2}9\cos^2 t\,dt=2(\pi/4+9\sqrt{3}/16+3\pi/8-9\sqrt{3}/16)=5\pi/4.$$

15. $\displaystyle A=2\int_{\pi/6}^{\pi/2}\frac{1}{2}[25\sin^2\theta-(2+\sin\theta)^2]d\theta=8\pi/3+\sqrt{3}$

16. $A = 2\int_0^{\pi} \frac{1}{2}[16 - (2 - 2\cos\theta)^2]d\theta = 10\pi$

17. $A = 2\int_0^{\pi/3} \frac{1}{2}[(2+2\cos\theta)^2 - 9]d\theta = 9\sqrt{3}/2 - \pi$

18. $A = 2\int_0^{\pi/4} \frac{1}{2}(16\sin^2\theta)d\theta = 2\pi - 4$

19. $A = 2\left[\int_0^{2\pi/3} \frac{1}{2}(1/2 + \cos\theta)^2 d\theta - \int_{2\pi/3}^{\pi} \frac{1}{2}(1/2+\cos\theta)^2 d\theta\right] = (\pi + 3\sqrt{3})/4$

20. $A = 2\int_0^{\pi/3} \frac{1}{2}\left[(2+2\cos\theta)^2 - \frac{9}{4}\sec^2\theta\right] d\theta = 2\pi + \frac{9}{4}\sqrt{3}$

21. $A = 2\int_0^{\cos^{-1}(3/5)} \frac{1}{2}(100 - 36\sec^2\theta)d\theta = 100\cos^{-1}(3/5) - 48$

22. $A = 8\int_0^{\pi/8} \frac{1}{2}(4a^2\cos^2 2\theta - 2a^2)d\theta = 2a^2$

23. **(a)** r is not real for $\pi/4 < \theta < 3\pi/4$ and $5\pi/4 < \theta < 7\pi/4$

(b) $A = 4\int_0^{\pi/4} \frac{1}{2}a^2\cos 2\theta\, d\theta = a^2$

(c) $A = 4\int_0^{\pi/6} \frac{1}{2}\left[4\cos 2\theta - 2\right] d\theta = 2\sqrt{3} - \frac{2\pi}{3}$

24. $A = 2\int_0^{\pi/2} \frac{1}{2}\sin 2\theta\, d\theta = 1$

25. $A = \int_{2\pi}^{4\pi} \frac{1}{2}a^2\theta^2\, d\theta - \int_0^{2\pi} \frac{1}{2}a^2\theta^2\, d\theta = 8\pi^3 a^2$

26. **(a)** $x = r\cos\theta, y = r\sin\theta,$

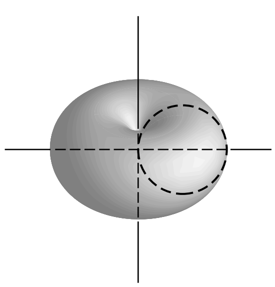
$(dx/d\theta)^2 + (dy/d\theta)^2 = (f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2 = f'(\theta)^2 + f(\theta)^2;$

$S = \int_\alpha^\beta 2\pi f(\theta)\sin\theta\sqrt{f'(\theta)^2 + f(\theta)^2}\, d\theta$ if about $\theta = 0$; similarly for $\theta = \pi/2$

(b) f', g' are continuous and no segment of the curve is traced more than once.

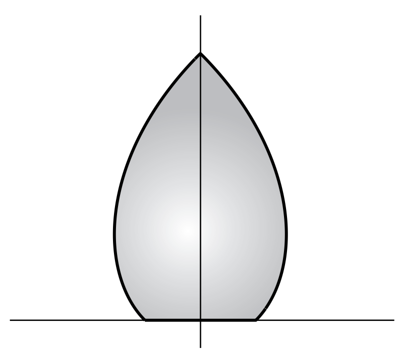
27. $r^2 + \left(\frac{dr}{d\theta}\right)^2 = \cos^2\theta + \sin^2\theta = 1,$

so $S = \int_{-\pi/2}^{\pi/2} 2\pi\cos^2\theta\, d\theta = \pi^2.$



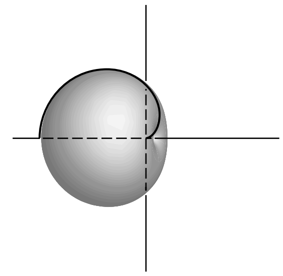
28. $S = \int_0^{\pi/2} 2\pi e^\theta \cos\theta\sqrt{2e^{2\theta}}\, d\theta$

$= 2\sqrt{2}\pi\int_0^{\pi/2} e^{2\theta}\cos\theta\, d\theta = \frac{2\sqrt{2}\pi}{5}(e^\pi - 2)$

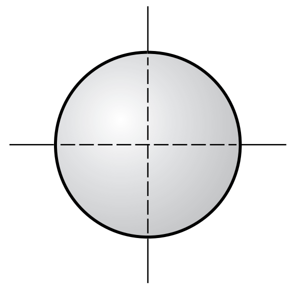


29. $S = \int_0^{\pi} 2\pi(1-\cos\theta)\sin\theta\sqrt{1-2\cos\theta+\cos^2\theta+\sin^2\theta}\,d\theta$

$= 2\sqrt{2}\pi\int_0^{\pi}\sin\theta(1-\cos\theta)^{3/2}\,d\theta = \frac{2}{5}2\sqrt{2}\pi(1-\cos\theta)^{5/2}\Big]_0^{\pi} = 32\pi/5$



30. $S = \int_0^{\pi} 2\pi a(\sin\theta)a\,d\theta = 4\pi a^2$



31. **(a)** $r^3\cos^3\theta - 3r^2\cos\theta\sin\theta + r^3\sin^3\theta = 0,\ r = \dfrac{3\cos\theta\sin\theta}{\cos^3\theta+\sin^3\theta}$

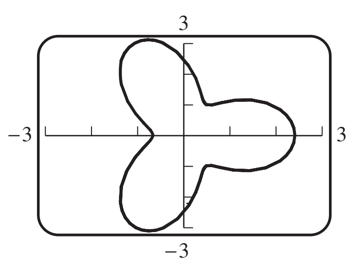
32. **(a)** $A = 2\int_0^{\pi/(2n)}\frac{1}{2}a^2\cos^2 n\theta\,d\theta = \dfrac{\pi a^2}{4n}$ **(b)** $A = 2\int_0^{\pi/(2n)}\frac{1}{2}a^2\cos^2 n\theta\,d\theta = \dfrac{\pi a^2}{4n}$

(c) $\dfrac{1}{2n}\times$ total area $= \dfrac{\pi a^2}{4n}$ **(d)** $\dfrac{1}{n}\times$ total area $= \dfrac{\pi a^2}{4n}$

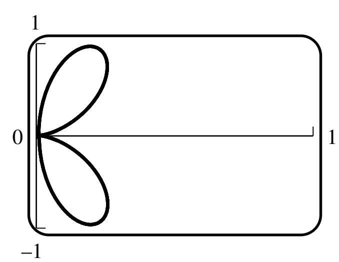
33. If the upper right corner of the square is the point (a,a) then the large circle has equation $r=\sqrt{2}a$ and the small circle has equation $(x-a)^2+y^2=a^2,\ r=2a\cos\theta$, so

area of crescent $= 2\int_0^{\pi/4}\frac{1}{2}\left[(2a\cos\theta)^2-(\sqrt{2}a)^2\right]d\theta = a^2 =$ area of square.

34. $A = \int_0^{2\pi}\frac{1}{2}(\cos 3\theta+2)^2\,d\theta = 9\pi/2$



35. $A = \int_0^{\pi/2}\frac{1}{2}4\cos^2\theta\sin^4\theta\,d\theta = \pi/16$



EXERCISE SET 11.4

1. **(a)** $4px = y^2$, point $(1,1), 4p=1, x=y^2$ **(b)** $-4py = x^2$, point $(3,-3), 12p=9, -3y=x^2$

(c) $a=3, b=2, \dfrac{x^2}{9}+\dfrac{y^2}{4}=1$ **(d)** $a=3, b=2, \dfrac{x^2}{4}+\dfrac{y^2}{9}=1$

(e) asymptotes: $y = \pm x$, so $a = b$; point $(0, 1)$, so $y^2 - x^2 = 1$

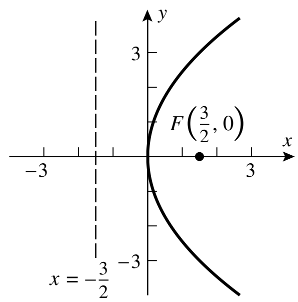
(f) asymptotes: $y = \pm x$, so $b = a$; point $(2, 0)$, so $\dfrac{x^2}{4} - \dfrac{y^2}{4} = 1$

2. **(a)** Part (a), vertex $(0, 0), p = 1/4$; focus $(1/4, 0)$, directrix: $x = -1/4$
Part (b), vertex $(0, 0), p = 3/4$; focus $(0, -3/4)$, directrix: $y = 3/4$

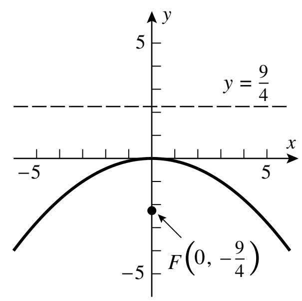
(b) Part (c), $c = \sqrt{a^2 - b^2} = \sqrt{5}$, foci $(\pm\sqrt{5}, 0)$
Part (d), $c = \sqrt{a^2 - b^2} = \sqrt{5}$, foci $(0, \pm\sqrt{5})$

(c) Part (e), $c = \sqrt{a^2 + b^2} = \sqrt{2}$, foci at $(0, \pm\sqrt{2})$; asymptotes: $y^2 - x^2 = 0, y = \pm x$
Part (f), $c = \sqrt{a^2 + b^2} = \sqrt{8} = 2\sqrt{2}$, foci at $(\pm 2\sqrt{2}, 0)$; asymptotes: $\dfrac{x^2}{4} - \dfrac{y^2}{4} = 0, y = \pm x$

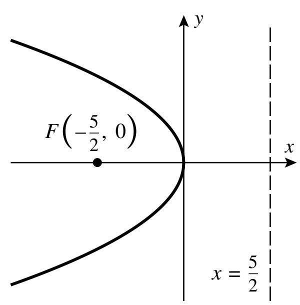
3. **(a)**



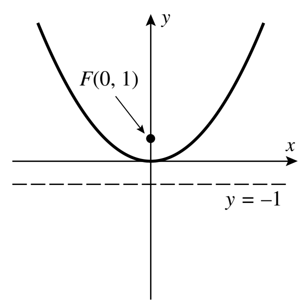
(b)



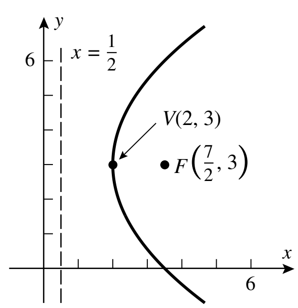
4. **(a)**



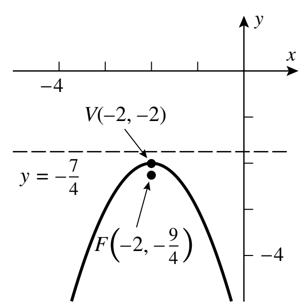
(b)



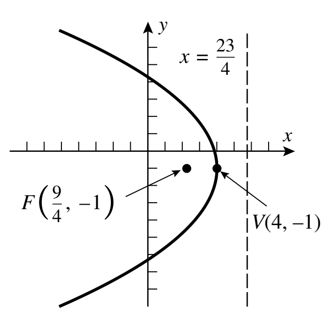
5. **(a)**



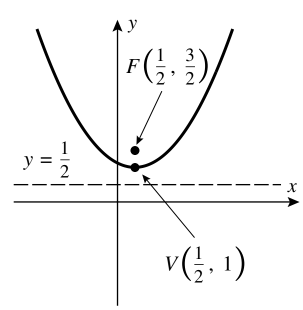
(b)



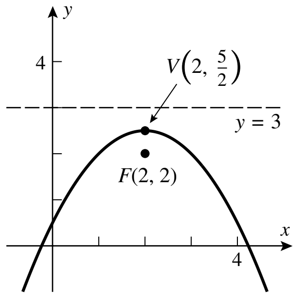
6. **(a)**



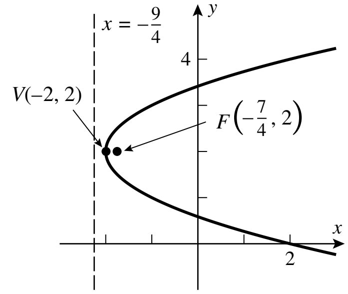
(b)



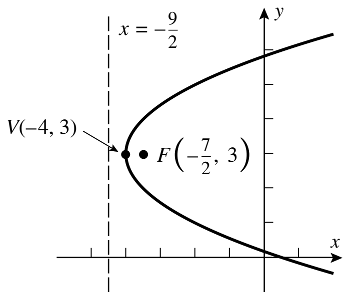
7. **(a)**



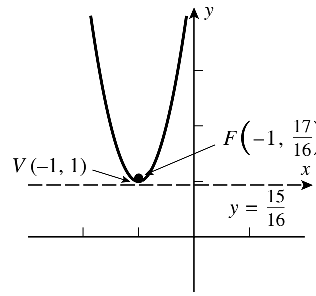
(b)



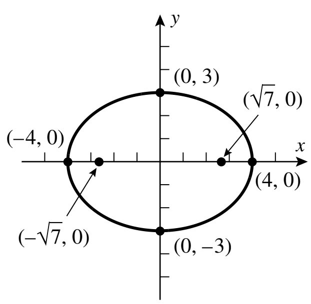
8. **(a)**



(b)

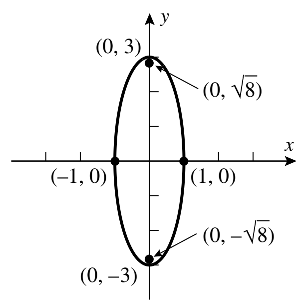


9. **(a)** $c^2 = 16 - 9 = 7,\ c = \sqrt{7}$

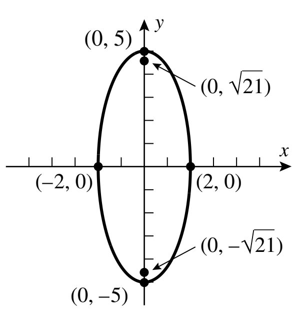


(b) $\dfrac{x^2}{1} + \dfrac{y^2}{9} = 1$

$c^2 = 9 - 1 = 8, c = \sqrt{8}$

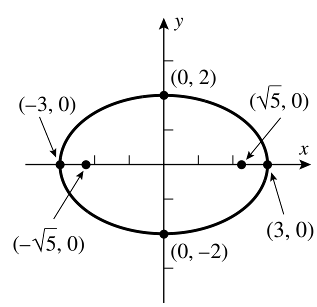


10. **(a)** $c^2 = 25 - 4 = 21,\ c = \sqrt{21}$



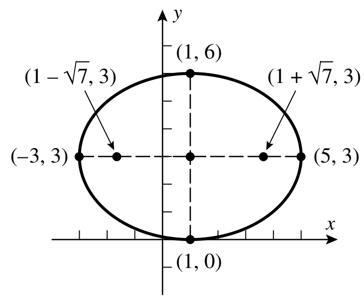
(b) $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

$c^2 = 9 - 4 = 5, c = \sqrt{5}$



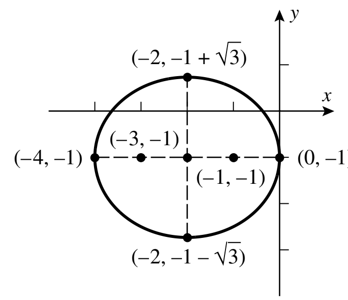
11. **(a)** $\dfrac{(x-1)^2}{16}+\dfrac{(y-3)^2}{9}=1$

$c^2=16-9=7, c=\sqrt{7}$



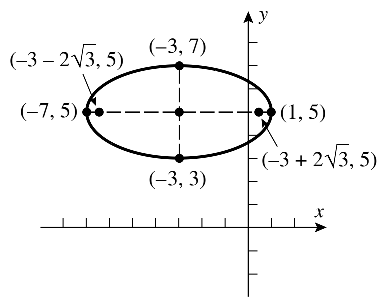
(b) $\dfrac{(x+2)^2}{4}+\dfrac{(y+1)^2}{3}=1$

$c^2=4-3=1, c=1$



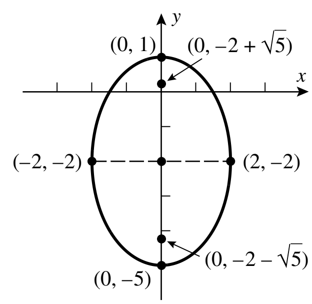
12. **(a)** $\dfrac{(x+3)^2}{16}+\dfrac{(y-5)^2}{4}=1$

$c^2=16-4=12, c=2\sqrt{3}$



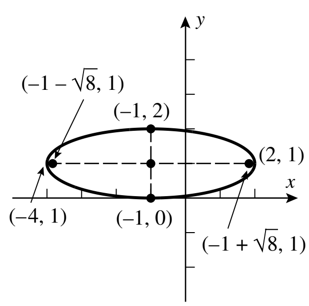
(b) $\dfrac{x^2}{4}+\dfrac{(y+2)^2}{9}=1$

$c^2=9-4=5, c=\sqrt{5}$



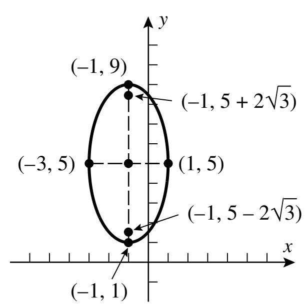
13. **(a)** $\dfrac{(x+1)^2}{9}+\dfrac{(y-1)^2}{1}=1$

$c^2=9-1=8, c=\sqrt{8}$



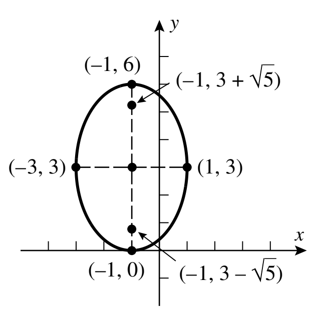
(b) $\dfrac{(x+1)^2}{4}+\dfrac{(y-5)^2}{16}=1$

$c^2=16-4=12, c=2\sqrt{3}$



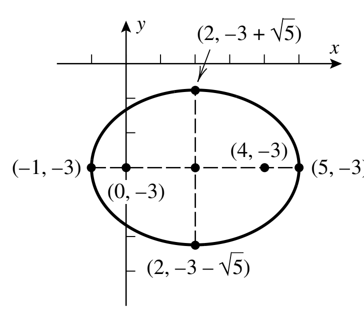
14. **(a)** $\dfrac{(x+1)^2}{4}+\dfrac{(y-3)^2}{9}=1$

$c^2=9-4=5, c=\sqrt{5}$

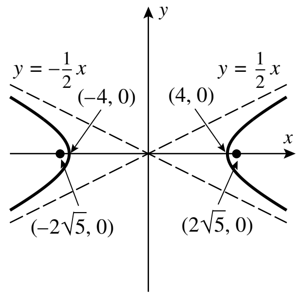


(b) $\dfrac{(x-2)^2}{9}+\dfrac{(y+3)^2}{5}=1$

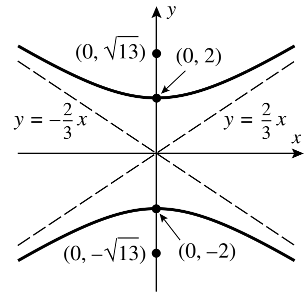
$c^2=9-5=4, c=2$



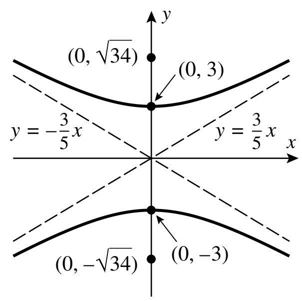
15. **(a)** $c^2 = a^2 + b^2 = 16 + 4 = 20, c = 2\sqrt{5}$



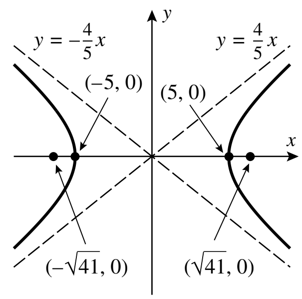
(b) $y^2/4 - x^2/9 = 1$
$c^2 = 4 + 9 = 13, c = \sqrt{13}$



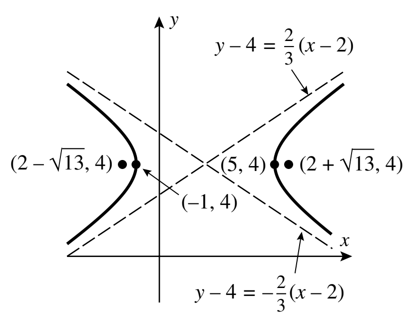
16. **(a)** $c^2 = a^2 + b^2 = 9 + 25 = 34, c = \sqrt{34}$



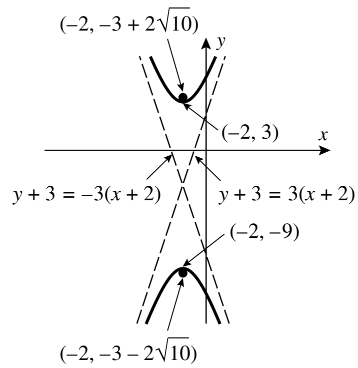
(b) $x^2/25 - y^2/16 = 1$
$c^2 = 25 + 16 = 41, c = \sqrt{41}$



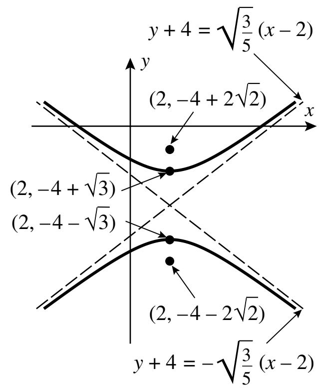
17. **(a)** $c^2 = 9 + 4 = 13, c = \sqrt{13}$



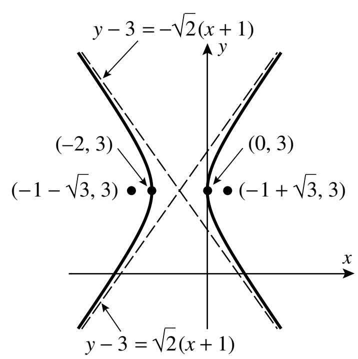
(b) $(y+3)^2/36 - (x+2)^2/4 = 1$
$c^2 = 36 + 4 = 40, c = 2\sqrt{10}$



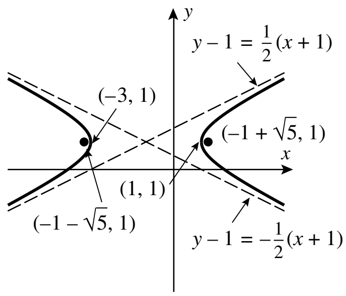
18. **(a)** $c^2 = 3 + 5 = 8, c = 2\sqrt{2}$



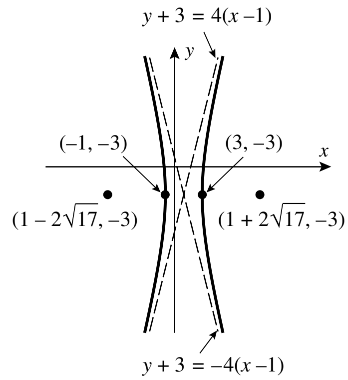
(b) $(x+1)^2/1 - (y-3)^2/2 = 1$
$c^2 = 1 + 2 = 3, c = \sqrt{3}$



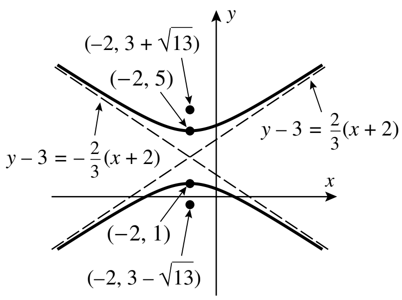
19. **(a)** $(x+1)^2/4 - (y-1)^2/1 = 1$
$c^2 = 4 + 1 = 5, c = \sqrt{5}$



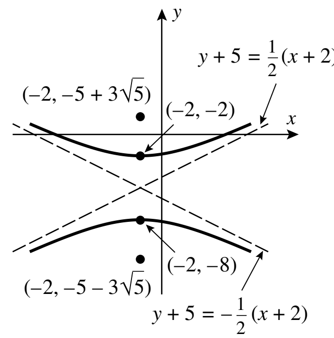
(b) $(x-1)^2/4 - (y+3)^2/64 = 1$
$c^2 = 4 + 64 = 68, c = 2\sqrt{17}$



20. **(a)** $(y-3)^2/4 - (x+2)^2/9 = 1$
$c^2 = 4 + 9 = 13, c = \sqrt{13}$



(b) $(y+5)^2/9 - (x+2)^2/36 = 1$
$c^2 = 9 + 36 = 45, c = 3\sqrt{5}$



21. **(a)** $y^2 = 4px$, $p = 3$, $y^2 = 12x$

(b) $y^2 = -4px$, $p = 7$, $y^2 = -28x$

22. **(a)** $x^2 = -4py$, $p = 4$, $x^2 = -16y$

(b) $x^2 = -4py$, $p = 1/2$, $x^2 = -2y$

23. **(a)** $x^2 = -4py$, $p = 3$, $x^2 = -12y$

(b) The vertex is 3 units above the directrix so $p = 3$, $(x-1)^2 = 12(y-1)$.

24. **(a)** $y^2 = 4px$, $p = 6$, $y^2 = 24x$

(b) The vertex is half way between the focus and directrix so the vertex is at $(2, 4)$, the focus is 3 units to the left of the vertex so $p = 3$, $(y-4)^2 = -12(x-2)$

25. $y^2 = a(x-h)$, $4 = a(3-h)$ and $9 = a(2-h)$, solve simultaneously to get $h = 19/5$, $a = -5$ so $y^2 = -5(x - 19/5)$

26. $(x-5)^2 = a(y+3)$, $(9-5)^2 = a(5+3)$ so $a = 2$, $(x-5)^2 = 2(y+3)$

27. **(a)** $x^2/9 + y^2/4 = 1$

(b) $a = 26/2 = 13$, $c = 5$, $b^2 = a^2 - c^2 = 169 - 25 = 144$; $x^2/169 + y^2/144 = 1$

28. **(a)** $x^2 + y^2/5 = 1$

(b) $b = 8$, $c = 6$, $a^2 = b^2 + c^2 = 64 + 36 = 100$; $x^2/64 + y^2/100 = 1$

29. **(a)** $c = 1$, $a^2 = b^2 + c^2 = 2 + 1 = 3$; $x^2/3 + y^2/2 = 1$

(b) $b^2 = 16 - 12 = 4$; $x^2/16 + y^2/4 = 1$ and $x^2/4 + y^2/16 = 1$

30. **(a)** $c = 3$, $b^2 = a^2 - c^2 = 16 - 9 = 7$; $x^2/16 + y^2/7 = 1$

(b) $a^2 = 9 + 16 = 25$; $x^2/25 + y^2/9 = 1$ and $x^2/9 + y^2/25 = 1$

31. **(a)** $a = 6$, $(2, 3)$ satisfies $x^2/36 + y^2/b^2 = 1$ so $4/36 + 9/b^2 = 1$, $b^2 = 81/8$; $x^2/36 + y^2/(81/8) = 1$

(b) The center is midway between the foci so it is at $(1, 3)$, thus $c = 1$, $b = 1$, $a^2 = 1 + 1 = 2$; $(x - 1)^2 + (y - 3)^2/2 = 1$

32. **(a)** Substitute $(3, 2)$ and $(1, 6)$ into $x^2/A + y^2/B = 1$ to get $9/A + 4/B = 1$ and $1/A + 36/B = 1$ which yields $A = 10$, $B = 40$; $x^2/10 + y^2/40 = 1$

(b) The center is at $(2, -1)$ thus $c = 2$, $a = 3$, $b^2 = 9 - 4 = 5$; $(x - 2)^2/5 + (y + 1)^2/9 = 1$

33. **(a)** $a = 2$, $c = 3$, $b^2 = 9 - 4 = 5$; $x^2/4 - y^2/5 = 1$

(b) $a = 1$, $b/a = 2$, $b = 2$; $x^2 - y^2/4 = 1$

34. **(a)** $a = 3$, $c = 5$, $b^2 = 25 - 9 = 16$; $y^2/9 - x^2/16 = 1$

(b) $a = 3$, $a/b = 1$, $b = 3$; $y^2/9 - x^2/9 = 1$

35. **(a)** vertices along x-axis: $b/a = 3/2$ so $a = 8/3$; $x^2/(64/9) - y^2/16 = 1$
vertices along y-axis: $a/b = 3/2$ so $a = 6$; $y^2/36 - x^2/16 = 1$

(b) $c = 5$, $a/b = 2$ and $a^2 + b^2 = 25$, solve to get $a^2 = 20$, $b^2 = 5$; $y^2/20 - x^2/5 = 1$

36. **(a)** foci along the x-axis: $b/a = 3/4$ and $a^2 + b^2 = 25$, solve to get $a^2 = 16$, $b^2 = 9$; $x^2/16 - y^2/9 = 1$ foci along the y-axis: $a/b = 3/4$ and $a^2 + b^2 = 25$ which results in $y^2/9 - x^2/16 = 1$

(b) $c = 3$, $b/a = 2$ and $a^2 + b^2 = 9$ so $a^2 = 9/5$, $b^2 = 36/5$; $x^2/(9/5) - y^2/(36/5) = 1$

37. **(a)** the center is at $(6, 4)$, $a = 4$, $c = 5$, $b^2 = 25 - 16 = 9$; $(x - 6)^2/16 - (y - 4)^2/9 = 1$

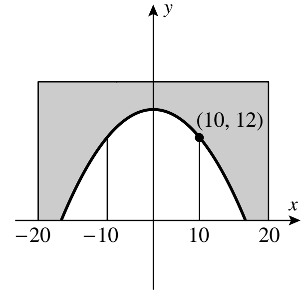
(b) The asymptotes intersect at $(1/2, 2)$ which is the center, $(y - 2)^2/a^2 - (x - 1/2)^2/b^2 = 1$ is the form of the equation because $(0, 0)$ is below both asymptotes, $4/a^2 - (1/4)/b^2 = 1$ and $a/b = 2$ which yields $a^2 = 3$, $b^2 = 3/4$; $(y - 2)^2/3 - (x - 1/2)^2/(3/4) = 1$.

38. **(a)** the center is at $(1, -2)$; $a = 2$, $c = 10$, $b^2 = 100 - 4 = 96$; $(y + 2)^2/4 - (x - 1)^2/96 = 1$

(b) the center is at $(1, -1)$; $2a = 5 - (-3) = 8, a = 4$, $\dfrac{(x-1)^2}{16} - \dfrac{(y+1)^2}{16} = 1$

39. **(a)** $y = ax^2 + b$, $(20, 0)$ and $(10, 12)$ are on the curve so $400a + b = 0$ and $100a + b = 12$. Solve for b to get $b = 16$ ft = height of arch.

(b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $\quad 400 = a^2, a = 20$; $\dfrac{100}{400} + \dfrac{144}{b^2} = 1$,
$b = 8\sqrt{3}$ ft = height of arch.



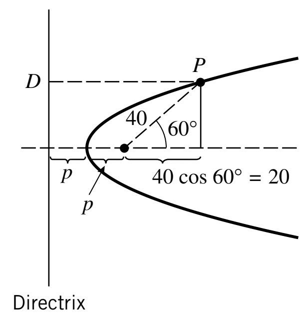
40. **(a)** $(x - b/2)^2 = a(y - h)$, but $(0, 0)$ is on the parabola so $b^2/4 = -ah$, $a = -\dfrac{b^2}{4h}$,

$(x - b/2)^2 = -\dfrac{b^2}{4h}(y - h)$

(b) As in Part (a), $y = -\dfrac{4h}{b^2}(x - b/2)^2 + h$, $A = \displaystyle\int_0^b \left[-\frac{4h}{b^2}(x - b/2)^2 + h\right] dx = \frac{2}{3}bh$

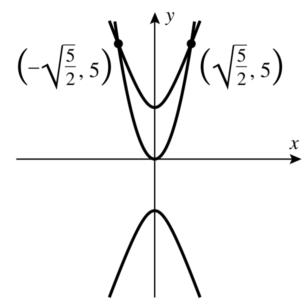
41. We may assume that the vertex is $(0, 0)$ and the parabola opens to the right. Let $P(x_0, y_0)$ be a point on the parabola $y^2 = 4px$, then by the definition of a parabola, PF = distance from P to directrix $x = -p$, so $PF = x_0 + p$ where $x_0 \geq 0$ and PF is a minimum when $x_0 = 0$ (the vertex).

42. Let p = distance (in millions of miles) between the vertex (closest point) and the focus F, then $PD = PF$, $2p + 20 = 40$, $p = 10$ million miles.



43. Use an xy-coordinate system so that $y^2 = 4px$ is an equation of the parabola, then $(1, 1/2)$ is a point on the curve so $(1/2)^2 = 4p(1)$, $p = 1/16$. The light source should be placed at the focus which is 1/16 ft. from the vertex.

44. **(a)** Substitute $x^2 = y/2$ into $y^2 - 8x^2 = 5$ to get $y^2 - 4y - 5 = 0$; $y = -1, 5$. Use $x^2 = y/2$ to find that there is no solution if $y = -1$ and that $x = \pm\sqrt{5/2}$ if $y = 5$. The curves intersect at $(\sqrt{5/2}, 5)$ and $(-\sqrt{5/2}, 5)$, and thus the area is

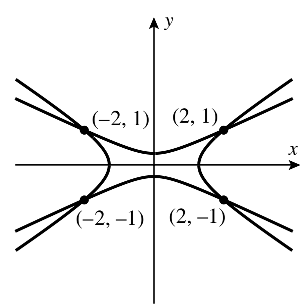


$$A = 2\int_0^{\sqrt{5/2}} (\sqrt{5 + 8x^2} - 2x^2)\, dx$$

$$= \left[x\sqrt{5 + 8x^2} + (5/4)\sqrt{2}\sinh^{-1}(2/5)\sqrt{10}x) - (4/3)x^3\right]_0^{5/2}$$

$$= \frac{5\sqrt{10}}{6} + \frac{5\sqrt{2}}{4}\ln(2 + \sqrt{5})$$

(b) Eliminate x to get $y^2 = 1$, $y = \pm 1$. Use either equation to find that $x = \pm 2$ if $y = 1$ or if $y = -1$. The curves intersect at $(2, 1)$, $(2, -1)$, $(-2, 1)$, and $(-2, -1)$, and thus the area is



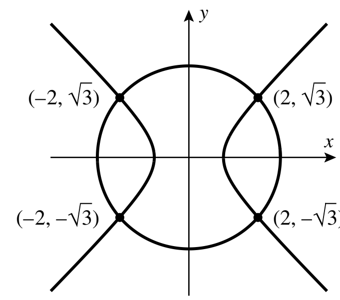
$$A = 4\int_0^{\sqrt{5/3}} \frac{1}{3}\sqrt{1 + 2x^2}\, dx$$

$$+ 4\int_{\sqrt{5/3}}^{2} \left[\frac{1}{3}\sqrt{1 + 2x^2} - \frac{1}{\sqrt{7}}\sqrt{3x^2 - 5}\right] dx$$

$$= \frac{1}{3}\sqrt{2}\ln(2\sqrt{2} + 3) + \frac{10}{21}\sqrt{21}\ln(2\sqrt{3} + \sqrt{7}) - \frac{5}{21}\ln 5$$

(c) Add both equations to get $x^2 = 4$, $x = \pm 2$.
Use either equation to find that $y = \pm\sqrt{3}$ if $x = 2$ or if $x = -2$. The curves intersect at $(2, \sqrt{3})$, $(2, -\sqrt{3})$, $(-2, \sqrt{3})$, $(-2, -\sqrt{3})$ and thus

$$A = 4\int_0^1 \sqrt{7 - x^2}\, dx + 4\int_1^2 \left[\sqrt{7 - x^2} - \sqrt{x^2 - 1}\right] dx$$
$$= 14 \sin^{-1}\left(\frac{2}{7}\sqrt{7}\right) + 2\ln(2 + \sqrt{3})$$



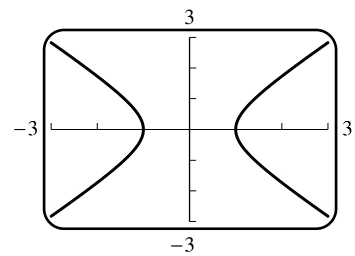
45. **(a)** $P : (b\cos t, b\sin t)$; $Q : (a\cos t, a\sin t)$; $R : (a\cos t, b\sin t)$

(b) For a circle, t measures the angle between the positive x-axis and the line segment joining the origin to the point. For an ellipse, t measures the angle between the x-axis and OPQ, not OR.

46. **(a)** For any point (x, y), the equation $y = b\sinh t$ has a unique solution t, $-\infty < t < +\infty$. On the hyperbola,

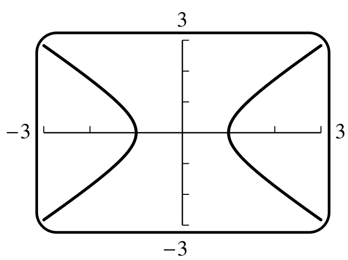
$$\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} = 1 + \sinh^2 t$$
$$= \cosh^2 t, \text{ so } x = \pm a\cosh t.$$

(b)



47. **(a)** For any point (x, y), the equation $y = b\tan t$ has a unique solution t where $-\pi/2 < t < \pi/2$. On the hyperbola, $\dfrac{x^2}{a^2} = 1 + \dfrac{y^2}{b^2} = 1 + \tan^2 t = \sec^2 t$, so $x = \pm a\sec t$.

(b)



48. By Definition 11.4.1, $(x + 1)^2 + (y - 4)^2 = (y - 1)^2, (x + 1)^2 = 6y - 15,\ \ (x + 1)^2 = 6(y - 5/2)$

49. $(4, 1)$ and $(4, 5)$ are the foci so the center is at $(4, 3)$ thus $c = 2$, $a = 12/2 = 6$, $b^2 = 36 - 4 = 32$; $(x - 4)^2/32 + (y - 3)^2/36 = 1$

50. From the definition of a hyperbola, $\left|\sqrt{(x - 1)^2 + (y - 1)^2} - \sqrt{x^2 + y^2}\right| = 1$, $\sqrt{(x - 1)^2 + (y - 1)^2} - \sqrt{x^2 + y^2} = \pm 1$, transpose the second radical to the right hand side of the equation and square and simplify to get $\pm 2\sqrt{x^2 + y^2} = -2x - 2y + 1$, square and simplify again to get $8xy - 4x - 4y + 1 = 0$.

51. Let the ellipse have equation $\dfrac{4}{81}x^2 + \dfrac{y^2}{4} = 1$, then $A(x) = (2y)^2 = 16\left(1 - \dfrac{4x^2}{81}\right)$,

$$V = 2\int_0^{9/2} 16\left(1 - \frac{4x^2}{81}\right) dx = 96$$

52. See Exercise 51, $A(y) = \sqrt{3}x^2 = \sqrt{3}\dfrac{81}{4}\left(1 - \dfrac{y^2}{4}\right), V = \sqrt{3}\dfrac{81}{2}\displaystyle\int_0^2 \left(1 - \dfrac{y^2}{4}\right) dy = 54\sqrt{3}$

53. Assume $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, A = 4\displaystyle\int_0^a b\sqrt{1 - x^2/a^2}\, dx = \pi ab$

54. **(a)** Assume $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, V = 2\displaystyle\int_0^a \pi b^2\left(1 - x^2/a^2\right) dx = \dfrac{4}{3}\pi ab^2$

(b) In Part (a) interchange a and b to obtain the result.

55. Assume $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, \dfrac{dy}{dx} = -\dfrac{bx}{a\sqrt{a^2 - x^2}}, 1 + \left(\dfrac{dy}{dx}\right)^2 = \dfrac{a^4 - (a^2 - b^2)x^2}{a^2(a^2 - x^2)},$

$$S = 2\int_0^a \frac{2\pi b}{a}\sqrt{1 - x^2/a^2}\sqrt{\frac{a^4 - (a^2 - b^2)x^2}{a^2 - x^2}}\, dx = 2\pi ab\left(\frac{b}{a} + \frac{a}{c}\sin^{-1}\frac{c}{a}\right), c = \sqrt{a^2 - b^2}$$

56. As in Exercise 55, $1 + \left(\dfrac{dx}{dy}\right)^2 = \dfrac{b^4 + (a^2 - b^2)y^2}{b^2(b^2 - y^2)},$

$$S = 2\int_0^b 2\pi a\sqrt{1 - y^2/b^2}\sqrt{\frac{b^4 + (a^2 - b^2)y^2}{b^2(b^2 - y^2)}}\, dy = 2\pi ab\left(\frac{a}{b} + \frac{b}{c}\ln\frac{a + c}{b}\right), c = \sqrt{a^2 - b^2}$$

57. Open the compass to the length of half the major axis, place the point of the compass at an end of the minor axis and draw arcs that cross the major axis to both sides of the center of the ellipse. Place the tacks where the arcs intersect the major axis.

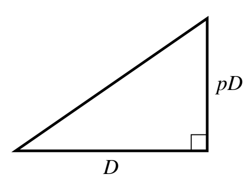
58. Let P denote the pencil tip, and let $R(x, 0)$ be the point below Q and P which lies on the line L. Then $QP + PF$ is the length of the string and $QR = QP + PR$ is the length of the side of the triangle. These two are equal, so $PF = PR$. But this is the definition of a parabola according to Definition 11.4.1.

59. Let P denote the pencil tip, and let k be the difference between the length of the ruler and that of the string. Then $QP + PF_2 + k = QF_1$, and hence $PF_2 + k = PF_1, PF_1 - PF_2 = k$. But this is the definition of a hyperbola according to Definition 11.4.3.

60. In the $x'y'$-plane an equation of the circle is $x'^2 + y'^2 = r^2$ where r is the radius of the cylinder. Let $P(x, y)$ be a point on the curve in the xy-plane, then $x' = x\cos\theta$ and $y' = y$ so $x^2\cos^2\theta + y^2 = r^2$ which is an equation of an ellipse in the xy-plane.

61. $L = 2a = \sqrt{D^2 + p^2D^2} = D\sqrt{1 + p^2}$ (see figure), so $a = \dfrac{1}{2}D\sqrt{1 + p^2}$, but $b = \dfrac{1}{2}D$,

$$T = c = \sqrt{a^2 - b^2} = \sqrt{\frac{1}{4}D^2(1 + p^2) - \frac{1}{4}D^2} = \frac{1}{2}pD.$$



62. $y = \dfrac{1}{4p}x^2$, $dy/dx = \dfrac{1}{2p}x$, $dy/dx\Big|_{x=x_0} = \dfrac{1}{2p}x_0$, the tangent line at (x_0, y_0) has the formula

$y - y_0 = \dfrac{x_0}{2p}(x - x_0) = \dfrac{x_0}{2p}x - \dfrac{x_0^2}{2p}$, but $\dfrac{x_0^2}{2p} = 2y_0$ because (x_0, y_0) is on the parabola $y = \dfrac{1}{4p}x^2$.

Thus the tangent line is $y - y_0 = \dfrac{x_0}{2p}x - 2y_0$, $y = \dfrac{x_0}{2p}x - y_0$.

63. By implicit differentiation, $\left.\dfrac{dy}{dx}\right|_{(x_0,y_0)} = -\dfrac{b^2}{a^2}\dfrac{x_0}{y_0}$ if $y_0 \neq 0$, the tangent line is

$y - y_0 = -\dfrac{b^2}{a^2}\dfrac{x_0}{y_0}(x - x_0)$, $a^2y_0y - a^2y_0^2 = -b^2x_0x + b^2x_0^2$, $b^2x_0x + a^2y_0y = b^2x_0^2 + a^2y_0^2$, but (x_0, y_0) is on the ellipse so $b^2x_0^2 + a^2y_0^2 = a^2b^2$; thus the tangent line is $b^2x_0x + a^2y_0y = a^2b^2$, $x_0x/a^2 + y_0y/b^2 = 1$. If $y_0 = 0$ then $x_0 = \pm a$ and the tangent lines are $x = \pm a$ which also follows from $x_0x/a^2 + y_0y/b^2 = 1$.

64. By implicit differentiation, $\left.\dfrac{dy}{dx}\right|_{(x_0,y_0)} = \dfrac{b^2}{a^2}\dfrac{x_0}{y_0}$ if $y_0 \neq 0$, the tangent line is $y - y_0 = \dfrac{b^2}{a^2}\dfrac{x_0}{y_0}(x - x_0)$, $b^2x_0x - a^2y_0y = b^2x_0^2 - a^2y_0^2 = a^2b^2$, $x_0x/a^2 - y_0y/b^2 = 1$. If $y_0 = 0$ then $x_0 = \pm a$ and the tangent lines are $x = \pm a$ which also follow from $x_0x/a^2 - y_0y/b^2 = 1$.

65. Use $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and $\dfrac{x^2}{A^2} - \dfrac{y^2}{B^2} = 1$ as the equations of the ellipse and hyperbola. If (x_0, y_0) is a point of intersection then $\dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} = 1 = \dfrac{x_0^2}{A^2} - \dfrac{y_0^2}{B^2}$, so $x_0^2\left(\dfrac{1}{A^2} - \dfrac{1}{a^2}\right) = y_0^2\left(\dfrac{1}{B^2} + \dfrac{1}{b^2}\right)$ and $a^2A^2y_0^2(b^2 + B^2) = b^2B^2x_0^2(a^2 - A^2)$. Since the conics have the same foci, $a^2 - b^2 = c^2 = A^2 + B^2$, so $a^2 - A^2 = b^2 + B^2$. Hence $a^2A^2y_0^2 = b^2B^2x_0^2$. From Exercises 63 and 64, the slopes of the tangent lines are $-\dfrac{b^2x_0}{a^2y_0}$ and $\dfrac{B^2x_0}{A^2y_0}$, whose product is $-\dfrac{b^2B^2x_0^2}{a^2A^2y_0^2} = -1$. Hence the tangent lines are perpendicular.

66. Use implicit differentiation on $x^2 + 4y^2 = 8$ to get $\left.\dfrac{dy}{dx}\right|_{(x_0,y_0)} = -\dfrac{x_0}{4y_0}$ where (x_0, y_0) is the point of tangency, but $-x_0/(4y_0) = -1/2$ because the slope of the line is $-1/2$ so $x_0 = 2y_0$. (x_0, y_0) is on the ellipse so $x_0^2 + 4y_0^2 = 8$ which when solved with $x_0 = 2y_0$ yields the points of tangency $(2, 1)$ and $(-2, -1)$. Substitute these into the equation of the line to get $k = \pm 4$.

67. Let (x_0, y_0) be such a point. The foci are at $(-\sqrt{5}, 0)$ and $(\sqrt{5}, 0)$, the lines are perpendicular if the product of their slopes is -1 so $\dfrac{y_0}{x_0 + \sqrt{5}} \cdot \dfrac{y_0}{x_0 - \sqrt{5}} = -1$, $y_0^2 = 5 - x_0^2$ and $4x_0^2 - y_0^2 = 4$. Solve to get $x_0 = \pm 3/\sqrt{5}$, $y_0 = \pm 4/\sqrt{5}$. The coordinates are $(\pm 3/\sqrt{5}, 4/\sqrt{5})$, $(\pm 3/\sqrt{5}, -4/\sqrt{5})$.

68. Let (x_0, y_0) be one of the points; then $\left.dy/dx\right|_{(x_0,y_0)} = 4x_0/y_0$, the tangent line is $y = (4x_0/y_0)x + 4$, but (x_0, y_0) is on both the line and the curve which leads to $4x_0^2 - y_0^2 + 4y_0 = 0$ and $4x_0^2 - y_0^2 = 36$, solve to get $x_0 = \pm 3\sqrt{13}/2$, $y_0 = -9$.

69. Let d_1 and d_2 be the distances of the first and second observers, respectively, from the point of the explosion. Then $t =$ (time for sound to reach the second observer) $-$ (time for sound to reach the first observer) $= d_2/v - d_1/v$ so $d_2 - d_1 = vt$. For constant v and t the difference of distances, d_2 and d_1 is constant so the explosion occurred somewhere on a branch of a hyperbola whose foci are where the observers are. Since $d_2 - d_1 = 2a$, $a = \dfrac{vt}{2}$, $b^2 = c^2 - \dfrac{v^2t^2}{4}$, and $\dfrac{x^2}{v^2t^2/4} - \dfrac{y^2}{c^2 - (v^2t^2/4)} = 1$.

70. As in Exercise 69, $d_2 - d_1 = 2a = vt = (299{,}792{,}458 \text{ m/s})(10^{-7} \text{ s}) \approx 29.9792$ m.
$a^2 = (vt/2)^2 \approx 449.3762 \text{ m}^2$; $c^2 = (50)^2 = 2500 \text{ m}^2$
$b^2 = c^2 - a^2 = 2050.6238$, $\dfrac{x^2}{449.3762} - \dfrac{y^2}{2050.6238} = 1$
But $y = 200$ km $= 200{,}000$ m, so $x \approx 93{,}625.05$ m $= 93.62505$ km. The ship is located at (93.62505,200).

71. **(a)** Use $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$, $x=\dfrac{3}{2}\sqrt{4-y^2}$,

$$V=\int_{-2}^{-2+h}(2)(3/2)\sqrt{4-y^2}(18)dy=54\int_{-2}^{-2+h}\sqrt{4-y^2}\,dy$$

$$=54\left[\frac{y}{2}\sqrt{4-y^2}+2\sin^{-1}\frac{y}{2}\right]_{-2}^{-2+h}=27\left[4\sin^{-1}\frac{h-2}{2}+(h-2)\sqrt{4h-h^2}+2\pi\right]\ \text{ft}^3$$

(b) When $h=4$ ft, $V_{\text{full}}=108\sin^{-1}1+54\pi=108\pi$ ft^3, so solve for h when $V=(k/4)V_{\text{full}}$, $k=1,2,3$, to get $h=1.19205, 2, 2.80795$ ft or $14.30465, 24, 33.69535$ in.

72. We may assume $A>0$, since if $A<0$ then one can multiply the equation by -1, and if $A=0$ then one can exchange A with C (C cannot be zero simultaneously with A). Then

$$Ax^2+Cy^2+Dx+Ey+F=A\left(x+\frac{D}{2A}\right)^2+C\left(y+\frac{E}{2C}\right)^2+F-\frac{D^2}{4A}-\frac{E^2}{4C}=0.$$

(a) Let $AC>0$. If $F<\dfrac{D^2}{4A}+\dfrac{E^2}{4C}$ the equation represents an ellipse (a circle if $A=C$); if $F=\dfrac{D^2}{4A}+\dfrac{E^2}{4C}$, the point $x=-D/(2A), y=-E/(2C)$; and if $F>\dfrac{D^2}{4A}+\dfrac{E^2}{4C}$ then there is no graph.

(b) If $AC<0$ and $F=\dfrac{D^2}{4A}+\dfrac{E^2}{4C}$, then

$$\left[\sqrt{A}\left(x+\frac{D}{2A}\right)+\sqrt{-C}\left(y+\frac{E}{2C}\right)\right]\left[\sqrt{A}\left(x+\frac{D}{2A}\right)-\sqrt{-C}\left(y+\frac{E}{2C}\right)\right]=0,$$

a pair of lines; otherwise a hyperbola

(c) Assume $C=0$, so $Ax^2+Dx+Ey+F=0$. If $E\neq 0$, parabola; if $E=0$ then $Ax^2+Dx+F=0$. If this polynomial has roots $x=x_1,x_2$ with $x_1\neq x_2$ then a pair of parallel lines; if $x_1=x_2$ then one line; if no roots, then no graph. If $A=0, C\neq 0$ then a similar argument applies.

73. **(a)** $(x-1)^2-5(y+1)^2=5$, hyperbola

(b) $x^2-3(y+1)^2=0, x=\pm\sqrt{3}(y+1)$, two lines

(c) $4(x+2)^2+8(y+1)^2=4$, ellipse

(d) $3(x+2)^2+(y+1)^2=0$, the point $(-2,-1)$ (degenerate case)

(e) $(x+4)^2+2y=2$, parabola

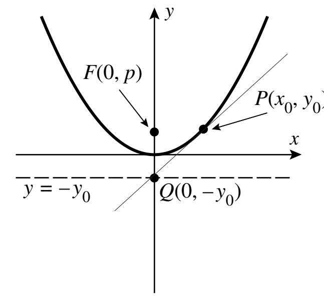
(f) $5(x+4)^2+2y=-14$, parabola

74. distance from the point (x,y) to the focus $(0,p)$ = distance to the directrix $y=-p$, so $x^2+(y-p)^2 = (y+p)^2, x^2=4py$

75. distance from the point (x,y) to the focus $(0,-c)$ plus distance to the focus $(0,c)$ = const = $2a$, $\sqrt{x^2+(y+c)^2}+\sqrt{x^2+(y-c)^2}=2a, x^2+(y+c)^2=4a^2+x^2+(y-c)^2-4a\sqrt{x^2+(y-c)^2}$, $\sqrt{x^2+(y-c)^2}=a-\dfrac{c}{a}y$, and since $a^2-c^2=b^2$, $\dfrac{x^2}{b^2}+\dfrac{y^2}{a^2}=1$

76. distance from the point (x,y) to the focus $(-c,0)$ less distance to the focus $(c,0)$ is equal to $2a$, $\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=\pm 2a, (x+c)^2+y^2=(x-c)^2+y^2+4a^2\pm 4a\sqrt{(x-c)^2+y^2}$, $\sqrt{(x-c)^2+y^2}=\pm\left(\dfrac{cx}{a}-a\right)$, and, since $c^2-a^2=b^2$, $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

77. Assume the equation of the parabola is $x^2 = 4py$. The tangent line at $P(x_0, y_0)$ (see figure) is given by $(y - y_0)/(x - x_0) = m = x_0/2p$. To find the y-intercept set $x = 0$ and obtain $y = -y_0$. Thus $Q : (0, -y_0)$. The focus is $(0, p) = (0, x_0^2/4y_0)$, the distance from P to the focus is $\sqrt{x_0^2 + (y_0 - p)^2} = \sqrt{4py_0 + (y_0 - p)^2} = \sqrt{(y_0 + p)^2} = y_0 + p$, and the distance from the focus to the y-intercept is $p + y_0$, so triangle FPQ is isosceles, and angles FPQ and FQP are equal.



78. **(a)** $\tan\theta = \tan(\phi_2 - \phi_1) = \dfrac{\tan\phi_2 - \tan\phi_1}{1 + \tan\phi_2\tan\phi_1} = \dfrac{m_2 - m_1}{1 + m_1 m_2}$

(b) By implicit differentiation, $m = dy/dx\Big|_{P(x_0,y_0)} = -\dfrac{b^2}{a^2}\dfrac{x_0}{y_0}$ if $y_0 \neq 0$. Let m_1 and m_2 be the slopes of the lines through P and the foci at $(-c, 0)$ and $(c, 0)$ respectively, then $m_1 = y_0/(x_0 + c)$ and $m_2 = y_0/(x_0 - c)$. For P in the first quadrant,

$$\tan\alpha = \frac{m - m_2}{1 + mm_2} = \frac{-(b^2x_0)/(a^2y_0) - y_0/(x_0 - c)}{1 - (b^2x_0)/[a^2(x_0 - c)]}$$

$$= \frac{-b^2x_0^2 - a^2y_0^2 + b^2cx_0}{[(a^2 - b^2)x_0 - a^2c]\,y_0} = \frac{-a^2b^2 + b^2cx_0}{(c^2x_0 - a^2c)y_0} = \frac{b^2}{cy_0}$$

similarly $\tan(\pi - \beta) = \dfrac{m - m_1}{1 + mm_1} = -\dfrac{b^2}{cy_0} = -\tan\beta$ so $\tan\alpha = \tan\beta$, $\alpha = \beta$. The proof for the case $y_0 = 0$ follows trivially. By symmetry, the result holds for P in the other three quadrants as well.

(c) Let $P(x_0, y_0)$ be in the third quadrant. Suppose $y_0 \neq 0$ and let $m =$ slope of the tangent line at P, $m_1 =$ slope of the line through P and $(-c, 0)$, $m_2 =$ slope of the line through P and $(c, 0)$ then $m = \dfrac{dy}{dx}\Big|_{(x_0,y_0)} = (b^2x_0)/(a^2y_0)$, $m_1 = y_0/(x_0 + c)$, $m_2 = y_0/(x_0 - c)$. Use $\tan\alpha = (m_1 - m)/(1 + m_1m)$ and $\tan\beta = (m - m_2)/(1 + mm_2)$ to get $\tan\alpha = \tan\beta = -b^2/(cy_0)$ so $\alpha = \beta$. If $y_0 = 0$ the result follows trivially and by symmetry the result holds for P in the other three quadrants as well.

EXERCISE SET 11.5

1. **(a)** $\sin\theta = \sqrt{3}/2$, $\cos\theta = 1/2$

$x' = (-2)(1/2) + (6)(\sqrt{3}/2) = -1 + 3\sqrt{3}$, $y' = -(-2)(\sqrt{3}/2) + 6(1/2) = 3 + \sqrt{3}$

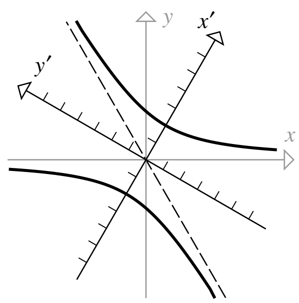
(b) $x = \dfrac{1}{2}x' - \dfrac{\sqrt{3}}{2}y' = \dfrac{1}{2}(x' - \sqrt{3}y')$, $y = \dfrac{\sqrt{3}}{2}x' + \dfrac{1}{2}y' = \dfrac{1}{2}(\sqrt{3}x' + y')$

$$\sqrt{3}\left[\frac{1}{2}(x' - \sqrt{3}y')\right]\left[\frac{1}{2}(\sqrt{3}x' + y')\right] + \left[\frac{1}{2}(\sqrt{3}x' + y')\right]^2 = 6$$

$$\frac{\sqrt{3}}{4}(\sqrt{3}x'^2 - 2x'y' - \sqrt{3}y'^2) + \frac{1}{4}(3x'^2 + 2\sqrt{3}x'y' + y'^2) = 6$$

$$\frac{3}{2}x'^2 - \frac{1}{2}y'^2 = 6,\ 3x'^2 - y'^2 = 12$$

(c)



2. **(a)** $\sin\theta = 1/2$, $\cos\theta = \sqrt{3}/2$

$x' = (1)(\sqrt{3}/2) + (-\sqrt{3})(1/2) = 0$, $y' = -(1)(1/2) + (-\sqrt{3})(\sqrt{3}/2) = -2$

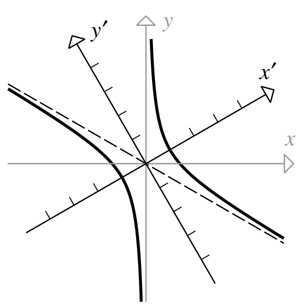
(b) $x = \dfrac{\sqrt{3}}{2}x' - \dfrac{1}{2}y' = \dfrac{1}{2}(\sqrt{3}x' - y')$, $y = \dfrac{1}{2}x' + \dfrac{\sqrt{3}}{2}y' = \dfrac{1}{2}(x' + \sqrt{3}y')$

$$2\left[\frac{1}{2}(\sqrt{3}x' - y')\right]^2 + 2\sqrt{3}\left[\frac{1}{2}(\sqrt{3}x' - y')\right]\left[\frac{1}{2}(x' + \sqrt{3}y')\right] = 3$$

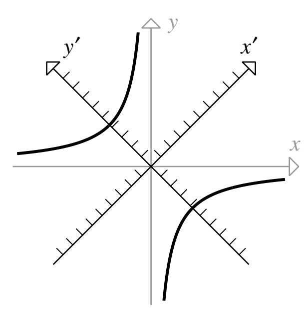
$$\frac{1}{2}(3x'^2 - 2\sqrt{3}x'y' + y'^2) + \frac{\sqrt{3}}{2}(\sqrt{3}x'^2 + 2x'y' - \sqrt{3}y'^2) = 3$$

$3x'^2 - y'^2 = 3$, $x'^2/1 - y'^2/3 = 1$

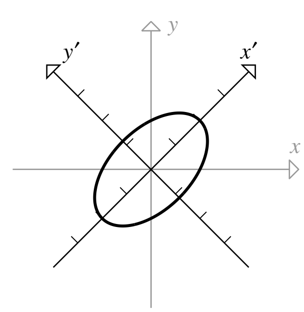
(c)



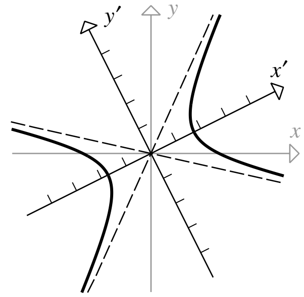
3. $\cot 2\theta = (0 - 0)/1 = 0$, $2\theta = 90°$, $\theta = 45°$
$x = (\sqrt{2}/2)(x' - y')$, $y = (\sqrt{2}/2)(x' + y')$
$y'^2/18 - x'^2/18 = 1$, hyperbola



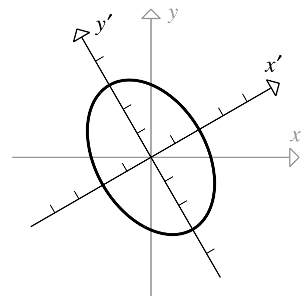
4. $\cot 2\theta = (1 - 1)/(-1) = 0$, $\theta = 45°$
$x = (\sqrt{2}/2)(x' - y')$, $y = (\sqrt{2}/2)(x' + y')$
$x'^2/4 + y'^2/(4/3) = 1$, ellipse



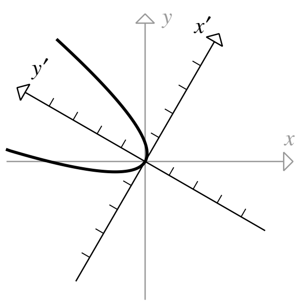
5. $\cot 2\theta = [1-(-2)]/4 = 3/4$
$\cos 2\theta = 3/5$
$\sin\theta = \sqrt{(1-3/5)/2} = 1/\sqrt{5}$
$\cos\theta = \sqrt{(1+3/5)/2} = 2/\sqrt{5}$
$x = (1/\sqrt{5})(2x' - y')$
$y = (1/\sqrt{5})(x' + 2y')$
$x'^2/3 - y'^2/2 = 1$, hyperbola



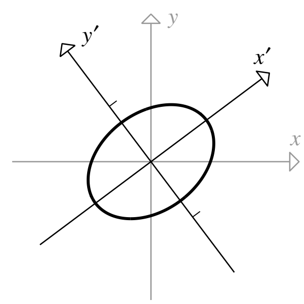
6. $\cot 2\theta = (31-21)/(10\sqrt{3}) = 1/\sqrt{3}$,
$2\theta = 60°, \theta = 30°$
$x = (1/2)(\sqrt{3}x' - y')$,
$y = (1/2)(x' + \sqrt{3}y')$
$x'^2/4 + y'^2/9 = 1$, ellipse



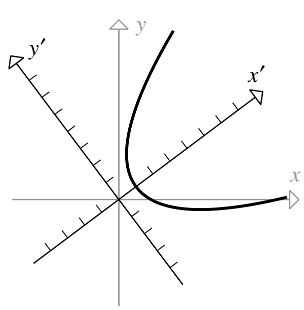
7. $\cot 2\theta = (1-3)/(2\sqrt{3}) = -1/\sqrt{3}$,
$2\theta = 120°, \theta = 60°$
$x = (1/2)(x' - \sqrt{3}y')$
$y = (1/2)(\sqrt{3}x' + y')$
$y' = x'^2$, parabola



8. $\cot 2\theta = (34-41)/(-24) = 7/24$
$\cos 2\theta = 7/25$
$\sin\theta = \sqrt{(1-7/25)/2} = 3/5$
$\cos\theta = \sqrt{(1+7/25)/2} = 4/5$
$x = (1/5)(4x' - 3y')$,
$y = (1/5)(3x' + 4y')$
$x'^2 + y'^2/(1/2) = 1$, ellipse



9. $\cot 2\theta = (9-16)/(-24) = 7/24$
$\cos 2\theta = 7/25$,
$\sin\theta = 3/5, \qquad \cos\theta = 4/5$
$x = (1/5)(4x' - 3y')$,
$y = (1/5)(3x' + 4y')$
$y'^2 = 4(x'-1)$, parabola



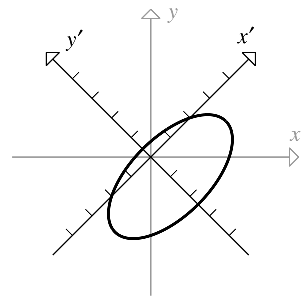
10. $\cot 2\theta = (5-5)/(-6) = 0,$

$\theta = 45°$

$x = (\sqrt{2}/2)(x' - y'),$

$y = (\sqrt{2}/2)(x' + y'),$

$x'^2/8 + (y'+1)^2/2 = 1$, ellipse



11. $\cot 2\theta = (52-73)/(-72) = 7/24$

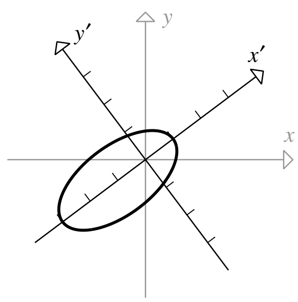
$\cos 2\theta = 7/25,\qquad \sin\theta = 3/5,$

$\cos\theta = 4/5$

$x = (1/5)(4x' - 3y'),$

$y = (1/5)(3x' + 4y')$

$(x'+1)^2/4 + y'^2 = 1$, ellipse



12. $\cot 2\theta = [6-(-1)]/24 = 7/24$

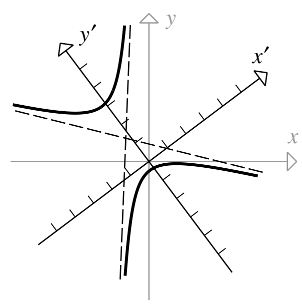
$\cos 2\theta = 7/25,\qquad \sin\theta = 3/5,$

$\cos\theta = 4/5$

$x = (1/5)(4x' - 3y'),$

$y = (1/5)(3x' + 4y')$

$(y' - 7/5)^2/3 - (x' + 1/5)^2/2 = 1$, hyperbola



13. Let $x = x'\cos\theta - y'\sin\theta$, $y = x'\sin\theta + y'\cos\theta$ then $x^2 + y^2 = r^2$ becomes $(\sin^2\theta + \cos^2\theta)x'^2 + (\sin^2\theta + \cos^2\theta)y'^2 = r^2$, $x'^2 + y'^2 = r^2$. Under a rotation transformation the center of the circle stays at the origin of both coordinate systems.

14. Multiply the first equation through by $\cos\theta$ and the second by $\sin\theta$ and add to get $x\cos\theta + y\sin\theta = (\cos^2\theta + \sin^2\theta)x' = x'$. Multiply the first by $-\sin\theta$ and the second by $\cos\theta$ and add to get y'.

15. $x' = (\sqrt{2}/2)(x+y)$, $y' = (\sqrt{2}/2)(-x+y)$ which when substituted into $3x'^2 + y'^2 = 6$ yields $x^2 + xy + y^2 = 3$.

16. From (5), $x = \dfrac{1}{2}(\sqrt{3}x' - y')$ and $y = \dfrac{1}{2}(x' + \sqrt{3}y')$ so $y = x^2$ becomes $\dfrac{1}{2}(x' + \sqrt{3}y') = \dfrac{1}{4}(\sqrt{3}x' - y')^2$; simplify to get $3x'^2 - 2\sqrt{3}x'y' + y'^2 - 2x' - 2\sqrt{3}y' = 0$.

17. $\sqrt{x} + \sqrt{y} = 1$, $\sqrt{x} = 1 - \sqrt{y}$, $x = 1 - 2\sqrt{y} + y$, $2\sqrt{y} = 1 - x + y$, $4y = 1 + x^2 + y^2 - 2x + 2y - 2xy$, $x^2 - 2xy + y^2 - 2x - 2y + 1 = 0$. $\cot 2\theta = \dfrac{1-1}{-2} = 0$, $2\theta = \pi/2$, $\theta = \pi/4$. Let $x = x'/\sqrt{2} - y'/\sqrt{2}$, $y = x'/\sqrt{2} + y'/\sqrt{2}$ to get $2y'^2 - 2\sqrt{2}x' + 1 = 0$, which is a parabola. From $\sqrt{x} + \sqrt{y} = 1$ we see that $0 \le x \le 1$ and $0 \le y \le 1$, so the graph is just a portion of a parabola.

18. Let $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$ in (7), expand and add all the coefficients of the terms that contain $x'y'$ to get B'.

19. Use (9) to express $B' - 4A'C'$ in terms of A, B, C, and θ, then simplify.

20. Use (9) to express $A' + C'$ in terms of A, B, C, and θ and then simplify.

21. $\cot 2\theta = (A - C)/B = 0$ if $A = C$ so $2\theta = 90°$, $\theta = 45°$.

22. If $F = 0$ then $x^2 + Bxy = 0$, $x(x + By) = 0$ so $x = 0$ or $x + By = 0$ which are lines that intersect at $(0, 0)$. Suppose $F \neq 0$, rotate through an angle θ where $\cot 2\theta = 1/B$ eliminating the cross product term to get $A'x'^2 + C'y'^2 + F' = 0$, and note that $F' = F$ so $F' \neq 0$. From (9),
$A' = \cos^2\theta + B\cos\theta\sin\theta = \cos\theta(\cos\theta + B\sin\theta)$ and
$C' = \sin^2\theta - B\sin\theta\cos\theta = \sin\theta(\sin\theta - B\cos\theta)$ so

$$\begin{aligned} A'C' &= \sin\theta\cos\theta[\sin\theta\cos\theta - B(\cos^2\theta - \sin^2\theta) - B^2\sin\theta\cos\theta] \\ &= \frac{1}{2}\sin 2\theta\left[\frac{1}{2}\sin 2\theta - B\cos 2\theta - \frac{1}{2}B^2\sin 2\theta\right] = \frac{1}{4}\sin^2 2\theta[1 - 2B\cot 2\theta - B^2] \\ &= \frac{1}{4}\sin^2 2\theta[1 - 2B(1/B) - B^2] = -\frac{1}{4}\sin^2 2\theta(1 + B^2) < 0 \end{aligned}$$

thus A' and C' have unlike signs so the graph is a hyperbola.

23. $B^2 - 4AC = (-1)^2 - 4(1)(1) = -3 < 0$; ellipse, point, or no graph. By inspection $(0, \pm\sqrt{2})$ lie on the curve, so it's an ellipse.

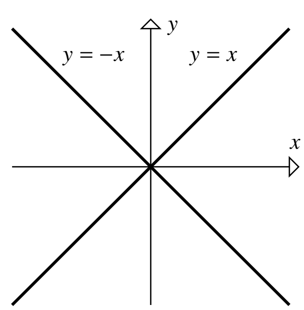
24. $B^2 - 4AC = (4)^2 - 4(1)(-2) = 24 > 0$; hyperbola or pair of intersecting lines

25. $B^2 - 4AC = (2\sqrt{3})^2 - 4(1)(3) = 0$; parabola, line, pair of parallel lines, or no graph. By inspection $(-\sqrt{3}, 3), (-\sqrt{3}, -1/3), (0, 0), (-2\sqrt{3}, 0), (0, 2/3)$ lie on the graph; since no three of these points are collinear, it's a parabola.

26. $B^2 - 4AC = (24)^2 - 4(6)(-1) = 600 > 0$; hyperbola or pair of intersecting lines

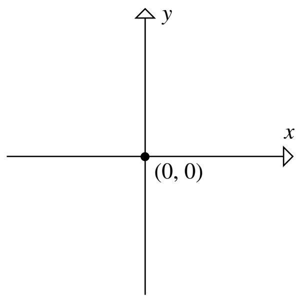
27. $B^2 - 4AC = (-24)^2 - 4(34)(41) = -5000 < 0$; ellipse, point, or no graph. By inspection $x = \pm 5/\sqrt{34}, y = 0$ satisfy the equation, so it's an ellipse.

28. **(a)** $(x - y)(x + y) = 0$
$y = x$ or $y = -x$
(two intersecting lines)

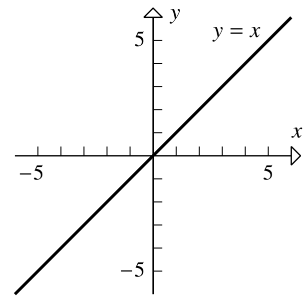


(b) $x^2 + 3y^2 = -7$ which has no real solutions, no graph

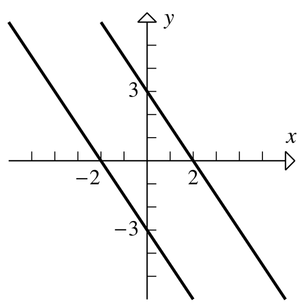
(c) $8x^2 + 7y^2 = 0$
$x = 0$ and $y = 0$, (a point)



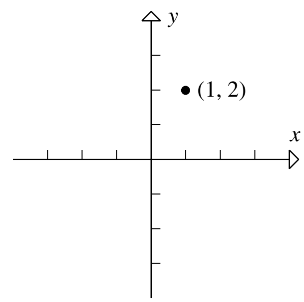
(d) $(x - y)^2 = 0$,
$y = x$ (a line)



(e) $(3x + 2y)^2 = 36$,
$3x + 2y = 6$ or $3x + 2y = -6$
(a pair of parallel lines)



(f) $(x - 1)^2 + (y - 2)^2 = 0$,
the point $(1, 2)$



29. Part (b): from (15), $A'C' < 0$ so A' and C' have opposite signs. By multiplying (14) through by -1, if necessary, assume that $A' < 0$ and $C' > 0$ so $(x' - h)^2/C' - (y' - k)^2/|A'| = K$. If $K \neq 0$ then the graph is a hyperbola (divide both sides by K), if $K = 0$ then we get the pair of intersecting lines $(x' - h)/\sqrt{C'} = \pm(y' - k)/\sqrt{|A'|}$.

Part (c): from (15), $A'C' = 0$ so either $A' = 0$ or $C' = 0$ but not both (this would imply that $A = B = C = 0$ which results in (14) being linear). Suppose $A' \neq 0$ and $C' = 0$ then complete the square to get $(x' - h)^2 = -E'y'/A' + K$. If $E' \neq 0$ the graph is a parabola, if $E' = 0$ and $K = 0$ the graph is the line $x' = h$, if $E' = 0$ and $K > 0$ the graph is the pair of parallel lines $x' = h \pm \sqrt{K}$, if $E' = 0$ and $K < 0$ there is no graph.

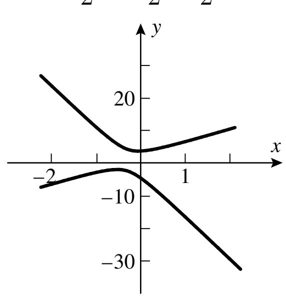
30. **(a)** $B^2 - 4AC = (1)^2 - 4(1)(2) < 0$ so it is an ellipse (it contains the points $x = 0, y = -1, -1/2$).

(b) $y = -\frac{1}{4}x - \frac{3}{4} - \frac{1}{4}\sqrt{1 + 14x - 7x^2}$ or $y = -\frac{1}{4}x - \frac{3}{4} + \frac{1}{4}\sqrt{1 + 14x - 7x^2}$

31. **(a)** $B^2 - 4AC = (9)^2 - 4(2)(1) > 0$ so the conic is a hyperbola (it contains the points $(2, -1), (2, -3/2)$).

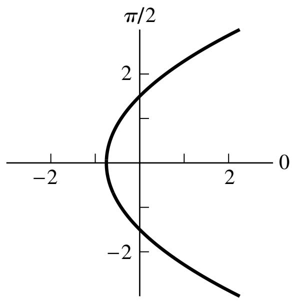
(b) $y = -\frac{9}{2}x - \frac{1}{2} - \frac{1}{2}\sqrt{73x^2 + 42x + 17}$ or $y = -\frac{9}{2}x - \frac{1}{2} + \frac{1}{2}\sqrt{73x^2 + 42x + 17}$

(c)

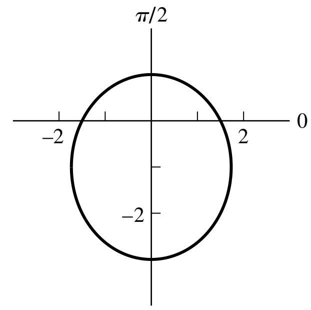


EXERCISE SET 11.6

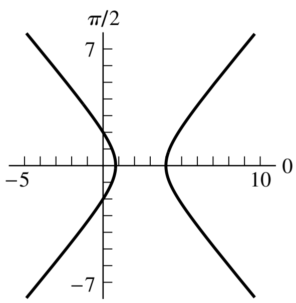
1. **(a)** $r = \dfrac{3/2}{1-\cos\theta}, e = 1, d = 3/2$



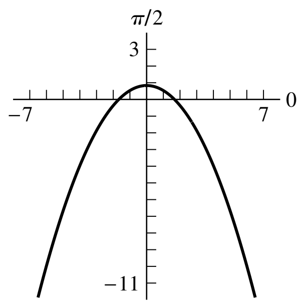
(b) $r = \dfrac{3/2}{1+\frac{1}{2}\sin\theta}, e = 1/2, d = 3$



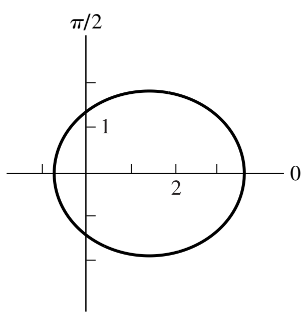
(c) $r = \dfrac{2}{1+\frac{3}{2}\cos\theta}, e = 3/2, d = 4/3$



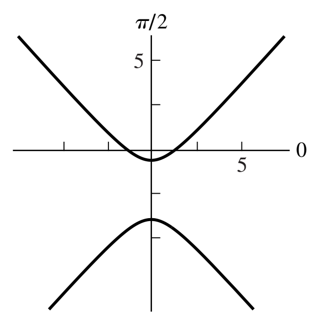
(d) $r = \dfrac{5/3}{1+\sin\theta}, e = 1, d = 5/3$



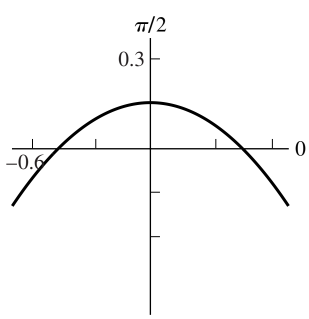
2. **(a)** $r = \dfrac{4/3}{1-\frac{2}{3}\cos\theta}, e = 2/3, d = 2$



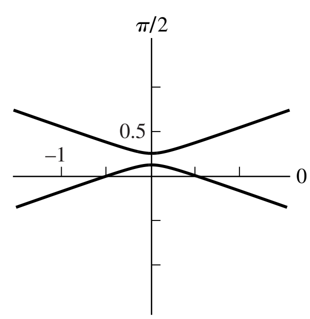
(b) $r = \dfrac{1}{1-\frac{4}{3}\sin\theta}, e = 4/3, d = 3/4$



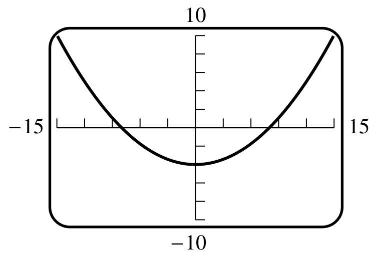
(c) $r = \dfrac{1/3}{1+\sin\theta}, e = 1, d = 1/3$



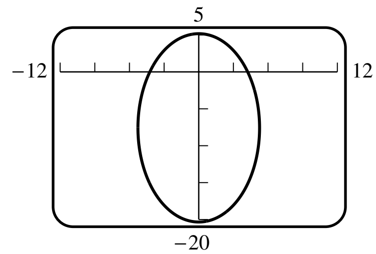
(d) $r = \dfrac{1/2}{1+3\sin\theta}, e = 3, d = 1/6$



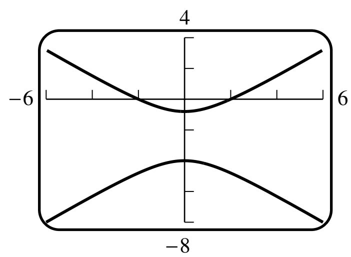
3. **(a)** $e = 1, d = 8,$ parabola, opens up



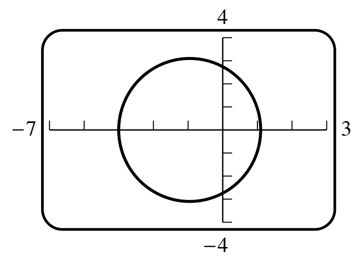
(b) $r = \dfrac{4}{1 + \frac{3}{4}\sin\theta}, e = 3/4, d = 16/3,$
ellipse, directrix 16/3 units above the pole



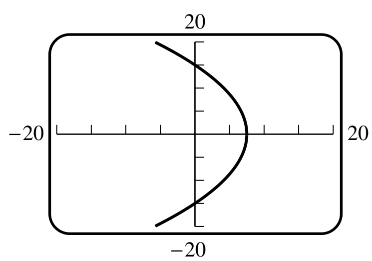
(c) $r = \dfrac{2}{1 - \frac{3}{2}\sin\theta}, e = 3/2, d = 4/3,$
hyperbola, directrix 4/3 units below the pole



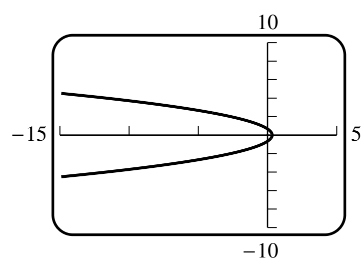
(d) $r = \dfrac{3}{1 + \frac{1}{4}\cos\theta}, e = 1/4, d = 12,$
ellipse, directrix 12 units to the right of the pole



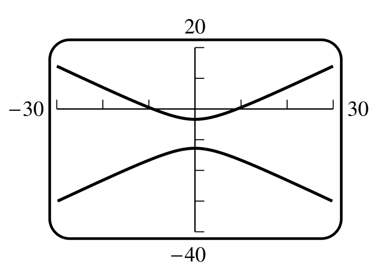
4. **(a)** $e = 1, d = 15,$ parabola, opens left



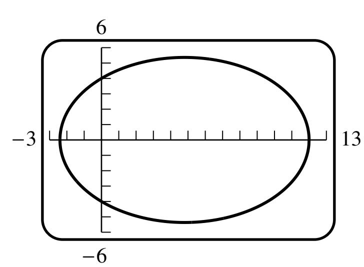
(b) $r = \dfrac{2/3}{1 + \cos\theta}, e = 1,$
$d = 2/3,$ parabola, opens left



(c) $r = \dfrac{64/7}{1 - \frac{12}{7}\sin\theta}, e = 12/7, d = 16/3,$
hyperbola, directrix 16/3 units below pole



(d) $r = \dfrac{4}{1 - \frac{2}{3}\cos\theta}, e = 2/3, d = 6,$
ellipse, directrix 6 units left of the pole



5. **(a)** $d = 1, r = \dfrac{ed}{1 + e\cos\theta} = \dfrac{2/3}{1 + \frac{2}{3}\cos\theta} = \dfrac{2}{3 + 2\cos\theta}$

(b) $e = 1, d = 1, r = \dfrac{ed}{1 - e\cos\theta} = \dfrac{1}{1 - \cos\theta}$

(c) $e = 3/2, d = 1, r = \dfrac{ed}{1 + e\sin\theta} = \dfrac{3/2}{1 + \frac{3}{2}\sin\theta} = \dfrac{3}{2 + 3\sin\theta}$

6. **(a)** $e = 2/3, d = 1, r = \dfrac{ed}{1 - e\sin\theta} = \dfrac{2/3}{1 - \frac{2}{3}\sin\theta} = \dfrac{2}{3 - 2\sin\theta}$

(b) $e = 1, d = 1, r = \dfrac{ed}{1 + e\sin\theta} = \dfrac{1}{1 + \sin\theta}$

(c) $e = 4/3, d = 1, r = \dfrac{ed}{1 - e\cos\theta} = \dfrac{4/3}{1 - \frac{4}{3}\cos\theta} = \dfrac{4}{3 - 4\cos\theta}$

7. **(a)** $r = \dfrac{ed}{1 \pm e\cos\theta}, \theta = 0 : 6 = \dfrac{ed}{1 \pm e}, \theta = \pi : 4 = \dfrac{ed}{1 \mp e}, 6 \pm 6e = 4 \mp 4e, 2 = \mp 10e$, use bottom sign to get $e = 1/5, d = 24,\ \ r = \dfrac{24/5}{1 - \cos\theta} = \dfrac{24}{5 - 5\cos\theta}$

(b) $e = 1, r = \dfrac{d}{1 - \sin\theta}, 1 = \dfrac{d}{2}, d = 2, r = \dfrac{2}{1 - \sin\theta}$

(c) $r = \dfrac{ed}{1 \pm e\sin\theta}, \theta = \pi/2 : 3 = \dfrac{ed}{1 \pm e}, \theta = 3\pi/2 : -7 = \dfrac{ed}{1 \mp e}, ed = 3 \pm 3e = -7 \pm 7e, 10 = \pm 4e,$

$e = 5/2, d = 21/5, r = \dfrac{21/2}{1 + (5/2)\sin\theta} = \dfrac{21}{2 + 5\sin\theta}$

8. **(a)** $r = \dfrac{ed}{1 \pm e\sin\theta}, 1 = \dfrac{ed}{1 \pm e}, 4 = \dfrac{ed}{1 \mp e}, 1 \pm e = 4 \mp 4e$, upper sign yields $e = 3/5, d = 8/3$,

$r = \dfrac{8/5}{1 + \frac{3}{5}\sin\theta} = \dfrac{8}{5 + 3\sin\theta}$

(b) $e = 1, r = \dfrac{d}{1 - \cos\theta}, 3 = \dfrac{d}{2}, d = 6,\ r = \dfrac{6}{1 - \cos\theta}$

(c) $a = b = 5, e = c/a = \sqrt{50}/5 = \sqrt{2}, r = \dfrac{\sqrt{2}d}{1 + \sqrt{2}\cos\theta}; r = 5$ when $\theta = 0$, so $d = 5 + \dfrac{5}{\sqrt{2}}$,

$r = \dfrac{5\sqrt{2} + 5}{1 + \sqrt{2}\cos\theta}$.

9. **(a)** $r = \dfrac{3}{1 + \frac{1}{2}\sin\theta}, e = 1/2, d = 6$, directrix 6 units above pole; if $\theta = \pi/2 : r_0 = 2$;

if $\theta = 3\pi/2 : r_1 = 6, a = (r_0 + r_1)/2 = 4, b = \sqrt{r_0 r_1} = 2\sqrt{3}$, center $(0, -2)$ (rectangular coordinates), $\dfrac{x^2}{12} + \dfrac{(y+2)^2}{16} = 1$

(b) $r = \dfrac{1/2}{1 - \frac{1}{2}\cos\theta}, e = 1/2, d = 1$, directrix 1 unit left of pole; if $\theta = \pi : r_0 = \dfrac{1/2}{3/2} = 1/3$;

if $\theta = 0 : r_1 = 1, a = 2/3, b = 1/\sqrt{3}$, center $= (1/3, 0)$ (rectangular coordinates),

$\dfrac{9}{4}(x - 1/3)^2 + 3y^2 = 1$

10. **(a)** $r=\dfrac{6/5}{1+\frac{2}{5}\cos\theta}$, $e=2/5, d=3$, directrix 3 units right of pole, if $\theta=0: r_0=6/7$,

if $\theta=\pi: r_1=2, a=10/7, b=2\sqrt{3}/\sqrt{7}$, center $(-4/7,0)$ (rectangular coordinates),

$\dfrac{49}{100}(x+4/7)^2+\dfrac{7}{12}y^2=1$

(b) $r=\dfrac{2}{1-\frac{3}{4}\sin\theta}$, $e=3/4, d=8/3$, directrix 8/3 units below pole, if $\theta=3\pi/2: r_0=8/7$,

if $\theta=\pi/2;\ \ r_1=8, a=32/7, b=8/\sqrt{7}$, center: $(0,24/7)$ (rectangular coordinates),

$\dfrac{7}{64}x^2+\dfrac{49}{1024}\left(y-\dfrac{24}{7}\right)^2=1$

11. **(a)** $r=\dfrac{2}{1+3\sin\theta}$, $e=3, d=2/3$, hyperbola, directrix 2/3 units above pole, if $\theta=\pi/2:$

$r_0=1/2; \theta=3\pi/2: r_1=1$, center $(0,3/4), a=1/4, b=1/\sqrt{2}, -2x^2+16\left(y-\dfrac{3}{4}\right)^2=1$

(b) $r=\dfrac{5/3}{1-\frac{3}{2}\cos\theta}$, $e=3/2, d=10/9$, hyperbola, directrix 10/9 units left of pole, if $\theta=\pi:$

$r_0=2/3; \theta=0: r_1=\dfrac{5/3}{1/2}=10/3$, center $(-2,0)$, $\ a=4/3, b=\sqrt{20/9}, \dfrac{9}{16}(x+2)^2-\dfrac{9}{20}y^2=1$

12. **(a)** $r=\dfrac{4}{1-2\sin\theta}$, $e=2, d=2$, hyperbola, directrix 2 units below pole, if $\theta=3\pi/2: r_0=4/3$;

$\theta=\pi/2: r_1=\left|\dfrac{4}{1-2}\right|=4$, center $(0,-8/3), a=4/3, b=4/\sqrt{3}$, $\ \dfrac{9}{16}\left(y+\dfrac{8}{3}\right)^2-\dfrac{3}{16}x^2=1$

(b) $r=\dfrac{15/2}{1+4\cos\theta}$, $e=4, d=15/8$, hyperbola, directrix 15/8 units right of pole, if $\theta=0:$

$r_0=3/2; \theta=\pi: r_1=\left|-\dfrac{5}{2}\right|=5/2, a=1/2, b=\dfrac{\sqrt{15}}{2}$, center $(2,0)$, $\ 4(x-2)^2-\dfrac{4}{15}y^2=1$

13. **(a)** $r=\dfrac{\frac{1}{2}d}{1+\frac{1}{2}\cos\theta}=\dfrac{d}{2+\cos\theta}$, if $\theta=0: r_0=d/3; \theta=\pi, r_1=d$,

$8=a=\dfrac{1}{2}(r_1+r_0)=\dfrac{2}{3}d, d=12,\ \ r=\dfrac{12}{2+\cos\theta}$

(b) $r=\dfrac{\frac{3}{5}d}{1-\frac{3}{5}\sin\theta}=\dfrac{3d}{5-3\sin\theta}$, if $\theta=3\pi/2: r_0=\dfrac{3}{8}d; \theta=\pi/2, r_1=\dfrac{3}{2}d$,

$4=a=\dfrac{1}{2}(r_1+r_0)=\dfrac{15}{16}d, d=\dfrac{64}{15}, r=\dfrac{3(64/15)}{5-3\sin\theta}=\dfrac{64}{25-15\sin\theta}$

(c) $r=\dfrac{\frac{3}{5}d}{1-\frac{3}{5}\cos\theta}=\dfrac{3d}{5-3\cos\theta}$, if $\theta=\pi: r_0=\dfrac{3}{8}d; \theta=0, r_1=\dfrac{3}{2}d, 4=b=\dfrac{3}{4}d$,

$d=16/3,\ \ r=\dfrac{16}{5-3\cos\theta}$

(d) $r=\dfrac{\frac{1}{5}d}{1+\frac{1}{5}\sin\theta}=\dfrac{d}{5+\sin\theta}$, if $\theta=\pi/2: r_0=d/6; \theta=3\pi/2, r_1=d/4$,

$5=c=\dfrac{1}{2}d\left(\dfrac{1}{4}-\dfrac{1}{6}\right)=\dfrac{1}{24}d, d=120,\ \ r=\dfrac{120}{5+\sin\theta}$

14. **(a)** $r = \dfrac{\frac{1}{2}d}{1 + \frac{1}{2}\sin\theta} = \dfrac{d}{2 + \sin\theta}$, if $\theta = \pi/2 : r_0 = d/3; \theta = 3\pi/2 : r_1 = d$,

$10 = a = \dfrac{1}{2}(r_0 + r_1) = \dfrac{2}{3}d, d = 15, \quad r = \dfrac{15}{2 + \sin\theta}$

(b) $r = \dfrac{\frac{1}{5}d}{1 - \frac{1}{5}\cos\theta} = \dfrac{d}{5 - \cos\theta}$, if $\theta = \pi : r_0 = d/6, \theta = 0 : r_1 = d/4$,

$6 = a = \dfrac{1}{2}(r_1 + r_0) = \dfrac{1}{2}d\left(\dfrac{1}{4} + \dfrac{1}{6}\right) = \dfrac{5}{24}d, d = 144/5, \quad r = \dfrac{144/5}{5 - \cos\theta} = \dfrac{144}{25 - 5\cos\theta}$

(c) $r = \dfrac{\frac{3}{4}d}{1 - \frac{3}{4}\sin\theta} = \dfrac{3d}{4 - 3\sin\theta}$, if $\theta = 3\pi/2 : r_0 = \dfrac{3}{7}d, \theta = \pi/2 : r_1 = 3d, 4 = b = 3d/\sqrt{7}$,

$d = \dfrac{4}{3}\sqrt{7}, \quad r = \dfrac{4\sqrt{7}}{4 - 3\sin\theta}$

(d) $r = \dfrac{\frac{4}{5}d}{1 + \frac{4}{5}\cos\theta} = \dfrac{4d}{5 + 4\cos\theta}$, if $\theta = 0 : r_0 = \dfrac{4}{9}d; \theta = \pi : r_1 = 4d$,

$c = 10 = \dfrac{1}{2}(r_1 - r_0) = \dfrac{1}{2}d\left(4 - \dfrac{4}{9}\right) = \dfrac{16}{9}d, d = \dfrac{45}{8}, \quad r = \dfrac{45/2}{5 + 4\cos\theta} = \dfrac{45}{10 + 8\cos\theta}$

15. **(a)** $e = c/a = \dfrac{\frac{1}{2}(r_1 - r_0)}{\frac{1}{2}(r_1 + r_0)} = \dfrac{r_1 - r_0}{r_1 + r_0}$

(b) $e = \dfrac{r_1/r_0 - 1}{r_1/r_0 + 1}, e(r_1/r_0 + 1) = r_1/r_0 - 1, \dfrac{r_1}{r_0} = \dfrac{1 + e}{1 - e}$

16. **(a)** $e = c/a = \dfrac{\frac{1}{2}(r_1 + r_0)}{\frac{1}{2}(r_1 - r_0)} = \dfrac{r_1 + r_0}{r_1 - r_0}$

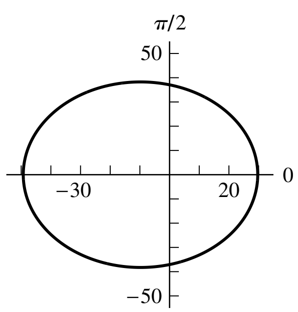
(b) $e = \dfrac{r_1/r_0 + 1}{r_1/r_0 - 1}, e(r_1/r_0 - 1) = r_1/r_0 + 1, \dfrac{r_1}{r_0} = \dfrac{e + 1}{e - 1}$

17. **(a)** $T = a^{3/2} = 39.5^{1.5} \approx 248$ yr

(b) $r_0 = a(1 - e) = 39.5(1 - 0.249) = 29.6645$ AU $\approx$ 4,449,675,000 km
$r_1 = a(1 + e) = 39.5(1 + 0.249) = 49.3355$ AU $\approx$ 7,400,325,000 km

(c) $r = \dfrac{a(1 - e^2)}{1 + e\cos\theta} \approx \dfrac{39.5(1 - (0.249)^2)}{1 + 0.249\cos\theta} \approx \dfrac{37.05}{1 + 0.249\cos\theta}$ AU

(d)



18. **(a)** In yr and AU, $T = a^{3/2}$; in days and km, $\dfrac{T}{365} = \left(\dfrac{a}{150 \times 10^6}\right)^{3/2}$,

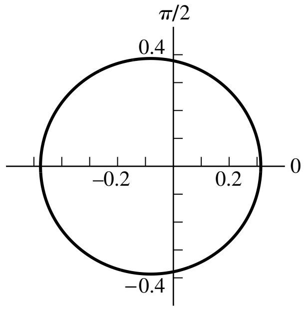
so $T = 365 \times 10^{-9}\left(\dfrac{a}{150}\right)^{3/2}$ days.

(b) $T = 365 \times 10^{-9} \left(\dfrac{57.95 \times 10^6}{150}\right)^{3/2} \approx 87.6$ days

(c) $r = \dfrac{55490833.8}{1 + .206\cos\theta}$

From (17) the polar equation of the orbit has the form $r = \dfrac{a(1-e^2)}{1+e\cos\theta} = \dfrac{55490833.8}{1+.206\cos\theta}$ km, or $r = \dfrac{0.3699}{1+0.206\cos\theta}$ AU.

(d)

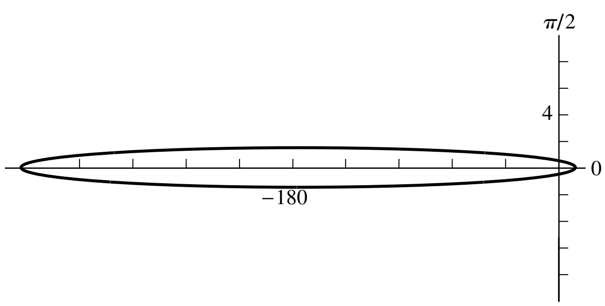


19. **(a)** $a = T^{2/3} = 2380^{2/3} \approx 178.26$ AU

(b) $r_0 = a(1-e) \approx 0.8735$ AU, $r_1 = a(1+e) \approx 355.64$ AU

(c) $r = \dfrac{a(1-e^2)}{1+e\cos\theta} \approx \dfrac{1.74}{1+0.9951\cos\theta}$ AU

(d)

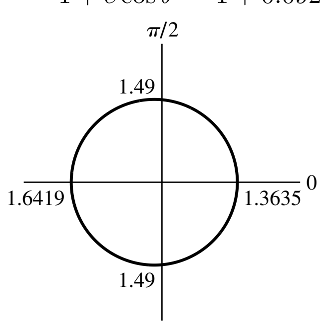


20. **(a)** By Exercise 15(a), $e = \dfrac{r_1 - r_0}{r_1 + r_0} \approx 0.092635$

(b) $a = \frac{1}{2}(r_0 + r_1) = 225{,}400{,}000$ km ≈ 1.503 AU, so $T = a^{3/2} \approx 1.84$ yr

(c) $r = \dfrac{a(1-e^2)}{1+e\cos\theta} \approx \dfrac{223465774.6}{1+0.092635\cos\theta}$ km, or $\approx \dfrac{1.48977}{1+0.092635\cos\theta}$ AU

(d)



21. $r_0 = a(1-e) \approx 7003$ km, $h_{\min} \approx 7003 - 6440 = 563$ km,
$r_1 = a(1+e) \approx 10{,}726$ km, $h_{\max} \approx 10{,}726 - 6440 = 4286$ km

22. $r_0 = a(1-e) \approx 651{,}736$ km, $h_{\min} \approx 581{,}736$ km; $r_1 = a(1+e) \approx 6{,}378{,}102$ km, $h_{\max} \approx 6{,}308{,}102$ km

23. Since the foci are fixed, c is constant; since $e \to 0$, the distance $\dfrac{a}{e} = \dfrac{c}{e^2} \to +\infty$.

24. **(a)** From Figure 11.4.22, $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, $\dfrac{x^2}{a^2} - \dfrac{y^2}{c^2 - a^2} = 1$, $\left(1 - \dfrac{c^2}{a^2}\right) x^2 + y^2 = a^2 - c^2$,

$c^2 + x^2 + y^2 = \left(\dfrac{c}{a}x\right) + a^2$, $(x-c)^2 + y^2 = \left(\dfrac{c}{a}x - a\right)^2$,

$\sqrt{(x-c)^2 + y^2} = \dfrac{c}{a}x - a$ for $x > a^2/c$.

(b) From Part (a) and Figure 11.6.1, $PF = \dfrac{c}{a}PD$, $\dfrac{PF}{PD} = c/a$.

CHAPTER 11 SUPPLEMENTARY EXERCISES

2. **(a)** $(\sqrt{2}, 3\pi/4)$ **(b)** $(-\sqrt{2}, 7\pi/4)$ **(c)** $(\sqrt{2}, 3\pi/4)$ **(d)** $(-\sqrt{2}, -\pi/4)$

3. **(a)** circle **(b)** rose **(c)** line **(d)** limaçon
(e) limaçon **(f)** none **(g)** none **(h)** spiral

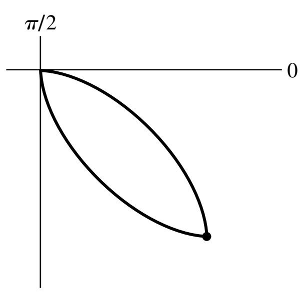
4. **(a)** $r = \dfrac{1/3}{1 + \frac{1}{3}\cos\theta}$, ellipse, right of pole, distance $= 1$

(b) hyperbola, left of pole, distance $= 1/3$

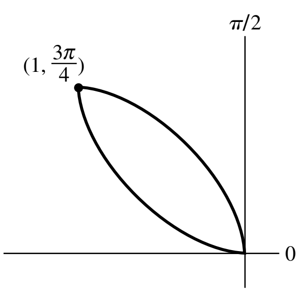
(c) $r = \dfrac{1/3}{1 + \sin\theta}$, parabola, above pole, distance $= 1/3$

(d) parabola, below pole, distance $= 3$

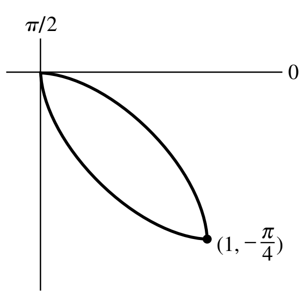
5. **(a)**



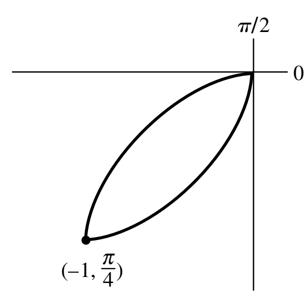
(b)



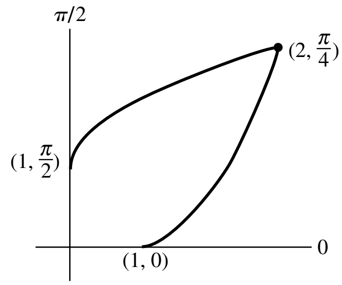
(c)



(d)



(e)



6. Family I: $x^2 + (y-b)^2 = b^2, b < 0$, or $r = 2b\sin\theta$; Family II: $(x-a)^2 + y^2 = a^2, a < 0$, or $r = 2a\cos\theta$

7. **(a)** $r = 2a/(1+\cos\theta), r + x = 2a, x^2 + y^2 = (2a-x)^2, y^2 = -4ax + 4a^2$, parabola

(b) $r^2(\cos^2\theta - \sin^2\theta) = x^2 - y^2 = a^2$, hyperbola

(c) $r\sin(\theta - \pi/4) = (\sqrt{2}/2)r(\sin\theta - \cos\theta) = 4, y - x = 4\sqrt{2}$, line

(d) $r^2 = 4r\cos\theta + 8r\sin\theta, x^2 + y^2 = 4x + 8y, (x-2)^2 + (y-4)^2 = 20$, circle

9. **(a)** $\dfrac{c}{a} = e = \dfrac{2}{7}$ and $2b = 6, b = 3, a^2 = b^2 + c^2 = 9 + \dfrac{4}{49}a^2, \dfrac{45}{49}a^2 = 9, a = \dfrac{7}{\sqrt{5}},\ \dfrac{5}{49}x^2 + \dfrac{1}{9}y^2 = 1$

(b) $x^2 = -4py$, directrix $y = 4$, focus $(-4, 0), 2p = 8, x^2 = -16y$

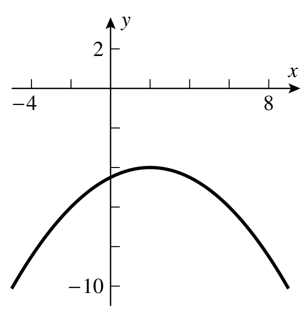
(c) For the ellipse, $a = 4, b = \sqrt{3}, c^2 = a^2 - b^2 = 16 - 3 = 13$, foci $(\pm\sqrt{13}, 0)$;
for the hyperbola, $c = \sqrt{13}, b/a = 2/3, b = 2a/3, 13 = c^2 = a^2 + b^2 = a^2 + \dfrac{4}{9}a^2 = \dfrac{13}{9}a^2$,
$a = 3, b = 2,\ \dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$

10. **(a)** $e = 4/5 = c/a, c = 4a/5$, but $a = 5$ so $c = 4, b = 3, \dfrac{(x+3)^2}{25} + \dfrac{(y-2)^2}{9} = 1$

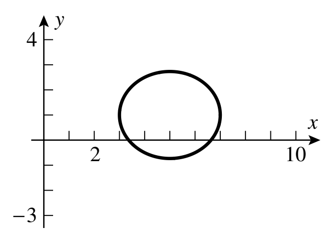
(b) directrix $y = 2, p = 2, (x+2)^2 = -8y$

(c) center $(-1, 5)$, vertices $(-1, 7)$ and $(-1, 3), a = 2, a/b = 8, b = 1/4,\ \dfrac{(y-5)^2}{4} - 16(x+1)^2 = 1$

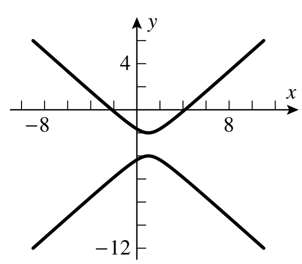
11. **(a)**



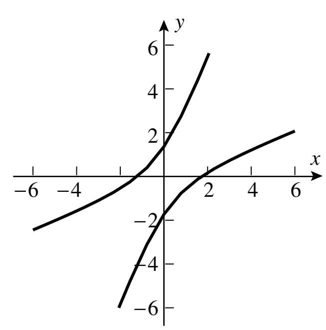
(b)



(c)



(d)



13. **(a)** The equation of the parabola is $y = ax^2$ and it passes through $(2100, 470)$, thus $a = \dfrac{470}{2100^2}$, $y = \dfrac{470}{2100^2}x^2$.

(b) $L = 2\displaystyle\int_0^{2100} \sqrt{1 + \left(2\frac{470}{2100^2}x\right)^2}\, dx$
$= \dfrac{x}{220500}\sqrt{48620250000 + 2209x^2} + \dfrac{220500}{47}\sinh^{-1}\left(\dfrac{47}{220500}x\right) \approx 4336.3$ ft

14. **(a)** As t runs from 0 to π, the upper portion of the curve is traced out from right to left; as t runs from π to 2π the bottom portion of the curve is traced out from right to left. The loop occurs for $\pi + \sin^{-1}\dfrac{1}{4} < t < 2\pi - \sin^{-1}\dfrac{1}{4}$.

(b) $\lim\limits_{t\to 0^+} x = +\infty$, $\lim\limits_{t\to 0^+} y = 1$; $\lim\limits_{t\to \pi^-} x = -\infty$, $\lim\limits_{t\to \pi^-} y = 1$; $\lim\limits_{t\to \pi^+} x = +\infty$, $\lim\limits_{t\to \pi^+} y = 1$; $\lim\limits_{t\to 2\pi^-} x = -\infty$, $\lim\limits_{t\to 2\pi^-} y = 1$; the horizontal asymptote is $y = 1$.

(c) horizontal tangent line when $dy/dx = 0$, or $dy/dt = 0$, so $\cos t = 0, t = \pi/2, 3\pi/2$;
vertical tangent line when $dx/dt = 0$, so $-\csc^2 t - 4\sin t = 0, t = \pi + \sin^{-1}\dfrac{1}{\sqrt[3]{4}}, 2\pi - \sin^{-1}\dfrac{1}{\sqrt[3]{4}}$,
$t = 3.823, 5.602$

(d) $r^2 = x^2 + y^2 = (\cot t + 4\cos t)^2 + (1 + 4\sin t)^2 = (4 + \csc t)^2, r = 4 + \csc t$; with $t = \theta$, $f(\theta) = 4 + \csc\theta; m = dy/dx = (f(\theta)\cos\theta + f'(\theta)\sin\theta)/(-f(\theta)\sin\theta + f'(\theta)\cos\theta)$; when $\theta = \pi + \sin^{-1}(1/4), m = \sqrt{15}/15$, when $\theta = 2\pi - \sin^{-1}(1/4), m = -\sqrt{15}/15$, so the tangent lines to the conchoid at the pole have polar equations $\theta = \pm\tan^{-1}\dfrac{1}{\sqrt{15}}$.

15. $= \displaystyle\int_0^{\pi/6} 4\sin^2\theta\, d\theta + \int_{\pi/6}^{\pi/4} 1\, d\theta = \int_0^{\pi/6} 2(1 - \cos 2\theta)\, d\theta + \frac{\pi}{12} = (2\theta - \sin 2\theta)\Big]_0^{\pi/6} + \frac{\pi}{12}$
$= \dfrac{\pi}{3} - \dfrac{\sqrt{3}}{2} + \dfrac{\pi}{12} = \dfrac{5\pi}{12} - \dfrac{\sqrt{3}}{2}$

16. The circle has radius $a/2$ and lies entirely inside the cardioid, so
$A = \displaystyle\int_0^{2\pi} \frac{1}{2}a^2(1 + \sin\theta)^2\, d\theta - \pi a^2/4 = \frac{3a^2}{2}\pi - \frac{a^2}{4}\pi = \frac{5a^2}{4}\pi$

17. **(a)** $r = 1/\theta, dr/d\theta = -1/\theta^2, r^2 + (dr/d\theta)^2 = 1/\theta^2 + 1/\theta^4$, $L = \displaystyle\int_{\pi/4}^{\pi/2} \frac{1}{\theta^2}\sqrt{1 + \theta^2}\, d\theta \approx 0.9457$ by Endpaper Table Formula 93.

(b) The integral $\displaystyle\int_1^{+\infty} \frac{1}{\theta^2}\sqrt{1 + \theta^2}\, d\theta$ diverges by the comparison test (with $1/\theta$), and thus the arc length is infinite.

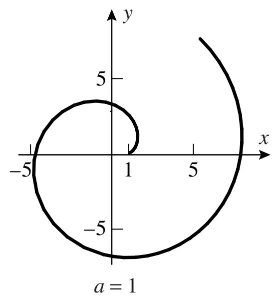
18. **(a)** When the point of departure of the thread from the circle has traversed an angle θ, the amount of thread that has been unwound is equal to the arc length traversed by the point of departure, namely $a\theta$. The point of departure is then located at $(a\cos\theta, a\sin\theta)$, and the tip of the string, located at (x, y), satisfies the equations $x - a\cos\theta = a\theta\sin\theta, y - a\sin\theta = -a\theta\cos\theta$; hence $x = a(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$.

(b) Assume for simplicity that $a = 1$. Then $dx/d\theta = \theta\cos\theta, dy/d\theta = \theta\sin\theta; dx/d\theta = 0$ has solutions $\theta = 0, \pi/2, 3\pi/2$; and $dy/d\theta = 0$ has solutions $\theta = 0, \pi, 2\pi$. At $\theta = \pi/2, dy/d\theta > 0$, so the direction is North; at $\theta = \pi, dx/d\theta < 0$, so West; at $\theta = 3\pi/2, dy/d\theta < 0$, so South; at $\theta = 2\pi, dx/d\theta > 0$, so East. Finally, $\lim\limits_{\theta\to 0^+} dy/dx = \lim\limits_{\theta\to 0^+} \tan\theta = 0$, so East.

(c)

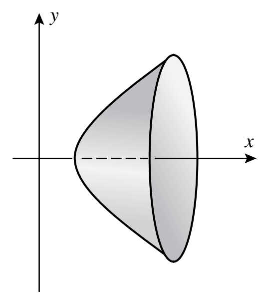
θ	0	$\pi/2$	π	$3\pi/2$	2π
x	1	$\pi/2$	-1	$-3\pi/2$	1
y	0	1	π	-1	-2π

Note that the parameter θ in these equations does not satisfy equations (1) and (2) of Section 11.1, since it measures the angle of the point of departure and not the angle of the tip of the thread.

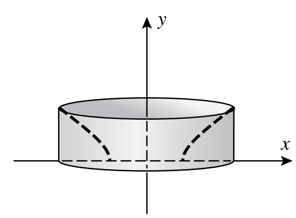


19. **(a)** $V = \int_a^{\sqrt{a^2+b^2}} \pi\left(b^2x^2/a^2 - b^2\right)\,dx$

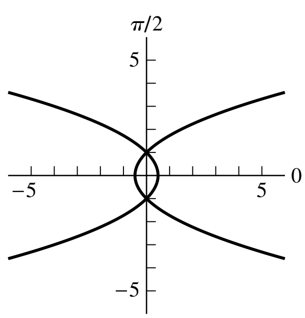
$= \dfrac{\pi b^2}{3a^2}(b^2 - 2a^2)\sqrt{a^2+b^2} + \dfrac{2}{3}ab^2\pi$



(b) $V = 2\pi \int_a^{\sqrt{a^2+b^2}} x\sqrt{b^2x^2/a^2 - b^2}\,dx = (2b^4/3a)\pi$



20. **(a)**



(b) $\theta = \pi/2, 3\pi/2, r = 1$

(c) $dy/dx = \dfrac{r\cos\theta + (dr/d\theta)\sin\theta}{-r\sin\theta + (dr/d\theta)\cos\theta}$; at $\theta = \pi/2, m_1 = (-1)/(-1) = 1, m_2 = 1/(-1) = -1$, $m_1m_2 = -1$; and at $\theta = 3\pi/2, m_1 = -1, m_2 = 1, m_1m_2 = -1$

22. The tips are located at $r = 1, \theta = \pi/6, 5\pi/6, 3\pi/2$ and, for example,
$d = \sqrt{1 + 1 - 2\cos(5\pi/6 - \pi/6)} = \sqrt{2(1 - \cos(2\pi/3))} = \sqrt{3}$

23. **(a)** $x = r\cos\theta = \cos\theta + \cos^2\theta, dx/d\theta = -\sin\theta - 2\sin\theta\cos\theta = -\sin\theta(1+2\cos\theta) = 0$ if $\sin\theta = 0$ or $\cos\theta = -1/2$, so $\theta = 0, \pi, 2\pi/3, 4\pi/3$; maximum $x = 2$ at $\theta = 0$, minimum $x = -1/4$ at $\theta = 2\pi/3, 4\pi/3$; $\theta = \pi$ is a local maximum for x

(b) $y = r\sin\theta = \sin\theta + \sin\theta\cos\theta, dy/d\theta = 2\cos^2\theta + \cos\theta - 1 = 0$ at $\cos\theta = 1/2, -1$, so $\theta = \pi/3, 5\pi/3, \pi$; maximum $y = 3\sqrt{3}/4$ at $\theta = \pi/3$, minimum $y = -3\sqrt{3}/4$ at $\theta = 5\pi/3$

24. **(a)** $y = r\sin\theta = (\sin\theta)/\sqrt{\theta}, dy/d\theta = \dfrac{2\theta\cos\theta - \sin\theta}{2\theta^{3/2}} = 0$ if $2\theta\cos\theta = \sin\theta, \tan\theta = 2\theta$ which only happens once on $(0, \pi]$. Since $\lim\limits_{\theta\to 0^+} y = 0$ and $y = 0$ at $\theta = \pi$, y has a maximum when $\tan\theta = 2\theta$.

(b) $\theta \approx 1.16556$

(c) $y_{\max} = (\sin\theta)/\sqrt{\theta} \approx 0.85124$

25. The width is twice the maximum value of y for $0 \le \theta \le \pi/4$:
$y = r\sin\theta = \sin\theta\cos 2\theta = \sin\theta - 2\sin^3\theta, dy/d\theta = \cos\theta - 6\sin^2\theta\cos\theta = 0$ when $\cos\theta = 0$ or $\sin\theta = 1/\sqrt{6}, y = 1/\sqrt{6} - 2/(6\sqrt{6}) = \sqrt{6}/9$, so the width of the petal is $2\sqrt{6}/9$.

26. **(a)** $\dfrac{x^2}{225} - \dfrac{y^2}{1521} = 1$, so $V = 2\displaystyle\int_0^{h/2} 225\pi\left(1 + \frac{y^2}{1521}\right) dy = \frac{25}{2028}\pi h^3 + 225\pi h$ ft^3.

(b)
$$S = 2\int_0^{h/2} 2\pi x\sqrt{1 + (dx/dy)^2}\, dy = 4\pi\int_0^{h/2}\sqrt{225 + y^2\left(\frac{225}{1521} + \left(\frac{225}{1521}\right)^2\right)}\, dy$$
$$= \frac{5\pi h}{338}\sqrt{1028196 + 194h^2} + \frac{7605\sqrt{194}}{97}\pi\ln\left[\frac{\sqrt{194}h + \sqrt{1028196 + 194h^2}}{1014}\right] \text{ ft}^2$$

27. **(a)** The end of the inner arm traces out the circle $x_1 = \cos t, y_1 = \sin t$. Relative to the end of the inner arm, the outer arm traces out the circle $x_2 = \cos 2t, y_2 = -\sin 2t$. Add to get the motion of the center of the rider cage relative to the center of the inner arm:
$x = \cos t + \cos 2t, y = \sin t - \sin 2t$.

(b) Same as Part (a), except $x_2 = \cos 2t, y_2 = \sin 2t$, so $x = \cos t + \cos 2t, y = \sin t + \sin 2t$

(c) $L_1 = \displaystyle\int_0^{2\pi}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{1/2} dt = \int_0^{2\pi}\sqrt{5 - 4\cos 3t}\, dt \approx 13.36489321$,

$L_2 = \displaystyle\int_0^{2\pi}\sqrt{5 + 4\cos t}\, dt \approx 13.36489322$; L_1 and L_2 appear to be equal, and indeed, with the substitution $u = 3t - \pi$ and the periodicity of $\cos u$,

$$L_1 = \frac{1}{3}\int_{-\pi}^{5\pi}\sqrt{5 - 4\cos(u + \pi)}\, du = \int_0^{2\pi}\sqrt{5 + 4\cos u}\, du = L_2.$$

29.
$$C = 4\int_0^{\pi/2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{1/2} dt = 4\int_0^{\pi/2}(a^2\sin^2 t + b^2\cos^2 t)^{1/2}\, dt$$
$$= 4\int_0^{\pi/2}(a^2\sin^2 t + (a^2 - c^2)\cos^2 t)^{1/2}\, dt = 4a\int_0^{\pi/2}(1 - e^2\cos^2 t)^{1/2}\, dt$$

Set $u = \dfrac{\pi}{2} - t$, $C = 4a\displaystyle\int_0^{\pi/2}(1 - e^2\sin^2 t)^{1/2}\, dt$

30. $a = 3, b = 2, c = \sqrt{5},\ \ C = 4(3)\int_0^{\pi/2}\sqrt{1-(5/9)\cos^2 u}\,du \approx 15.86543959$

31. **(a)** $\dfrac{r_0}{r_1} = \dfrac{59}{61} = \dfrac{1-e}{1+e}, e = \dfrac{1}{60}$

(b) $a = 93\times 10^6, r_0 = a(1-e) = \dfrac{59}{60}\left(93\times 10^6\right) = 91{,}450{,}000$ mi

(c) $C = 4\times 93\times 10^6\displaystyle\int_0^{\pi/2}\left[1-\left(\frac{\cos\theta}{60}\right)^2\right]^{1/2} d\theta \approx 584{,}295{,}652.5$ mi

32. **(a)** $y = y_0 + (v_0\sin\alpha)\dfrac{x}{v_0\cos\alpha} - \dfrac{g}{2}\left(\dfrac{x}{v_0\cos\alpha}\right)^2 = y_0 + x\tan\alpha - \dfrac{g}{2v_0^2\cos^2\alpha}x^2$

(b) $\dfrac{dy}{dx} = \tan\alpha - \dfrac{g}{v_0^2\cos^2\alpha}x, dy/dx = 0$ at $x = \dfrac{v_0^2}{g}\sin\alpha\cos\alpha,$

$y = y_0 + \dfrac{v_0^2}{g}\sin^2\alpha - \dfrac{g}{2v_0^2\cos^2\alpha}\left(\dfrac{v_0^2\sin\alpha\cos\alpha}{g}\right)^2 = y_0 + \dfrac{v_0^2}{2g}\sin^2\alpha$

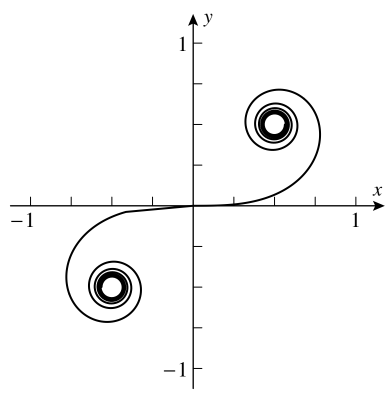
33. $\alpha = \pi/4, y_0 = 3, x = v_0 t/\sqrt{2}, y = 3 + v_0 t/\sqrt{2} - 16t^2$

(a) Assume the ball passes through $x = 391, y = 50$, then $391 = v_0 t/\sqrt{2}, 50 = 3 + 391 - 16t^2$, $16t^2 = 344,\ t = \sqrt{21.5},\ v_0 = \sqrt{2}x/t \approx 119.2538820$ ft/s

(b) $\dfrac{dy}{dt} = \dfrac{v_0}{\sqrt{2}} - 32t = 0$ at $t = \dfrac{v_0}{32\sqrt{2}},\ y_{\max} = 3 + \dfrac{v_0}{\sqrt{2}}\dfrac{v_0}{32\sqrt{2}} - 16\dfrac{v_0^2}{2^{11}} = 3 + \dfrac{v_0^2}{128} \approx 114.1053779$ ft

(c) $y = 0$ when $t = \dfrac{-v_0/\sqrt{2}\pm\sqrt{v_0^2/2+192}}{-32},\ t \approx -0.035339577$ (discard) and 5.305666365, dist $= 447.4015292$ ft

34. **(a)**



(c) $L = \displaystyle\int_{-1}^{1}\left[\cos^2\left(\frac{\pi t^2}{2}\right)+\sin^2\left(\frac{\pi t^2}{2}\right)\right] dt = 2$

35. $\tan\psi = \tan(\phi-\theta) = \dfrac{\tan\phi - \tan\theta}{1+\tan\phi\tan\theta} = \dfrac{\dfrac{dy}{dx}-\dfrac{y}{x}}{1+\dfrac{y}{x}\dfrac{dy}{dx}}$

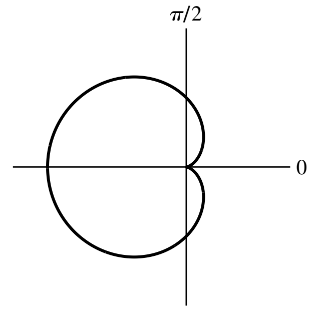
$$= \frac{\dfrac{r\cos\theta + (dr/d\theta)\sin\theta}{-r\sin\theta + (dr/d\theta)\cos\theta} - \dfrac{\sin\theta}{\cos\theta}}{1 + \left(\dfrac{r\cos\theta + (dr/d\theta)\sin\theta)}{-r\sin\theta + (dr/d\theta)\cos\theta)}\right)\left(\dfrac{\sin\theta}{\cos\theta}\right)} = \frac{r}{dr/d\theta}$$

36. **(a)** From Exercise 35,

$$\tan\psi = \frac{r}{dr/d\theta} = \frac{1-\cos\theta}{\sin\theta} = \tan\frac{\theta}{2},$$

so $\psi = \theta/2$.

(b)



(c) At $\theta = \pi/2, \psi = \theta/2 = \pi/4$. At $\theta = 3\pi/2, \psi = \theta/2 = 3\pi/4$.

37. $\tan\psi = \dfrac{r}{dr/d\theta} = \dfrac{ae^{b\theta}}{abe^{b\theta}} = \dfrac{1}{b}$ is constant, so ψ is constant.

CHAPTER 11 HORIZON MODULE

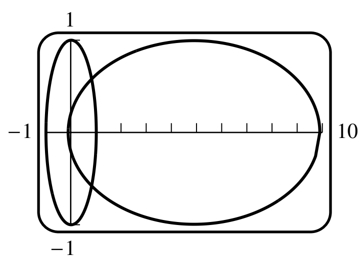
1. For the Earth, $a_E(1-e_E^2) = 1(1-0.017^2) = 0.999711$, so the polar equation is

$$r = \frac{a_E(1-e_E^2)}{1-e_E\cos\theta} = \frac{0.999711}{1-0.017\cos\theta}.$$

For Rogue 2000, $a_R(1-e_R^2) = 5(1-0.98^2) = 0.198$, so the polar equation is

$$r = \frac{a_R(1-e_R^2)}{1-e_R\cos\theta} = \frac{0.198}{1-0.98\cos\theta}.$$

2.



3. At the intersection point A, $\dfrac{k_E}{1-e_E\cos\theta} = \dfrac{k_R}{1-e_R\cos\theta}$, so $k_E - k_E e_R\cos\theta = k_R - k_R e_E\cos\theta$.

Solving for $\cos\theta$ gives $\cos\theta = \dfrac{k_E - k_R}{k_E e_R - k_R e_E}$.

4. From Exercise 1, $k_E = 0.999711$ and $k_R = 0.198$, so

$$\cos\theta = \frac{k_E - k_R}{k_E e_R - k_R e_E} = \frac{0.999711 - 0.198}{0.999711(0.98) - 0.198(0.017)} \approx 0.821130$$

and $\theta = \cos^{-1} 0.821130 \approx 0.607408$ radian.

5. Substituting $\cos\theta \approx 0.821130$ into the polar equation for the Earth gives

$$r \approx \frac{0.999711}{1-0.017(0.821130)} \approx 1.013864,$$

so the polar coordinates of intersection A are approximately $(1.013864, 0.607408)$.

6. By Theorem 11.3.2 the area of the elliptic sector is $\int_{\theta_I}^{\theta_F} \frac{1}{2}r^2\,d\theta$. By Exercise 11.4.53 the area of the entire ellipse is πab, where a is the semimajor axis and b is the semiminor axis. But
$b = \sqrt{a^2 - c^2} = \sqrt{a^2 - (ea)^2} = a\sqrt{1 - e^2}$,

so Formula (1) becomes $\dfrac{t}{T} = \dfrac{\int_{\theta_I}^{\theta_F} r^2\,d\theta}{2\pi a^2\sqrt{1-e^2}}$, which implies Formula (2).

7. In Formula (2) substitute $T = 1, \theta_I = 0$, and $\theta_F \approx 0.607408$, and use the polar equation of the Earth's orbit found in Exercise 1:

$$t = \frac{\int_0^{\theta_F}\left(\dfrac{k_E}{1 - e_E\cos\theta}\right)^2 d\theta}{2\pi\sqrt{1 - e_E^2}} \approx \frac{\int_0^{0.607408}\left(\dfrac{0.999711}{1 - 0.017\cos\theta}\right)^2 d\theta}{2\pi\sqrt{0.999711}} \approx 0.099793 \text{ yr}.$$

Note: This calculation can be done either by numerical integration or by using the integration formula

$$\int \frac{d\theta}{(1 - e\cos\theta)^2} = \frac{2\tan^{-1}\left(\sqrt{\dfrac{1+e}{1-e}}\tan\dfrac{\theta}{2}\right)}{(1-e^2)^{3/2}} + \frac{e\sin\theta}{(1-e^2)(1-e\cos\theta)} + C,$$

obtained by using a CAS or by the substitution $u = \tan(\theta/2)$.

8. In Formula (2) we substitute $T = 5\sqrt{5}$ and $\theta_I = 0.45$, and use the polar equation of Rogue 2000's orbit found in Exercise 1:

$$t = \frac{T\int_{\theta_I}^{\theta_F}\left(\dfrac{a_R(1-e_R^2)}{1 - e_R\cos\theta}\right)^2 d\theta}{2\pi a_R^2\sqrt{1-e_R^2}} = \frac{5\sqrt{5}\int_{0.45}^{\theta_F}\left(\dfrac{a_R(1-e_R^2)}{1 - e_R\cos\theta}\right)^2 d\theta}{2\pi a_R^2\sqrt{1-e_R^2}},$$

so

$$\int_{0.45}^{\theta_F}\left(\frac{a_R(1-e_R^2)}{1 - e_R\cos\theta}\right)^2 d\theta = \frac{2t\pi a_R^2\sqrt{1-e_R^2}}{5\sqrt{5}}.$$

9. **(a)** A CAS shows that

$$\int\left(\frac{a_R(1-e_R^2)}{1 - e_R\cos\theta}\right)^2 d\theta = a_R^2\left(2\sqrt{1-e_R^2}\tan^{-1}\left(\sqrt{\frac{1+e_R}{1-e_R}}\tan\frac{\theta}{2}\right) + \frac{e_R(1-e_R^2)\sin\theta}{1-e_R\cos\theta}\right) + C$$

(b) Evaluating the integral above from $\theta = 0.45$ to $\theta = \theta_F$, setting the result equal to the right side of (3), and simplifying gives

$$\tan^{-1}\left(\sqrt{\frac{1+e_R}{1-e_R}}\tan\frac{\theta}{2}\right) + \frac{e_R\sqrt{1-e_R^2}\sin\theta}{2(1-e_R\cos\theta)}\Bigg]_{0.45}^{\theta_F} = \frac{t\pi}{5\sqrt{5}}.$$

Using the known values of e_R and t, and solving numerically, $\theta_F \approx 0.611346$.

10. Substituting $\theta_F \approx 0.611346$ in the equation for Rogue 2000's orbit gives $r \approx 1.002525$ AU. So the polar coordinates of Rogue 2000 when the Earth is at intersection A are about $(1.002525, 0.611346)$.

11. Substituting the values found in Exercises 5 and 10 into the distance formula in Exercise 67 of Section 11.1 gives $d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)} \approx 0.012014$ AU $\approx 1.797201 \times 10^6$ km.
Since this is less than 4 million kilometers, a notification should be issued. (Incidentally, Rogue 2000's closest approach to the Earth does not occur when the Earth is at A, but about 9 hours earlier, at $t \approx 0.098768$ yr, at which time the distance is about 1.219435 million kilometers.)