CHAPTER 10
Infinite Series

EXERCISE SET 10.1

1. **(a)** $f^{(k)}(x) = (-1)^k e^{-x},\ f^{(k)}(0) = (-1)^k;\ e^{-x} \approx 1 - x + x^2/2$ (quadratic), $e^{-x} \approx 1 - x$ (linear)

(b) $f'(x) = -\sin x, f''(x) = -\cos x, f(0) = 1, f'(0) = 0, f''(0) = -1,$
$\cos x \approx 1 - x^2/2$ (quadratic), $\cos x \approx 1$ (linear)

(c) $f'(x) = \cos x, f''(x) = -\sin x, f(\pi/2) = 1, f'(\pi/2) = 0, f''(\pi/2) = -1,$
$\sin x \approx 1 - (x - \pi/2)^2/2$ (quadratic), $\sin x \approx 1$ (linear)

(d) $f(1) = 1, f'(1) = 1/2, f''(1) = -1/4;$
$\sqrt{x} = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2$ (quadratic), $\sqrt{x} \approx 1 + \frac{1}{2}(x-1)$ (linear)

2. **(a)** $p_2(x) = 1 + x + x^2/2,\ p_1(x) = 1 + x$

(b) $p_2(x) = 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2,\ p_1(x) = 3 + \frac{1}{6}(x-9)$

(c) $p_2(x) = \frac{\pi}{3} + \frac{\sqrt{3}}{6}(x-2) - \frac{7}{72}\sqrt{3}(x-2)^2,\ p_1(x) = \frac{\pi}{3} + \frac{\sqrt{3}}{6}(x-2)$

(d) $p_2(x) = x,\ p_1(x) = x$

3. **(a)** $f'(x) = \frac{1}{2}x^{-1/2},\ f''(x) = -\frac{1}{4}x^{-3/2};\ f(1) = 1, f'(1) = \frac{1}{2}, f''(1) = -\frac{1}{4};$
$\sqrt{x} \approx 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2$

(b) $x = 1.1, x_0 = 1, \sqrt{1.1} \approx 1 + \frac{1}{2}(0.1) - \frac{1}{8}(0.1)^2 = 1.04875$, calculator value ≈ 1.0488088

4. **(a)** $\cos x \approx 1 - x^2/2$

(b) $2^\circ = \pi/90$ rad, $\cos 2^\circ = \cos(\pi/90) \approx 1 - \frac{\pi^2}{2 \cdot 90^2} \approx 0.99939077$, calculator value ≈ 0.99939083

5. $f(x) = \tan x,\ 61^\circ = \pi/3 + \pi/180$ rad; $x_0 = \pi/3,\ f'(x) = \sec^2 x,\ f''(x) = 2\sec^2 x \tan x;$
$f(\pi/3) = \sqrt{3}, f'(\pi/3) = 4, f''(x) = 8\sqrt{3};\ \tan x \approx \sqrt{3} + 4(x - \pi/3) + 4\sqrt{3}(x - \pi/3)^2,$
$\tan 61^\circ = \tan(\pi/3 + \pi/180) \approx \sqrt{3} + 4\pi/180 + 4\sqrt{3}(\pi/180)^2 \approx 1.80397443,$
calculator value ≈ 1.80404776

6. $f(x) = \sqrt{x},\ x_0 = 36, f'(x) = \frac{1}{2}x^{-1/2},\ f''(x) = -\frac{1}{4}x^{-3/2};$
$f(36) = 6, f'(36) = \frac{1}{12}, f''(36) = -\frac{1}{864};\ \sqrt{x} \approx 6 + \frac{1}{12}(x-36) - \frac{1}{1728}(x-36)^2;$
$\sqrt{36.03} \approx 6 + \frac{0.03}{12} - \frac{(0.03)^2}{1728} \approx 6.00249947917$, calculator value ≈ 6.00249947938

7. $f^{(k)}(x) = (-1)^k e^{-x},\ f^{(k)}(0) = (-1)^k; p_0(x) = 1,\ p_1(x) = 1 - x,\ p_2(x) = 1 - x + \frac{1}{2}x^2,$
$p_3(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{3!}x^3,\ p_4(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4;\ \sum_{k=0}^{n} \frac{(-1)^k}{k!}x^k$

8. $f^{(k)}(x) = a^k e^{ax},\ f^{(k)}(0) = a^k;\ p_0(x) = 1,\ p_1(x) = 1 + ax,\ p_2(x) = 1 + ax + \frac{a^2}{2}x^2,$

$p_3(x) = 1 + ax + \frac{a^2}{2}x^2 + \frac{a^3}{3!}x^3,\ p_4(x) = 1 + ax + \frac{a^2}{2}x^2 + \frac{a^3}{3!}x^3 + \frac{a^4}{4!}x^4;\ \sum_{k=0}^{n} \frac{a^k}{k!}x^k$

9. $f^{(k)}(0) = 0$ if k is odd, $f^{(k)}(0)$ is alternately π^k and $-\pi^k$ if k is even; $p_0(x) = 1,\ p_1(x) = 1,$

$p_2(x) = 1 - \frac{\pi^2}{2!}x^2;\ p_3(x) = 1 - \frac{\pi^2}{2!}x^2,\ p_4(x) = 1 - \frac{\pi^2}{2!}x^2 + \frac{\pi^4}{4!}x^4;\ \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k \pi^{2k}}{(2k)!}x^{2k}$

NB: The function $[x]$ defined for real x indicates the greatest integer which is $\leq x$.

10. $f^{(k)}(0) = 0$ if k is even, $f^{(k)}(0)$ is alternately π^k and $-\pi^k$ if k is odd; $p_0(x) = 0,\ p_1(x) = \pi x,$

$p_2(x) = \pi x;\ p_3(x) = \pi x - \frac{\pi^3}{3!}x^3,\ p_4(x) = \pi x - \frac{\pi^3}{3!}x^3;\ \sum_{k=0}^{[\frac{n-1}{2}]} \frac{(-1)^k \pi^{2k+1}}{(2k+1)!}x^{2k+1}$

NB If $n = 0$ then $[\frac{n-1}{2}] = -1$; by definition any sum which runs from $k = 0$ to $k = -1$ is called the 'empty sum' and has value 0.

11. $f^{(0)}(0) = 0$; for $k \geq 1$, $f^{(k)}(x) = \frac{(-1)^{k+1}(k-1)!}{(1+x)^k},\ f^{(k)}(0) = (-1)^{k+1}(k-1)!;\ p_0(x) = 0,$

$p_1(x) = x,\ p_2(x) = x - \frac{1}{2}x^2,\ p_3(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3,\ p_4(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4;\ \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}x^k$

12. $f^{(k)}(x) = (-1)^k \frac{k!}{(1+x)^{k+1}};\ f^{(k)}(0) = (-1)^k k!;\ p_0(x) = 1,\ p_1(x) = 1 - x,$

$p_2(x) = 1 - x + x^2,\ p_3(x) = 1 - x + x^2 - x^3,\ p_4(x) = 1 - x + x^2 - x^3 + x^4;\ \sum_{k=0}^{n}(-1)^k x^k$

13. $f^{(k)}(0) = 0$ if k is odd, $f^{(k)}(0) = 1$ if k is even; $p_0(x) = 1, p_1(x) = 1,$

$p_2(x) = 1 + x^2/2,\ p_3(x) = 1 + x^2/2,\ p_4(x) = 1 + x^2/2 + x^4/4!;\ \sum_{k=0}^{[\frac{n}{2}]} \frac{1}{(2k)!}x^{2k}$

14. $f^{(k)}(0) = 0$ if k is even, $f^{(k)}(0) = 1$ if k is odd; $p_0(x) = 0,\ p_1(x) = x,\ p_2(x) = x,$

$p_3(x) = x + x^3/3!,\ p_4(x) = x + x^3/3!;\ \sum_{k=0}^{[\frac{n-1}{2}]} \frac{1}{(2k+1)!}x^{2k+1}$

15. $f^{(k)}(x) = \begin{cases} (-1)^{k/2}(x\sin x - k\cos x) & k \text{ even} \\ (-1)^{(k-1)/2}(x\cos x + k\sin x) & k \text{ odd} \end{cases}, \qquad f^{(k)}(0) = \begin{cases} (-1)^{1+k/2}k & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$

$p_0(x) = 0,\ p_1(x) = 0,\ p_2(x) = x^2, p_3(x) = x^2,\ p_4(x) = x^2 - \frac{1}{6}x^4;\ \sum_{k=0}^{[\frac{n}{2}]-1} \frac{(-1)^k}{(2k+1)!}x^{2k+2}$

16. $f^{(k)}(x) = (k + x)e^x,\ f^{(k)}(0) = k;\ p_0(x) = 0,\ p_1(x) = x,\ p_2(x) = x + x^2,$

$p_3(x) = x + x^2 + \frac{1}{2}x^3,\ p_4(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{3!}x^4;\ \sum_{k=1}^{n} \frac{1}{(k-1)!}x^k$

17. $f^{(k)}(x_0) = e;\ p_0(x) = e,\ p_1(x) = e + e(x-1),$

$p_2(x) = e + e(x-1) + \frac{e}{2}(x-1)^2,\ p_3(x) = e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{3!}(x-1)^3,$

$p_4(x) = e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{3!}(x-1)^3 + \frac{e}{4!}(x-1)^4;\ \sum_{k=0}^{n} \frac{e}{k!}(x-1)^k$

18. $f^{(k)}(x) = (-1)^k e^{-x},\ f^{(k)}(\ln 2) = (-1)^k \frac{1}{2};\ p_0(x) = \frac{1}{2},\ p_1(x) = \frac{1}{2} - \frac{1}{2}(x - \ln 2),$

$p_2(x) = \frac{1}{2} - \frac{1}{2}(x-\ln 2) + \frac{1}{2\cdot 2}(x-\ln 2)^2,\ p_3(x) = \frac{1}{2} - \frac{1}{2}(x-\ln 2) + \frac{1}{2\cdot 2}(x-\ln 2)^2 - \frac{1}{2\cdot 3!}(x-\ln 2)^3,$

$p_4(x) = \frac{1}{2} - \frac{1}{2}(x-\ln 2) + \frac{1}{2\cdot 2}(x-\ln 2)^2 - \frac{1}{2\cdot 3!}(x-\ln 2)^3 + \frac{1}{2\cdot 4!}(x-\ln 2)^4;$

$\sum_{k=0}^{n} \frac{(-1)^k}{2\cdot k!}(x-\ln 2)^k$

19. $f^{(k)}(x) = \frac{(-1)^k k!}{x^{k+1}},\ f^{(k)}(-1) = -k!;\ p_0(x) = -1;\ p_1(x) = -1 - (x+1);$

$p_2(x) = -1 - (x+1) - (x+1)^2;\ p_3(x) = -1 - (x+1) - (x+1)^2 - (x+1)^3;$

$p_4(x) = -1 - (x+1) - (x+1)^2 - (x+1)^3 - (x+1)^4;\ \sum_{k=0}^{n}(-1)(x+1)^k$

20. $f^{(k)}(x) = \frac{(-1)^k k!}{(x+2)^{k+1}},\ f^{(k)}(3) = \frac{(-1)^k k!}{5^{k+1}};\ p_0(x) = \frac{1}{5};\ p_1(x) = \frac{1}{5} - \frac{1}{25}(x-3);$

$p_2(x) = \frac{1}{5} - \frac{1}{25}(x-3) + \frac{1}{125}(x-3)^2;\ p_3(x) = \frac{1}{5} - \frac{1}{25}(x-3) + \frac{1}{125}(x-3)^2 - \frac{1}{625}(x-3)^3;$

$p_4(x) = \frac{1}{5} - \frac{1}{25}(x-3) + \frac{1}{125}(x-3)^2 - \frac{1}{625}(x-3)^3 + \frac{1}{3125}(x-3)^4;\ \sum_{k=0}^{n} \frac{(-1)^k}{5^{k+1}}(x-3)^k$

21. $f^{(k)}(1/2) = 0$ if k is odd, $f^{(k)}(1/2)$ is alternately π^k and $-\pi^k$ if k is even;

$p_0(x) = p_1(x) = 1, p_2(x) = p_3(x) = 1 - \frac{\pi^2}{2}(x - 1/2)^2,$

$p_4(x) = 1 - \frac{\pi^2}{2}(x-1/2)^2 + \frac{\pi^4}{4!}(x-1/2)^4;\ \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k \pi^{2k}}{(2k)!}(x-1/2)^{2k}$

22. $f^{(k)}(\pi/2) = 0$ if k is even, $f^{(k)}(\pi/2)$ is alternately -1 and 1 if k is odd; $p_0(x) = 0,$

$p_1(x) = -(x-\pi/2),\ p_2(x) = -(x-\pi/2),\ p_3(x) = -(x-\pi/2) + \frac{1}{3!}(x-\pi/2)^3,$

$p_4(x) = -(x-\pi/2) + \frac{1}{3!}(x-\pi/2)^3;\ \sum_{k=0}^{[\frac{n-1}{2}]} \frac{(-1)^{k+1}}{(2k+1)!}(x-\pi/2)^{2k+1}$

23. $f(1) = 0$, for $k \geq 1$, $f^{(k)}(x) = \dfrac{(-1)^{k-1}(k-1)!}{x^k}$; $f^{(k)}(1) = (-1)^{k-1}(k-1)!$;

$p_0(x) = 0,\ p_1(x) = (x-1);\ p_2(x) = (x-1) - \dfrac{1}{2}(x-1)^2;\ p_3(x) = (x-1) - \dfrac{1}{2}(x-1)^2 + \dfrac{1}{3}(x-1)^3,$

$p_4(x) = (x-1) - \dfrac{1}{2}(x-1)^2 + \dfrac{1}{3}(x-1)^3 - \dfrac{1}{4}(x-1)^4;\ \displaystyle\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}(x-1)^k$

24. $f(e) = 1$, for $k \geq 1$, $f^{(k)}(x) = \dfrac{(-1)^{k-1}(k-1)!}{x^k}$; $f^{(k)}(e) = \dfrac{(-1)^{k-1}(k-1)!}{e^k}$;

$p_0(x) = 1,\ p_1(x) = 1 + \dfrac{1}{e}(x-e);\ p_2(x) = 1 + \dfrac{1}{e}(x-e) - \dfrac{1}{2e^2}(x-e)^2;$

$p_3(x) = 1 + \dfrac{1}{e}(x-e) - \dfrac{1}{2e^2}(x-e)^2 + \dfrac{1}{3e^3}(x-e)^3,$

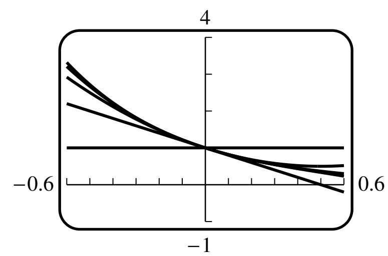
$p_4(x) = 1 + \dfrac{1}{e}(x-e) - \dfrac{1}{2e^2}(x-e)^2 + \dfrac{1}{3e^3}(x-e)^3 - \dfrac{1}{4e^4}(x-e)^4;\ 1 + \displaystyle\sum_{k=1}^{n} \frac{(-1)^{k-1}}{ke^k}(x-e)^k$

25. **(a)** $f(0) = 1, f'(0) = 2, f''(0) = -2, f'''(0) = 6$, the third MacLaurin polynomial for $f(x)$ is $f(x)$.
(b) $f(1) = 1, f'(1) = 2, f''(1) = -2, f'''(1) = 6$, the third Taylor polynomial for $f(x)$ is $f(x)$.

26. **(a)** $f^{(k)}(0) = k!c_k$ for $k \leq n$; the nth Maclaurin polynomial for $f(x)$ is $f(x)$.
(b) $f^{(k)}(x_0) = k!c_k$ for $k \leq n$; the nth Taylor polynomial about $x = 1$ for $f(x)$ is $f(x)$.

27. $f^{(k)}(0) = (-2)^k;\ p_0(x) = 1,\ p_1(x) = 1 - 2x,$

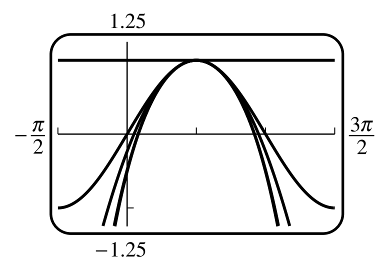
$p_2(x) = 1 - 2x + 2x^2,\ p_3(x) = 1 - 2x + 2x^2 - \dfrac{4}{3}x^3$



28. $f^{(k)}(\pi/2) = 0$ if k is odd, $f^{(k)}(\pi/2)$ is alternately 1 and -1 if k is even; $p_0(x) = 1,\ p_2(x) = 1 - \dfrac{1}{2}(x - \pi/2)^2,$

$p_4(x) = 1 - \dfrac{1}{2}(x - \pi/2)^2 + \dfrac{1}{24}(x - \pi/2)^4,$

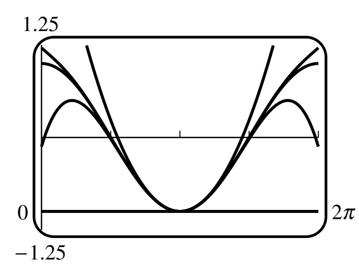
$p_6(x) = 1 - \dfrac{1}{2}(x - \pi/2)^2 + \dfrac{1}{24}(x - \pi/2)^4 - \dfrac{1}{720}(x - \pi/2)^6$



29. $f^{(k)}(\pi) = 0$ if k is odd, $f^{(k)}(\pi)$ is alternately -1 and 1 if k is even; $p_0(x) = -1,\ p_2(x) = -1 + \dfrac{1}{2}(x-\pi)^2,$

$p_4(x) = -1 + \dfrac{1}{2}(x-\pi)^2 - \dfrac{1}{24}(x-\pi)^4,$

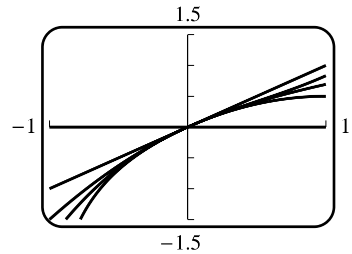
$p_6(x) = -1 + \dfrac{1}{2}(x-\pi)^2 - \dfrac{1}{24}(x-\pi)^4 + \dfrac{1}{720}(x-\pi)^6$



30. $f(0) = 0$; for $k \geq 1$, $f^{(k)}(x) = \dfrac{(-1)^{k-1}(k-1)!}{(x+1)^k}$,

$f^{(k)}(0) = (-1)^{k-1}(k-1)!$; $p_0(x) = 0,\ p_1(x) = x$,

$p_2(x) = x - \dfrac{1}{2}x^2,\ p_3(x) = x - \dfrac{1}{2}x^2 + \dfrac{1}{3}x^3$



31. $f^{(k)}(x) = e^x, |f^{(k)}(x)| \leq e^{1/2} < 2$ on $[0, 1/2]$, let $M = 2$,

$e^{1/2} = 1 + \dfrac{1}{2} + \dfrac{1}{8} + \dfrac{1}{48} + \dfrac{1}{24 \cdot 16} + \cdots + \dfrac{1}{n!2^n} + R_n(1/2)$;

$|R_n(1/2)| \leq \dfrac{M}{(n+1)!}(1/2)^{n+1} \leq \dfrac{2}{(n+1)!}(1/2)^{n+1} \leq 0.00005$ for $n = 5$;

$e^{1/2} \approx 1 + \dfrac{1}{2} + \dfrac{1}{8} + \dfrac{1}{48} + \dfrac{1}{24 \cdot 16} + \dfrac{1}{120 \cdot 32} \approx 1.64870$, calculator value 1.64872

32. $f(x) = e^x, f^{(k)}(x) = e^x, |f^{(k)}(x)| \leq 1$ on $[-1, 0]$, $|R_n(x)| \leq \dfrac{1}{(n+1)!}(1)^{n+1} = \dfrac{1}{(n+1)!} < 0.5 \times 10^{-3}$

if $n = 6$, so $e^{-1} \approx 1 - 1 + \dfrac{1}{2!} - \dfrac{1}{3!} + \dfrac{1}{4!} - \dfrac{1}{5!} + \dfrac{1}{6!} \approx 0.3681$, calculator value 0.3679

33. $p(0) = 1$, $p(x)$ has slope -1 at $x = 0$, and $p(x)$ is concave up at $x = 0$, eliminating I, II and III respectively and leaving IV.

34. Let $p_0(x) = 2,\ p_1(x) = p_2(x) = 2 - 3(x-1),\ p_3(x) = 2 - 3(x-1) + (x-1)^3$.

35. $f^{(k)}(\ln 4) = 15/8$ for k even, $f^{(k)}(\ln 4) = 17/8$ for k odd, which can be written as

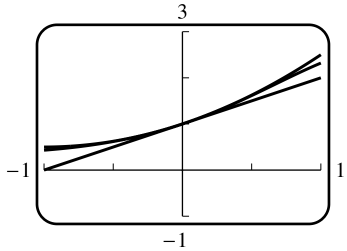
$f^{(k)}(\ln 4) = \dfrac{16 - (-1)^k}{8}$; $\displaystyle\sum_{k=0}^{n} \frac{16 - (-1)^k}{8k!}(x - \ln 4)^k$

36. **(a)** $\cos\alpha \approx 1 - \alpha^2/2$; $x = r - r\cos\alpha = r(1 - \cos\alpha) \approx r\alpha^2/2$

(b) In Figure Ex-36 let $r = 4000$ mi and $\alpha = 1/80$ so that the arc has length $2r\alpha = 100$ mi. Then $x \approx r\alpha^2/2 = \dfrac{4000}{2 \cdot 80^2} = 5/16$ mi.

37. From Exercise 2(a), $p_1(x) = 1 + x$, $p_2(x) = 1 + x + x^2/2$

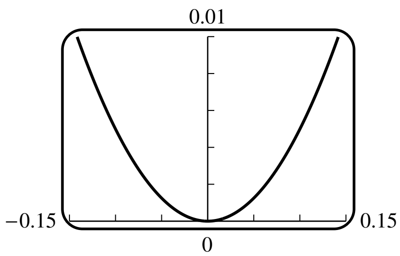
(a)



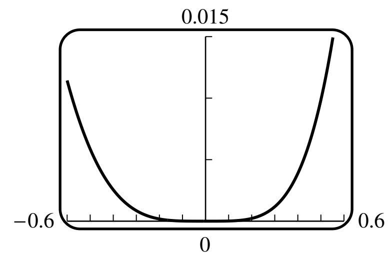
(b)

x	-1.000	-0.750	-0.500	-0.250	0.000	0.250	0.500	0.750	1.000
$f(x)$	0.431	0.506	0.619	0.781	1.000	1.281	1.615	1.977	2.320
$p_1(x)$	0.000	0.250	0.500	0.750	1.000	1.250	1.500	1.750	2.000
$p_2(x)$	0.500	0.531	0.625	0.781	1.000	1.281	1.625	2.031	2.500

(c) $|e^{\sin x} - (1 + x)| < 0.01$ for $-0.14 < x < 0.14$



(d) $|e^{\sin x} - (1 + x + x^2/2)| < 0.01$ for $-0.50 < x < 0.50$

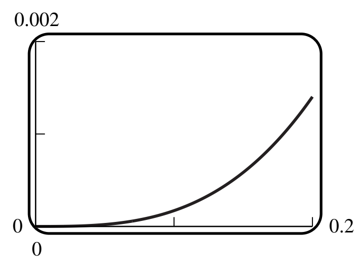


38. **(a)** $f^{(k)}(x) = e^x \le e^b$,

$|R_2(x)| \le \dfrac{e^b b^3}{3!} < 0.0005$,

$e^b b^3 < 0.003$ if $b \le 0.137$ (by trial and error with a hand calculator), so $[0, 0.137]$.

(b)

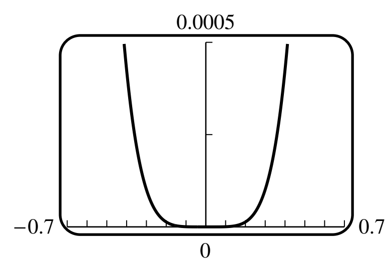


39. **(a)** $\sin x = x - \dfrac{x^3}{3!} + 0 \cdot x^4 + R_4(x)$,

$|R_4(x)| \le \dfrac{|x|^5}{5!} < 0.5 \times 10^{-3}$ if $|x|^5 < 0.06$,

$|x| < (0.06)^{1/5} \approx 0.569, (-0.569, 0.569)$

(b)



EXERCISE SET 10.2

1. **(a)** $\dfrac{1}{3^{n-1}}$ **(b)** $\dfrac{(-1)^{n-1}}{3^{n-1}}$ **(c)** $\dfrac{2n-1}{2n}$ **(d)** $\dfrac{n^2}{\pi^{1/(n+1)}}$

2. **(a)** $(-r)^{n-1}$; $(-r)^n$ **(b)** $(-1)^{n+1}r^n$; $(-1)^n r^{n+1}$

3. **(a)** $2, 0, 2, 0$ **(b)** $1, -1, 1, -1$ **(c)** $2(1 + (-1)^n)$; $2 + 2\cos n\pi$

4. **(a)** $(2n)!$ **(b)** $(2n - 1)!$

5. $1/3,\ 2/4,\ 3/5,\ 4/6,\ 5/7, \ldots$; $\lim_{n\to+\infty} \dfrac{n}{n+2} = 1$, converges

6. $1/3,\ 4/5,\ 9/7,\ 16/9,\ 25/11, \ldots;\ \lim_{n\to+\infty} \dfrac{n^2}{2n+1} = +\infty$, diverges

7. $2, 2, 2, 2, 2, \ldots;\ \lim_{n\to+\infty} 2 = 2$, converges

8. $\ln 1,\ \ln\dfrac{1}{2},\ \ln\dfrac{1}{3},\ \ln\dfrac{1}{4},\ \ln\dfrac{1}{5}, \ldots;\ \lim_{n\to+\infty} \ln(1/n) = -\infty$, diverges

9. $\dfrac{\ln 1}{1},\ \dfrac{\ln 2}{2},\ \dfrac{\ln 3}{3},\ \dfrac{\ln 4}{4},\ \dfrac{\ln 5}{5}, \ldots;$

$\lim_{n\to+\infty} \dfrac{\ln n}{n} = \lim_{n\to+\infty} \dfrac{1}{n} = 0 \left(\text{apply L'Hôpital's Rule to } \dfrac{\ln x}{x}\right)$, converges

10. $\sin\pi,\ 2\sin(\pi/2),\ 3\sin(\pi/3),\ 4\sin(\pi/4),\ 5\sin(\pi/5), \ldots;$

$\lim_{n\to+\infty} n\sin(\pi/n) = \lim_{n\to+\infty} \dfrac{\sin(\pi/n)}{1/n} = \lim_{n\to+\infty} \dfrac{(-\pi/n^2)\cos(\pi/n)}{-1/n^2} = \pi$, converges

11. $0, 2, 0, 2, 0, \ldots;$ diverges

12. $1,\ -1/4,\ 1/9,\ -1/16,\ 1/25, \ldots;\ \lim_{n\to+\infty} \dfrac{(-1)^{n+1}}{n^2} = 0$, converges

13. $-1,\ 16/9,\ -54/28,\ 128/65,\ -250/126, \ldots;$ diverges because odd-numbered terms approach -2, even-numbered terms approach 2.

14. $1/2,\ 2/4,\ 3/8,\ 4/16,\ 5/32, \ldots;\ \lim_{n\to+\infty} \dfrac{n}{2^n} = \lim_{n\to+\infty} \dfrac{1}{2^n \ln 2} = 0$, converges

15. $6/2,\ 12/8,\ 20/18,\ 30/32,\ 42/50, \ldots;\ \lim_{n\to+\infty} \dfrac{1}{2}(1+1/n)(1+2/n) = 1/2$, converges

16. $\pi/4,\ \pi^2/4^2,\ \pi^3/4^3,\ \pi^4/4^4,\ \pi^5/4^5, \ldots;\ \lim_{n\to+\infty} (\pi/4)^n = 0$, converges

17. $\cos(3),\ \cos(3/2),\ \cos(1),\ \cos(3/4),\ \cos(3/5), \ldots;\ \lim_{n\to+\infty} \cos(3/n) = 1$, converges

18. $0,\ -1,\ 0,\ 1,\ 0, \ldots;$ diverges

19. $e^{-1},\ 4e^{-2},\ 9e^{-3},\ 16e^{-4},\ 25e^{-5}, \ldots;\ \lim_{x\to+\infty} x^2 e^{-x} = \lim_{x\to+\infty} \dfrac{x^2}{e^x} = 0$, so $\lim_{n\to+\infty} n^2 e^{-n} = 0$, converges

20. $1,\ \sqrt{10}-2,\ \sqrt{18}-3,\ \sqrt{28}-4,\ \sqrt{40}-5, \ldots;$

$\lim_{n\to+\infty} (\sqrt{n^2+3n} - n) = \lim_{n\to+\infty} \dfrac{3n}{\sqrt{n^2+3n}+n} = \lim_{n\to+\infty} \dfrac{3}{\sqrt{1+3/n}+1} = \dfrac{3}{2}$, converges

21. $2,\ (5/3)^2,\ (6/4)^3,\ (7/5)^4,\ (8/6)^5, \ldots;$ let $y = \left[\dfrac{x+3}{x+1}\right]^x$, converges because

$$\lim_{x\to+\infty} \ln y = \lim_{x\to+\infty} \frac{\ln \dfrac{x+3}{x+1}}{1/x} = \lim_{x\to+\infty} \frac{2x^2}{(x+1)(x+3)} = 2, \text{ so } \lim_{n\to+\infty} \left[\frac{n+3}{n+1}\right]^n = e^2$$

22. $-1,\ 0,\ (1/3)^3,\ (2/4)^4,\ (3/5)^5, \ldots$; let $y = (1-2/x)^x$, converges because

$$\lim_{x\to+\infty} \ln y = \lim_{x\to+\infty} \frac{\ln(1-2/x)}{1/x} = \lim_{x\to+\infty} \frac{-2}{1-2/x} = -2,\ \lim_{n\to+\infty}(1-2/n)^n = \lim_{x\to+\infty} y = e^{-2}$$

23. $\left\{\dfrac{2n-1}{2n}\right\}_{n=1}^{+\infty}$; $\lim\limits_{n\to+\infty} \dfrac{2n-1}{2n} = 1$, converges

24. $\left\{\dfrac{n-1}{n^2}\right\}_{n=1}^{+\infty}$; $\lim\limits_{n\to+\infty} \dfrac{n-1}{n^2} = 0$, converges

25. $\left\{\dfrac{1}{3^n}\right\}_{n=1}^{+\infty}$; $\lim\limits_{n\to+\infty} \dfrac{1}{3^n} = 0$, converges

26. $\{(-1)^n n\}_{n=1}^{+\infty}$; diverges because odd-numbered terms tend toward $-\infty$, even-numbered terms tend toward $+\infty$.

27. $\left\{\dfrac{1}{n} - \dfrac{1}{n+1}\right\}_{n=1}^{+\infty}$; $\lim\limits_{n\to+\infty}\left(\dfrac{1}{n} - \dfrac{1}{n+1}\right) = 0$, converges

28. $\{3/2^{n-1}\}_{n=1}^{+\infty}$; $\lim\limits_{n\to+\infty} 3/2^{n-1} = 0$, converges

29. $\{\sqrt{n+1} - \sqrt{n+2}\}_{n=1}^{+\infty}$; converges because

$$\lim_{n\to+\infty}(\sqrt{n+1} - \sqrt{n+2}) = \lim_{n\to+\infty} \frac{(n+1)-(n+2)}{\sqrt{n+1}+\sqrt{n+2}} = \lim_{n\to+\infty} \frac{-1}{\sqrt{n+1}+\sqrt{n+2}} = 0$$

30. $\{(-1)^{n+1}/3^{n+4}\}_{n=1}^{+\infty}$; $\lim\limits_{n\to+\infty}(-1)^{n+1}/3^{n+4} = 0$, converges

31. **(a)** $1, 2, 1, 4, 1, 6$ **(b)** $a_n = \begin{cases} n, & n \text{ odd} \\ 1/2^n, & n \text{ even} \end{cases}$ **(c)** $a_n = \begin{cases} 1/n, & n \text{ odd} \\ 1/(n+1), & n \text{ even} \end{cases}$

(d) In Part (a) the sequence diverges, since the even terms diverge to $+\infty$ and the odd terms equal 1; in Part (b) the sequence diverges, since the odd terms diverge to $+\infty$ and the even terms tend to zero; in Part (c) $\lim\limits_{n\to+\infty} a_n = 0$.

32. The even terms are zero, so the odd terms must converge to zero, and this is true if and only if $\lim\limits_{n\to+\infty} b^n = 0$, or $-1 < b < 1$.

33. $\lim\limits_{n\to+\infty} \sqrt[n]{n} = 1$, so $\lim\limits_{n\to+\infty} \sqrt[n]{n^3} = 1^3 = 1$

35. $\lim\limits_{n\to+\infty} x_{n+1} = \dfrac{1}{2}\lim\limits_{n\to+\infty}\left(x_n + \dfrac{a}{x_n}\right)$ or $L = \dfrac{1}{2}\left(L + \dfrac{a}{L}\right), 2L^2 - L^2 - a = 0, L = \sqrt{a}$ (we reject $-\sqrt{a}$ because $x_n > 0$, thus $L \ge 0$.

36. **(a)** $a_{n+1} = \sqrt{6 + a_n}$

(b) $\lim\limits_{n\to+\infty} a_{n+1} = \lim\limits_{n\to+\infty} \sqrt{6+a_n}$, $L = \sqrt{6+L}$, $L^2 - L - 6 = 0$, $(L-3)(L+2) = 0$,

$L = -2$ (reject, because the terms in the sequence are positive) or $L = 3$; $\lim\limits_{n\to+\infty} a_n = 3$.

37. **(a)** $1, \frac{1}{4}+\frac{2}{4}, \frac{1}{9}+\frac{2}{9}+\frac{3}{9}, \frac{1}{16}+\frac{2}{16}+\frac{3}{16}+\frac{4}{16} = 1, \frac{3}{4}, \frac{2}{3}, \frac{5}{8}$

(c) $a_n = \frac{1}{n^2}(1+2+\cdots+n) = \frac{1}{n^2}\frac{1}{2}n(n+1) = \frac{1}{2}\frac{n+1}{n}, \lim_{n\to+\infty} a_n = 1/2$

38. **(a)** $1, \frac{1}{8}+\frac{4}{8}, \frac{1}{27}+\frac{4}{27}+\frac{9}{27}, \frac{1}{64}+\frac{4}{64}+\frac{9}{64}+\frac{16}{64} = 1, \frac{5}{8}, \frac{14}{27}, \frac{15}{32}$

(c) $a_n = \frac{1}{n^3}(1^2+2^2+\cdots+n^2) = \frac{1}{n^3}\frac{1}{6}n(n+1)(2n+1) = \frac{1}{6}\frac{(n+1)(2n+1)}{n^2},$

$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty}\frac{1}{6}(1+1/n)(2+1/n) = 1/3$

39. Let $a_n = 0, b_n = \frac{\sin^2 n}{n}, c_n = \frac{1}{n}$; then $a_n \le b_n \le c_n$, $\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} c_n = 0$, so $\lim_{n\to+\infty} b_n = 0$.

40. Let $a_n = 0, b_n = \left(\frac{1+n}{2n}\right)^n, c_n = \left(\frac{3}{4}\right)^n$; then (for $n \ge 2$), $a_n \le b_n \le \left(\frac{n/2+n}{2n}\right)^n = c_n$,
$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} c_n = 0$, so $\lim_{n\to+\infty} b_n = 0$.

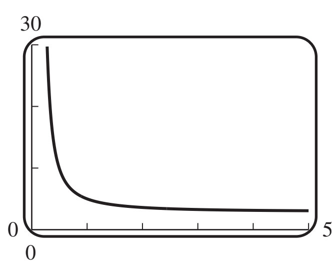
41. **(a)** $a_1 = (0.5)^2, a_2 = a_1^2 = (0.5)^4, \ldots, a_n = (0.5)^{2^n}$

(c) $\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} e^{2^n \ln(0.5)} = 0$, since $\ln(0.5) < 0$.

(d) Replace 0.5 in Part (a) with a_0; then the sequence converges for $-1 \le a_0 \le 1$, because if $a_0 = \pm 1$, then $a_n = 1$ for $n \ge 1$; if $a_0 = 0$ then $a_n = 0$ for $n \ge 1$; and if $0 < |a_0| < 1$ then $a_1 = a_0^2 > 0$ and $\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} e^{2^{n-1}\ln a_1} = 0$ since $0 < a_1 < 1$. This same argument proves divergence to $+\infty$ for $|a| > 1$ since then $\ln a_1 > 0$.

42. $f(0.2) = 0.4,\ f(0.4) = 0.8,\ f(0.8) = 0.6,\ f(0.6) = 0.2$ and then the cycle repeats, so the sequence does not converge.

43. **(a)**



(b) Let $y = (2^x+3^x)^{1/x}$, $\lim_{x\to+\infty} \ln y = \lim_{x\to+\infty}\frac{\ln(2^x+3^x)}{x} = \lim_{x\to+\infty}\frac{2^x\ln 2 + 3^x \ln 3}{2^x+3^x}$

$= \lim_{x\to+\infty}\frac{(2/3)^x\ln 2 + \ln 3}{(2/3)^x + 1} = \ln 3$, so $\lim_{n\to+\infty}(2^n+3^n)^{1/n} = e^{\ln 3} = 3$

Alternate proof: $3 = (3^n)^{1/n} < (2^n+3^n)^{1/n} < (2\cdot 3^n)^{1/n} = 3\cdot 2^{1/n}$. Then apply the Squeezing Theorem.

44. Let $f(x) = 1/(1+x)$, $0 \le x \le 1$. Take $\Delta x_k = 1/n$ and $x_k^* = k/n$ then

$$a_n = \sum_{k=1}^{n}\frac{1}{1+(k/n)}(1/n) = \sum_{k=1}^{n}\frac{1}{1+x_k^*}\Delta x_k \text{ so } \lim_{n\to+\infty} a_n = \int_0^1 \frac{1}{1+x}dx = \ln(1+x)\Big]_0^1 = \ln 2$$

45. $a_n = \dfrac{1}{n-1}\displaystyle\int_1^n \dfrac{1}{x}\,dx = \dfrac{\ln n}{n-1}$, $\displaystyle\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} \dfrac{\ln n}{n-1} = \lim_{n\to+\infty} \dfrac{1}{n} = 0$,
$\left(\text{apply L'Hôpital's Rule to } \dfrac{\ln n}{n-1}\right)$, converges

46. **(a)** If $n \geq 1$, then $a_{n+2} = a_{n+1} + a_n$, so $\dfrac{a_{n+2}}{a_{n+1}} = 1 + \dfrac{a_n}{a_{n+1}}$.

(c) With $L = \displaystyle\lim_{n\to+\infty}(a_{n+2}/a_{n+1}) = \lim_{n\to+\infty}(a_{n+1}/a_n)$, $L = 1 + 1/L$, $L^2 - L - 1 = 0$,
$L = (1 \pm \sqrt{5})/2$, so $L = (1+\sqrt{5})/2$ because the limit cannot be negative.

47. $\left|\dfrac{1}{n} - 0\right| = \dfrac{1}{n} < \epsilon$ if $n > 1/\epsilon$

(a) $1/\epsilon = 1/0.5 = 2$, $N = 3$ **(b)** $1/\epsilon = 1/0.1 = 10$, $N = 11$

(c) $1/\epsilon = 1/0.001 = 1000$, $N = 1001$

48. $\left|\dfrac{n}{n+1} - 1\right| = \dfrac{1}{n+1} < \epsilon$ if $n + 1 > 1/\epsilon$, $n > 1/\epsilon - 1$

(a) $1/\epsilon - 1 = 1/0.25 - 1 = 3$, $N = 4$ **(b)** $1/\epsilon - 1 = 1/0.1 - 1 = 9$, $N = 10$

(c) $1/\epsilon - 1 = 1/0.001 - 1 = 999$, $N = 1000$

49. **(a)** $\left|\dfrac{1}{n} - 0\right| = \dfrac{1}{n} < \epsilon$ if $n > 1/\epsilon$, choose any $N > 1/\epsilon$.

(b) $\left|\dfrac{n}{n+1} - 1\right| = \dfrac{1}{n+1} < \epsilon$ if $n > 1/\epsilon - 1$, choose any $N > 1/\epsilon - 1$.

50. If $|r| < 1$ then $\displaystyle\lim_{n\to+\infty} r^n = 0$; if $r > 1$ then $\displaystyle\lim_{n\to+\infty} r^n = +\infty$, if $r < -1$ then r^n oscillates between positive and negative values that grow in magnitude so $\displaystyle\lim_{n\to+\infty} r^n$ does not exist for $|r| > 1$; if $r = 1$ then $\displaystyle\lim_{n\to+\infty} 1^n = 1$; if $r = -1$ then $(-1)^n$ oscillates between -1 and 1 so $\displaystyle\lim_{n\to+\infty}(-1)^n$ does not exist.

EXERCISE SET 10.3

1. $a_{n+1} - a_n = \dfrac{1}{n+1} - \dfrac{1}{n} = -\dfrac{1}{n(n+1)} < 0$ for $n \geq 1$, so strictly decreasing.

2. $a_{n+1} - a_n = (1 - \dfrac{1}{n+1}) - (1 - \dfrac{1}{n}) = \dfrac{1}{n(n+1)} > 0$ for $n \geq 1$, so strictly increasing.

3. $a_{n+1} - a_n = \dfrac{n+1}{2n+3} - \dfrac{n}{2n+1} = \dfrac{1}{(2n+1)(2n+3)} > 0$ for $n \geq 1$, so strictly increasing.

4. $a_{n+1} - a_n = \dfrac{n+1}{4n+3} - \dfrac{n}{4n-1} = -\dfrac{1}{(4n-1)(4n+3)} < 0$ for $n \geq 1$, so strictly decreasing.

5. $a_{n+1} - a_n = (n + 1 - 2^{n+1}) - (n - 2^n) = 1 - 2^n < 0$ for $n \geq 1$, so strictly decreasing.

6. $a_{n+1} - a_n = [(n+1) - (n+1)^2] - (n - n^2) = -2n < 0$ for $n \geq 1$, so strictly decreasing.

7. $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)/(2n+3)}{n/(2n+1)} = \dfrac{(n+1)(2n+1)}{n(2n+3)} = \dfrac{2n^2+3n+1}{2n^2+3n} > 1$ for $n \geq 1$, so strictly increasing.

8. $\dfrac{a_{n+1}}{a_n} = \dfrac{2^{n+1}}{1+2^{n+1}} \cdot \dfrac{1+2^n}{2^n} = \dfrac{2+2^{n+1}}{1+2^{n+1}} = 1 + \dfrac{1}{1+2^{n+1}} > 1$ for $n \geq 1$, so strictly increasing.

9. $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)e^{-(n+1)}}{ne^{-n}} = (1+1/n)e^{-1} < 1$ for $n \geq 1$, so strictly decreasing.

10. $\dfrac{a_{n+1}}{a_n} = \dfrac{10^{n+1}}{(2n+2)!} \cdot \dfrac{(2n)!}{10^n} = \dfrac{10}{(2n+2)(2n+1)} < 1$ for $n \geq 1$, so strictly decreasing.

11. $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)^{n+1}}{(n+1)!} \cdot \dfrac{n!}{n^n} = \dfrac{(n+1)^n}{n^n} = (1+1/n)^n > 1$ for $n \geq 1$, so strictly increasing.

12. $\dfrac{a_{n+1}}{a_n} = \dfrac{5^{n+1}}{2^{(n+1)^2}} \cdot \dfrac{2^{n^2}}{5^n} = \dfrac{5}{2^{2n+1}} < 1$ for $n \geq 1$, so strictly decreasing.

13. $f(x) = x/(2x+1)$, $f'(x) = 1/(2x+1)^2 > 0$ for $x \geq 1$, so strictly increasing.

14. $f(x) = 3 - 1/x$, $f'(x) = 1/x^2 > 0$ for $x \geq 1$, so strictly increasing.

15. $f(x) = 1/(x + \ln x)$, $f'(x) = -\dfrac{1+1/x}{(x+\ln x)^2} < 0$ for $x \geq 1$, so strictly decreasing.

16. $f(x) = xe^{-2x}$, $f'(x) = (1-2x)e^{-2x} < 0$ for $x \geq 1$, so strictly decreasing.

17. $f(x) = \dfrac{\ln(x+2)}{x+2}$, $f'(x) = \dfrac{1-\ln(x+2)}{(x+2)^2} < 0$ for $x \geq 1$, so strictly decreasing.

18. $f(x) = \tan^{-1} x$, $f'(x) = 1/(1+x^2) > 0$ for $x \geq 1$, so strictly increasing.

19. $f(x) = 2x^2 - 7x$, $f'(x) = 4x - 7 > 0$ for $x \geq 2$, so eventually strictly increasing.

20. $f(x) = x^3 - 4x^2$, $f'(x) = 3x^2 - 8x = x(3x-8) > 0$ for $x \geq 3$, so eventually strictly increasing.

21. $f(x) = \dfrac{x}{x^2+10}$, $f'(x) = \dfrac{10-x^2}{(x^2+10)^2} < 0$ for $x \geq 4$, so eventually strictly decreasing.

22. $f(x) = x + \dfrac{17}{x}$, $f'(x) = \dfrac{x^2-17}{x^2} > 0$ for $x \geq 5$, so eventually strictly increasing.

23. $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)!}{3^{n+1}} \cdot \dfrac{3^n}{n!} = \dfrac{n+1}{3} > 1$ for $n \geq 3$, so eventually strictly increasing.

24. $f(x) = x^5 e^{-x}$, $f'(x) = x^4(5-x)e^{-x} < 0$ for $x \geq 6$, so eventually strictly decreasing.

25. **(a)** Yes: a monotone sequence is increasing or decreasing; if it is increasing, then it is increasing and bounded above, so by Theorem 10.3.3 it converges; if decreasing, then use Theorem 10.3.4. The limit lies in the interval $[1, 2]$.

(b) Such a sequence may converge, in which case, by the argument in Part (a), its limit is ≤ 2. But convergence may not happen: for example, the sequence $\{-n\}_{n=1}^{+\infty}$ diverges.

26. **(a)** $a_{n+1} = \dfrac{|x|^{n+1}}{(n+1)!} = \dfrac{|x|}{n+1}\dfrac{|x|^n}{n!} = \dfrac{|x|}{n+1}a_n$

(b) $a_{n+1}/a_n = |x|/(n+1) < 1$ if $n > |x| - 1$.

(c) From Part (b) the sequence is eventually decreasing, and it is bounded below by 0, so by Theorem 10.3.4 it converges.

(d) If $\lim\limits_{n\to+\infty} a_n = L$ then from Part (a), $L = \dfrac{|x|}{\lim\limits_{n\to+\infty}(n+1)}L = 0$.

(e) $\lim\limits_{n\to+\infty} \dfrac{|x|^n}{n!} = \lim\limits_{n\to+\infty} a_n = 0$

27. **(a)** $\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}$

(b) $a_1 = \sqrt{2} < 2$ so $a_2 = \sqrt{2+a_1} < \sqrt{2+2} = 2$, $a_3 = \sqrt{2+a_2} < \sqrt{2+2} = 2$, and so on indefinitely.

(c) $a_{n+1}^2 - a_n^2 = (2+a_n) - a_n^2 = 2 + a_n - a_n^2 = (2-a_n)(1+a_n)$

(d) $a_n > 0$ and, from Part (b), $a_n < 2$ so $2 - a_n > 0$ and $1 + a_n > 0$ thus, from Part (c), $a_{n+1}^2 - a_n^2 > 0$, $a_{n+1} - a_n > 0$, $a_{n+1} > a_n$; $\{a_n\}$ is a strictly increasing sequence.

(e) The sequence is increasing and has 2 as an upper bound so it must converge to a limit L, $\lim\limits_{n\to+\infty} a_{n+1} = \lim\limits_{n\to+\infty} \sqrt{2+a_n}$, $L = \sqrt{2+L}$, $L^2 - L - 2 = 0$, $(L-2)(L+1) = 0$ thus $\lim\limits_{n\to+\infty} a_n = 2$.

28. **(a)** If $f(x) = \frac{1}{2}(x + 3/x)$, then $f'(x) = (x^2-3)/(2x^2)$ and $f'(x) = 0$ for $x = \sqrt{3}$; the minimum value of $f(x)$ for $x > 0$ is $f(\sqrt{3}) = \sqrt{3}$. Thus $f(x) \geq \sqrt{3}$ for $x > 0$ and hence $a_n \geq \sqrt{3}$ for $n \geq 2$.

(b) $a_{n+1} - a_n = (3 - a_n^2)/(2a_n) \leq 0$ for $n \geq 2$ since $a_n \geq \sqrt{3}$ for $n \geq 2$; $\{a_n\}$ is eventually decreasing.

(c) $\sqrt{3}$ is a lower bound for a_n so $\{a_n\}$ converges; $\lim\limits_{n\to+\infty} a_{n+1} = \lim\limits_{n\to+\infty} \frac{1}{2}(a_n + 3/a_n)$, $L = \frac{1}{2}(L + 3/L)$, $L^2 - 3 = 0$, $L = \sqrt{3}$.

29. **(a)** The altitudes of the rectangles are $\ln k$ for $k = 2$ to n, and their bases all have length 1 so the sum of their areas is $\ln 2 + \ln 3 + \cdots + \ln n = \ln(2\cdot 3\cdots n) = \ln n!$. The area under the curve $y = \ln x$ for x in the interval $[1,n]$ is $\int_1^n \ln x\,dx$, and $\int_1^{n+1} \ln x\,dx$ is the area for x in the interval $[1, n+1]$ so, from the figure, $\int_1^n \ln x\,dx < \ln n! < \int_1^{n+1} \ln x\,dx$.

(b) $\int_1^n \ln x\,dx = (x\ln x - x)\Big]_1^n = n\ln n - n + 1$ and $\int_1^{n+1} \ln x\,dx = (n+1)\ln(n+1) - n$ so from Part (a), $n\ln n - n + 1 < \ln n! < (n+1)\ln(n+1) - n$, $e^{n\ln n - n + 1} < n! < e^{(n+1)\ln(n+1) - n}$, $e^{n\ln n}e^{1-n} < n! < e^{(n+1)\ln(n+1)}e^{-n}$, $\dfrac{n^n}{e^{n-1}} < n! < \dfrac{(n+1)^{n+1}}{e^n}$

(c) From Part (b), $\left[\dfrac{n^n}{e^{n-1}}\right]^{1/n} < \sqrt[n]{n!} < \left[\dfrac{(n+1)^{n+1}}{e^n}\right]^{1/n}$,

$\dfrac{n}{e^{1-1/n}} < \sqrt[n]{n!} < \dfrac{(n+1)^{1+1/n}}{e}$, $\dfrac{1}{e^{1-1/n}} < \dfrac{\sqrt[n]{n!}}{n} < \dfrac{(1+1/n)(n+1)^{1/n}}{e}$,

but $\dfrac{1}{e^{1-1/n}} \to \dfrac{1}{e}$ and $\dfrac{(1+1/n)(n+1)^{1/n}}{e} \to \dfrac{1}{e}$ as $n \to +\infty$ (why?), so $\lim\limits_{n\to+\infty} \dfrac{\sqrt[n]{n!}}{n} = \dfrac{1}{e}$.

30. $n! > \dfrac{n^n}{e^{n-1}}$, $\sqrt[n]{n!} > \dfrac{n}{e^{1-1/n}}$, $\lim_{n\to+\infty} \dfrac{n}{e^{1-1/n}} = +\infty$ so $\lim_{n\to+\infty} \sqrt[n]{n!} = +\infty$.

EXERCISE SET 10.4

1. **(a)** $s_1 = 2$, $s_2 = 12/5$, $s_3 = \dfrac{62}{25}$, $s_4 = \dfrac{312}{125}$ $s_n = \dfrac{2-2(1/5)^n}{1-1/5} = \dfrac{5}{2} - \dfrac{5}{2}(1/5)^n$,

$\lim_{n\to+\infty} s_n = \dfrac{5}{2}$, converges

(b) $s_1 = \dfrac{1}{4}$, $s_2 = \dfrac{3}{4}$, $s_3 = \dfrac{7}{4}$, $s_4 = \dfrac{15}{4}$ $s_n = \dfrac{(1/4)-(1/4)2^n}{1-2} = -\dfrac{1}{4} + \dfrac{1}{4}(2^n)$,

$\lim_{n\to+\infty} s_n = +\infty$, diverges

(c) $\dfrac{1}{(k+1)(k+2)} = \dfrac{1}{k+1} - \dfrac{1}{k+2}$, $s_1 = \dfrac{1}{6}$, $s_2 = \dfrac{1}{4}$, $s_3 = \dfrac{3}{10}$, $s_4 = \dfrac{1}{3}$;

$s_n = \dfrac{1}{2} - \dfrac{1}{n+2}$, $\lim_{n\to+\infty} s_n = \dfrac{1}{2}$, converges

2. **(a)** $s_1 = 1/4, s_2 = 5/16, s_3 = 21/64, s_4 = 85/256$

$$s_n = \frac{1}{4}\left(1 + \frac{1}{4} + \cdots + \left(\frac{1}{4}\right)^{n-1}\right) = \frac{1}{4}\frac{1-(1/4)^n}{1-1/4} = \frac{1}{3}\left(1 - \left(\frac{1}{4}\right)^n\right);\ \lim_{n\to+\infty} s_n = \frac{1}{3}$$

(b) $s_1 = 1, s_2 = 5, s_3 = 21, s_4 = 85$; $s_n = \dfrac{4^n - 1}{3}$, diverges

(c) $s_1 = 1/20, s_2 = 1/12, s_3 = 3/28, s_4 = 1/8$;

$$s_n = \sum_{k=1}^{n}\left(\frac{1}{k+3} - \frac{1}{k+4}\right) = \frac{1}{4} - \frac{1}{n+4},\ \lim_{n\to+\infty} s_n = 1/4$$

3. geometric, $a = 1$, $r = -3/4$, sum $= \dfrac{1}{1-(-3/4)} = 4/7$

4. geometric, $a = (2/3)^3$, $r = 2/3$, sum $= \dfrac{(2/3)^3}{1-2/3} = 8/9$

5. geometric, $a = 7$, $r = -1/6$, sum $= \dfrac{7}{1+1/6} = 6$

6. geometric, $r = -3/2$, diverges

7. $s_n = \displaystyle\sum_{k=1}^{n}\left(\frac{1}{k+2} - \frac{1}{k+3}\right) = \frac{1}{3} - \frac{1}{n+3},\ \lim_{n\to+\infty} s_n = 1/3$

8. $s_n = \displaystyle\sum_{k=1}^{n}\left(\frac{1}{2^k} - \frac{1}{2^{k+1}}\right) = \frac{1}{2} - \frac{1}{2^{n+1}},\ \lim_{n\to+\infty} s_n = 1/2$

9. $s_n = \displaystyle\sum_{k=1}^{n}\left(\frac{1/3}{3k-1} - \frac{1/3}{3k+2}\right) = \frac{1}{6} - \frac{1/3}{3n+2},\ \lim_{n\to+\infty} s_n = 1/6$

10. $s_n = \sum_{k=2}^{n+1}\left[\frac{1/2}{k-1} - \frac{1/2}{k+1}\right] = \frac{1}{2}\left[\sum_{k=2}^{n+1}\frac{1}{k-1} - \sum_{k=2}^{n+1}\frac{1}{k+1}\right]$

$= \frac{1}{2}\left[\sum_{k=2}^{n+1}\frac{1}{k-1} - \sum_{k=4}^{n+3}\frac{1}{k-1}\right] = \frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right]$; $\lim_{n\to+\infty} s_n = \frac{3}{4}$

11. $\sum_{k=3}^{\infty}\frac{1}{k-2} = \sum_{k=1}^{\infty} 1/k$, the harmonic series, so the series diverges.

12. geometric, $a = (e/\pi)^4$, $r = e/\pi < 1$, sum $= \frac{(e/\pi)^4}{1 - e/\pi} = \frac{e^4}{\pi^3(\pi - e)}$

13. $\sum_{k=1}^{\infty}\frac{4^{k+2}}{7^{k-1}} = \sum_{k=1}^{\infty} 64\left(\frac{4}{7}\right)^{k-1}$; geometric, $a = 64$, $r = 4/7$, sum $= \frac{64}{1 - 4/7} = 448/3$

14. geometric, $a = 125, r = 125/7$, diverges

15. $0.4444\cdots = 0.4 + 0.04 + 0.004 + \cdots = \frac{0.4}{1 - 0.1} = 4/9$

16. $0.9999\cdots = 0.9 + 0.09 + 0.009 + \cdots = \frac{0.9}{1 - 0.1} = 1$

17. $5.373737\cdots = 5 + 0.37 + 0.0037 + 0.000037 + \cdots = 5 + \frac{0.37}{1 - 0.01} = 5 + 37/99 = 532/99$

18. $0.159159159\cdots = 0.159 + 0.000159 + 0.000000159 + \cdots = \frac{0.159}{1 - 0.001} = 159/999 = 53/333$

19. $0.782178217821\cdots = 0.7821 + 0.00007821 + 0.000000007821 + \cdots = \frac{0.7821}{1 - 0.0001} = \frac{7821}{9999} = \frac{79}{101}$

20. $0.451141414\cdots = 0.451 + 0.00014 + 0.0000014 + 0.000000014 + \cdots = 0.451 + \frac{0.00014}{1 - 0.01} = \frac{44663}{99000}$

21. $d = 10 + 2\cdot\frac{3}{4}\cdot 10 + 2\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot 10 + 2\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot 10 + \cdots$

$= 10 + 20\left(\frac{3}{4}\right) + 20\left(\frac{3}{4}\right)^2 + 20\left(\frac{3}{4}\right)^3 + \cdots = 10 + \frac{20(3/4)}{1 - 3/4} = 10 + 60 = 70$ meters

22. volume $= 1^3 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{4}\right)^3 + \cdots + \left(\frac{1}{2^n}\right)^3 + \cdots = 1 + \frac{1}{8} + \left(\frac{1}{8}\right)^2 + \cdots + \left(\frac{1}{8}\right)^n + \cdots$

$= \frac{1}{1 - (1/8)} = 8/7$

23. **(a)** $s_n = \ln\frac{1}{2} + \ln\frac{2}{3} + \ln\frac{3}{4} + \cdots + \ln\frac{n}{n+1} = \ln\left(\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdots\frac{n}{n+1}\right) = \ln\frac{1}{n+1} = -\ln(n+1)$,
$\lim_{n\to+\infty} s_n = -\infty$, series diverges.

(b) $\ln(1-1/k^2) = \ln\dfrac{k^2-1}{k^2} = \ln\dfrac{(k-1)(k+1)}{k^2} = \ln\dfrac{k-1}{k} + \ln\dfrac{k+1}{k} = \ln\dfrac{k-1}{k} - \ln\dfrac{k}{k+1}$,

$$s_n = \sum_{k=2}^{n+1}\left[\ln\frac{k-1}{k} - \ln\frac{k}{k+1}\right]$$

$$= \left(\ln\frac{1}{2} - \ln\frac{2}{3}\right) + \left(\ln\frac{2}{3} - \ln\frac{3}{4}\right) + \left(\ln\frac{3}{4} - \ln\frac{4}{5}\right) + \cdots + \left(\ln\frac{n}{n+1} - \ln\frac{n+1}{n+2}\right)$$

$$= \ln\frac{1}{2} - \ln\frac{n+1}{n+2},\ \lim_{n\to+\infty} s_n = \ln\frac{1}{2} = -\ln 2$$

24. (a) $\displaystyle\sum_{k=0}^{\infty}(-1)^k x^k = 1 - x + x^2 - x^3 + \cdots = \frac{1}{1-(-x)} = \frac{1}{1+x}$ if $|-x| < 1$, $|x| < 1$, $-1 < x < 1$.

(b) $\displaystyle\sum_{k=0}^{\infty}(x-3)^k = 1 + (x-3) + (x-3)^2 + \cdots = \frac{1}{1-(x-3)} = \frac{1}{4-x}$ if $|x-3| < 1$, $2 < x < 4$.

(c) $\displaystyle\sum_{k=0}^{\infty}(-1)^k x^{2k} = 1 - x^2 + x^4 - x^6 + \cdots = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}$ if $|-x^2| < 1$, $|x| < 1$, $-1 < x < 1$.

25. (a) Geometric series, $a = x$, $r = -x^2$. Converges for $|-x^2| < 1$, $|x| < 1$;
$S = \dfrac{x}{1-(-x^2)} = \dfrac{x}{1+x^2}$.

(b) Geometric series, $a = 1/x^2$, $r = 2/x$. Converges for $|2/x| < 1$, $|x| > 2$;
$S = \dfrac{1/x^2}{1-2/x} = \dfrac{1}{x^2-2x}$.

(c) Geometric series, $a = e^{-x}$, $r = e^{-x}$. Converges for $|e^{-x}| < 1$, $e^{-x} < 1$, $e^x > 1$, $x > 0$;
$S = \dfrac{e^{-x}}{1-e^{-x}} = \dfrac{1}{e^x - 1}$.

26. $\dfrac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k^2+k}} = \dfrac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k}\sqrt{k+1}} = \dfrac{1}{\sqrt{k}} - \dfrac{1}{\sqrt{k+1}}$,

$$s_n = \sum_{k=1}^{n}\left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right) = \left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}}\right)$$

$$+ \cdots + \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) = 1 - \frac{1}{\sqrt{n+1}};\ \lim_{n\to+\infty} s_n = 1$$

27. $s_n = (1-1/3) + (1/2-1/4) + (1/3-1/5) + (1/4-1/6) + \cdots + [1/n - 1/(n+2)]$

$= (1 + 1/2 + 1/3 + \cdots + 1/n) - (1/3 + 1/4 + 1/5 + \cdots + 1/(n+2))$

$= 3/2 - 1/(n+1) - 1/(n+2)$, $\lim_{n\to+\infty} s_n = 3/2$

28. $$s_n = \sum_{k=1}^{n}\frac{1}{k(k+2)} = \sum_{k=1}^{n}\left[\frac{1/2}{k} - \frac{1/2}{k+2}\right] = \frac{1}{2}\left[\sum_{k=1}^{n}\frac{1}{k} - \sum_{k=1}^{n}\frac{1}{k+2}\right]$$

$$= \frac{1}{2}\left[\sum_{k=1}^{n}\frac{1}{k} - \sum_{k=3}^{n+2}\frac{1}{k}\right] = \frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right];\ \lim_{n\to+\infty} s_n = \frac{3}{4}$$

29. $s_n = \sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \sum_{k=1}^{n} \left[\frac{1/2}{2k-1} - \frac{1/2}{2k+1}\right] = \frac{1}{2}\left[\sum_{k=1}^{n} \frac{1}{2k-1} - \sum_{k=1}^{n} \frac{1}{2k+1}\right]$

$= \frac{1}{2}\left[\sum_{k=1}^{n} \frac{1}{2k-1} - \sum_{k=2}^{n+1} \frac{1}{2k-1}\right] = \frac{1}{2}\left[1 - \frac{1}{2n+1}\right]; \lim_{n\to+\infty} s_n = \frac{1}{2}$

30. Geometric series, $a = \sin\ x$, $r = -\frac{1}{2}\sin x$. Converges for $|-\frac{1}{2}\sin x| < 1$, $|\sin x| < 2$, so converges for all values of x. $S = \frac{\sin x}{1 + \frac{1}{2}\sin x} = \frac{2\sin x}{2 + \sin x}$.

31. $a_2 = \frac{1}{2}a_1 + \frac{1}{2}$, $a_3 = \frac{1}{2}a_2 + \frac{1}{2} = \frac{1}{2^2}a_1 + \frac{1}{2^2} + \frac{1}{2}$, $a_4 = \frac{1}{2}a_3 + \frac{1}{2} = \frac{1}{2^3}a_1 + \frac{1}{2^3} + \frac{1}{2^2} + \frac{1}{2}$,

$a_5 = \frac{1}{2}a_4 + \frac{1}{2} = \frac{1}{2^4}a_1 + \frac{1}{2^4} + \frac{1}{2^3} + \frac{1}{2^2} + \frac{1}{2}, \ldots, a_n = \frac{1}{2^{n-1}}a_1 + \frac{1}{2^{n-1}} + \frac{1}{2^{n-2}} + \cdots + \frac{1}{2}$,

$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} \frac{a_1}{2^{n-1}} + \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = 0 + \frac{1/2}{1-1/2} = 1$

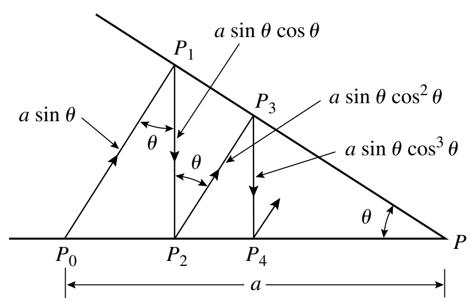
32. $0.a_1a_2\cdots a_n9999\cdots = 0.a_1a_2\cdots a_n + 0.9\,(10^{-n}) + 0.09\,(10^{-n}) + \cdots$

$= 0.a_1a_2\cdots a_n + \frac{0.9\,(10^{-n})}{1-0.1} = 0.a_1a_2\cdots a_n + 10^{-n}$

$= 0.a_1a_2\cdots(a_n+1) = 0.a_1a_2\cdots(a_n+1)\,0000\cdots$

33. The series converges to $1/(1-x)$ only if $-1 < x < 1$.

34. $P_0P_1 = a\sin\theta$,
$P_1P_2 = a\sin\theta\cos\theta$,
$P_2P_3 = a\sin\theta\cos^2\theta$,
$P_3P_4 = a\sin\theta\cos^3\theta, \ldots$
(see figure)
Each sum is a geometric series.



(a) $P_0P_1 + P_1P_2 + P_2P_3 + \cdots = a\sin\theta + a\sin\theta\cos\theta + a\sin\theta\cos^2\theta + \cdots = \frac{a\sin\theta}{1-\cos\theta}$

(b) $P_0P_1 + P_2P_3 + P_4P_5 + \cdots = a\sin\theta + a\sin\theta\cos^2\theta + a\sin\theta\cos^4\theta + \cdots$

$= \frac{a\sin\theta}{1-\cos^2\theta} = \frac{a\sin\theta}{\sin^2\theta} = a\csc\theta$

(c) $P_1P_2 + P_3P_4 + P_5P_6 + \cdots = a\sin\theta\cos\theta + a\sin\theta\cos^3\theta + \cdots$

$= \frac{a\sin\theta\cos\theta}{1-\cos^2\theta} = \frac{a\sin\theta\cos\theta}{\sin^2\theta} = a\cot\theta$

35. By inspection, $\frac{\theta}{2} - \frac{\theta}{4} + \frac{\theta}{8} - \frac{\theta}{16} + \cdots = \frac{\theta/2}{1-(-1/2)} = \theta/3$

36. $A_1 + A_2 + A_3 + \cdots = 1 + 1/2 + 1/4 + \cdots = \dfrac{1}{1-(1/2)} = 2$

37. **(b)** $$\frac{2^k A}{3^k - 2^k} + \frac{2^k B}{3^{k+1} - 2^{k+1}} = \frac{2^k\left(3^{k+1} - 2^{k+1}\right)A + 2^k\left(3^k - 2^k\right)B}{\left(3^k - 2^k\right)\left(3^{k+1} - 2^{k+1}\right)}$$
$$= \frac{\left(3\cdot 6^k - 2\cdot 2^{2k}\right)A + \left(6^k - 2^{2k}\right)B}{\left(3^k - 2^k\right)\left(3^{k+1} - 2^{k+1}\right)} = \frac{(3A+B)6^k - (2A+B)2^{2k}}{\left(3^k - 2^k\right)\left(3^{k+1} - 2^{k+1}\right)}$$
so $3A + B = 1$ and $2A + B = 0$, $A = 1$ and $B = -2$.

(c) $s_n = \displaystyle\sum_{k=1}^{n}\left[\frac{2^k}{3^k - 2^k} - \frac{2^{k+1}}{3^{k+1} - 2^{k+1}}\right] = \sum_{k=1}^{n}(a_k - a_{k+1})$ where $a_k = \dfrac{2^k}{3^k - 2^k}$.

But $s_n = (a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + \cdots + (a_n - a_{n+1})$ which is a telescoping sum,
$s_n = a_1 - a_{n+1} = 2 - \dfrac{2^{n+1}}{3^{n+1} - 2^{n+1}}$, $\displaystyle\lim_{n\to+\infty} s_n = \lim_{n\to+\infty}\left[2 - \frac{(2/3)^{n+1}}{1-(2/3)^{n+1}}\right] = 2.$

38. **(a)** geometric; 18/5 **(b)** geometric; diverges **(c)** $\displaystyle\sum_{k=1}^{\infty}\frac{1}{2}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right) = 1/2$

EXERCISE SET 10.5

1. **(a)** $\displaystyle\sum_{k=1}^{\infty}\frac{1}{2^k} = \frac{1/2}{1-1/2} = 1$; $\displaystyle\sum_{k=1}^{\infty}\frac{1}{4^k} = \frac{1/4}{1-1/4} = 1/3$; $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{2^k} + \frac{1}{4^k}\right) = 1 + 1/3 = 4/3$

(b) $\displaystyle\sum_{k=1}^{\infty}\frac{1}{5^k} = \frac{1/5}{1-1/5} = 1/4$; $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k(k+1)} = 1$ (Example 5, Section 10 .4);
$\displaystyle\sum_{k=1}^{\infty}\left[\frac{1}{5^k} - \frac{1}{k(k+1)}\right] = 1/4 - 1 = -3/4$

2. **(a)** $\displaystyle\sum_{k=2}^{\infty}\frac{1}{k^2-1} = 3/4$ (Exercise 10, Section 10.4); $\displaystyle\sum_{k=2}^{\infty}\frac{7}{10^{k-1}} = \frac{7/10}{1-1/10} = 7/9$;
so $\displaystyle\sum_{k=2}^{\infty}\left[\frac{1}{k^2-1} - \frac{7}{10^{k-1}}\right] = 3/4 - 7/9 = -1/36$

(b) with $a = 9/7, r = 3/7$, geometric, $\displaystyle\sum_{k=1}^{\infty}7^{-k}3^{k+1} = \frac{9/7}{1-(3/7)} = 9/4$;
with $a = 4/5, r = 2/5$, geometric, $\displaystyle\sum_{k=1}^{\infty}\frac{2^{k+1}}{5^k} = \frac{4/5}{1-(2/5)} = 4/3$;
$\displaystyle\sum_{k=1}^{\infty}\left[7^{-k}3^{k+1} - \frac{2^{k+1}}{5^k}\right] = 9/4 - 4/3 = 11/12$

3. **(a)** $p=3$, converges **(b)** $p=1/2$, diverges **(c)** $p=1$, diverges **(d)** $p=2/3$, diverges

4. **(a)** $p=4/3$, converges **(b)** $p=1/4$, diverges **(c)** $p=5/3$, converges **(d)** $p=\pi$, converges

5. **(a)** $\lim_{k\to+\infty} \dfrac{k^2+k+3}{2k^2+1} = \dfrac{1}{2}$; the series diverges.

(b) $\lim_{k\to+\infty} \left(1+\dfrac{1}{k}\right)^k = e$; the series diverges.

(c) $\lim_{k\to+\infty} \cos k\pi$ does not exist; the series diverges.

(d) $\lim_{k\to+\infty} \dfrac{1}{k!} = 0$; no information

6. **(a)** $\lim_{k\to+\infty} \dfrac{k}{e^k} = 0$; no information

(b) $\lim_{k\to+\infty} \ln k = +\infty$; the series diverges.

(c) $\lim_{k\to+\infty} \dfrac{1}{\sqrt{k}} = 0$; no information

(d) $\lim_{k\to+\infty} \dfrac{\sqrt{k}}{\sqrt{k}+3} = 1$; the series diverges.

7. **(a)** $\displaystyle\int_1^{+\infty} \frac{1}{5x+2} = \lim_{\ell\to+\infty} \frac{1}{5}\ln(5x+2)\Big]_1^\ell = +\infty$, the series diverges by the Integral Test.

(b) $\displaystyle\int_1^{+\infty} \frac{1}{1+9x^2}dx = \lim_{\ell\to+\infty} \frac{1}{3}\tan^{-1}3x\Big]_1^\ell = \frac{1}{3}\left(\pi/2 - \tan^{-1}3\right)$, the series converges by the Integral Test.

8. **(a)** $\displaystyle\int_1^{+\infty} \frac{x}{1+x^2}dx = \lim_{\ell\to+\infty} \frac{1}{2}\ln(1+x^2)\Big]_1^\ell = +\infty$, the series diverges by the Integral Test.

(b) $\displaystyle\int_1^{+\infty} (4+2x)^{-3/2}dx = \lim_{\ell\to+\infty} -1/\sqrt{4+2x}\Big]_1^\ell = 1/\sqrt{6}$, the series converges by the Integral Test.

9. $\displaystyle\sum_{k=1}^\infty \frac{1}{k+6} = \sum_{k=7}^\infty \frac{1}{k}$, diverges because the harmonic series diverges.

10. $\displaystyle\sum_{k=1}^\infty \frac{3}{5k} = \sum_{k=1}^\infty \frac{3}{5}\left(\frac{1}{k}\right)$, diverges because the harmonic series diverges.

11. $\displaystyle\sum_{k=1}^\infty \frac{1}{\sqrt{k+5}} = \sum_{k=6}^\infty \frac{1}{\sqrt{k}}$, diverges because the p-series with $p = 1/2 \le 1$ diverges.

12. $\displaystyle\lim_{k\to+\infty} \frac{1}{e^{1/k}} = 1$, the series diverges because $\displaystyle\lim_{k\to+\infty} u_k = 1 \ne 0$.

13. $\displaystyle\int_1^{+\infty} (2x-1)^{-1/3}dx = \lim_{\ell\to+\infty} \frac{3}{4}(2x-1)^{2/3}\Big]_1^\ell = +\infty$, the series diverges by the Integral Test.

14. $\dfrac{\ln x}{x}$ is decreasing for $x \ge e$, and $\displaystyle\int_3^{+\infty} \frac{\ln x}{x} = \lim_{\ell\to+\infty} \frac{1}{2}(\ln x)^2\Big]_3^\ell = +\infty$, so the series diverges by the Integral Test.

15. $\displaystyle\lim_{k\to+\infty} \frac{k}{\ln(k+1)} = \lim_{k\to+\infty} \frac{1}{1/(k+1)} = +\infty$, the series diverges because $\displaystyle\lim_{k\to+\infty} u_k \ne 0$.

16. $\displaystyle\int_1^{+\infty} xe^{-x^2}dx = \lim_{\ell\to+\infty} -\frac{1}{2}e^{-x^2}\Big]_1^\ell = e^{-1}/2$, the series converges by the Integral Test.

17. $\lim_{k\to+\infty}(1+1/k)^{-k} = 1/e \neq 0$, the series diverges.

18. $\lim_{k\to+\infty}\dfrac{k^2+1}{k^2+3} = 1 \neq 0$, the series diverges.

19. $\displaystyle\int_1^{+\infty}\frac{\tan^{-1}x}{1+x^2}dx = \lim_{\ell\to+\infty}\frac{1}{2}\left(\tan^{-1}x\right)^2\Big]_1^\ell = 3\pi^2/32$, the series converges by the Integral Test, since $\dfrac{d}{dx}\dfrac{\tan^{-1}x}{1+x^2} = \dfrac{1-2x\tan^{-1}x}{(1+x^2)^2} < 0$ for $x \geq 1$.

20. $\displaystyle\int_1^{+\infty}\frac{1}{\sqrt{x^2+1}}dx = \lim_{\ell\to+\infty}\sinh^{-1}x\Big]_1^\ell = +\infty$, the series diverges by the Integral Test.

21. $\lim_{k\to+\infty}k^2\sin^2(1/k) = 1 \neq 0$, the series diverges.

22. $\displaystyle\int_1^{+\infty}x^2e^{-x^3}dx = \lim_{\ell\to+\infty}-\frac{1}{3}e^{-x^3}\Big]_1^\ell = e^{-1}/3$,

the series converges by the Integral Test ($x^2e^{-x^3}$ is decreasing for $x \geq 1$).

23. $7\displaystyle\sum_{k=5}^{\infty}k^{-1.01}$, p-series with $p > 1$, converges

24. $\displaystyle\int_1^{+\infty}\operatorname{sech}^2 x\,dx = \lim_{\ell\to+\infty}\tanh x\Big]_1^\ell = 1-\tanh(1)$, the series converges by the Integral Test.

25. $\dfrac{1}{x(\ln x)^p}$ is decreasing for $x \geq e^p$, so use the Integral Test with $\displaystyle\int_{e^p}^{+\infty}\frac{dx}{x(\ln x)^p}$ to get

$$\lim_{\ell\to+\infty}\ln(\ln x)\Big]_{e^p}^\ell = +\infty \text{ if } p=1, \qquad \lim_{\ell\to+\infty}\frac{(\ln x)^{1-p}}{1-p}\Big]_{e^p}^\ell = \begin{cases} +\infty & \text{if } p<1 \\ \dfrac{p^{1-p}}{p-1} & \text{if } p>1 \end{cases}$$

Thus the series converges for $p > 1$.

26. If $p > 0$ set $g(x) = x(\ln x)[\ln(\ln x)]^p$, $g'(x) = (\ln(\ln x))^{p-1}\left[(1+\ln x)\ln(\ln x)+p\right]$, and, for $x > e^e$, $g'(x) > 0$, thus $1/g(x)$ is decreasing for $x > e^e$; use the Integral Test with $\displaystyle\int_{e^e}^{+\infty}\frac{dx}{x(\ln x)[\ln(\ln x)]^p}$ to get

$$\lim_{\ell\to+\infty}\ln[\ln(\ln x)]\Big]_{e^e}^\ell = +\infty \text{ if } p=1, \lim_{\ell\to+\infty}\frac{[\ln(\ln x)]^{1-p}}{1-p}\Big]_{e^e}^\ell = \begin{cases} +\infty & \text{if } p<1, \\ \dfrac{1}{p-1} & \text{if } p>1 \end{cases}$$

Thus the series converges for $p > 1$ and diverges for $0 < p \leq 1$. If $p \leq 0$ then $\dfrac{[\ln(\ln x)]^p}{x\ln x} \geq \dfrac{1}{x\ln x}$ for $x > e^e$ so the series diverges.

27. **(a)** $3\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2} - \sum_{k=1}^{\infty}\frac{1}{k^4} = \pi^2/2 - \pi^4/90$ **(b)** $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2} - 1 - \frac{1}{2^2} = \pi^2/6 - 5/4$

(c) $\displaystyle\sum_{k=2}^{\infty}\frac{1}{(k-1)^4} = \sum_{k=1}^{\infty}\frac{1}{k^4} = \pi^4/90$

28. **(a)** Suppose $\Sigma(u_k+v_k)$ converges; then so does $\Sigma[(u_k+v_k)-u_k]$, but $\Sigma[(u_k+v_k)-u_k]=\Sigma v_k$, so Σv_k converges which contradicts the assumption that Σv_k diverges. Suppose $\Sigma(u_k-v_k)$ converges; then so does $\Sigma[u_k-(u_k-v_k)]=\Sigma v_k$ which leads to the same contradiction as before.

(b) Let $u_k=2/k$ and $v_k=1/k$; then both $\Sigma(u_k+v_k)$ and $\Sigma(u_k-v_k)$ diverge; let $u_k=1/k$ and $v_k=-1/k$ then $\Sigma(u_k+v_k)$ converges; let $u_k=v_k=1/k$ then $\Sigma(u_k-v_k)$ converges.

29. **(a)** diverges because $\sum_{k=1}^{\infty}(2/3)^{k-1}$ converges and $\sum_{k=1}^{\infty}1/k$ diverges.

(b) diverges because $\sum_{k=1}^{\infty}1/(3k+2)$ diverges and $\sum_{k=1}^{\infty}1/k^{3/2}$ converges.

(c) converges because both $\sum_{k=2}^{\infty}\frac{1}{k(\ln k)^2}$ (Exercise 25) and $\sum_{k=2}^{\infty}1/k^2$ converge.

30. **(a)** If $S=\sum_{k=1}^{\infty}u_k$ and $s_n=\sum_{k=1}^{n}u_k$, then $S-s_n=\sum_{k=n+1}^{\infty}u_k$. Interpret u_k, $k=n+1,n+2,\ldots$, as the areas of inscribed or circumscribed rectangles with height u_k and base of length one for the curve $y=f(x)$ to obtain the result.

(b) Add $s_n=\sum_{k=1}^{n}u_k$ to each term in the conclusion of Part (a) to get the desired result:

$$s_n+\int_{n+1}^{+\infty}f(x)\,dx<\sum_{k=1}^{+\infty}u_k<s_n+\int_{n}^{+\infty}f(x)\,dx$$

31. **(a)** In Exercise 30 above let $f(x)=\frac{1}{x^2}$. Then $\int_n^{+\infty}f(x)\,dx=-\frac{1}{x}\Big]_n^{+\infty}=\frac{1}{n}$;
use this result and the same result with $n+1$ replacing n to obtain the desired result.

(b) $s_3=1+1/4+1/9=49/36$; $58/36=s_3+\frac{1}{4}<\frac{1}{6}\pi^2<s_3+\frac{1}{3}=61/36$

(d) $1/11<\frac{1}{6}\pi^2-s_{10}<1/10$

33. Apply Exercise 30 in each case:

(a) $f(x)=\frac{1}{(2x+1)^2}$, $\int_n^{+\infty}f(x)\,dx=\frac{1}{2(2n+1)}$, so $\frac{1}{46}<\sum_{k=1}^{\infty}\frac{1}{(2k+1)^2}-s_{10}<\frac{1}{42}$

(b) $f(x)=\frac{1}{k^2+1}$, $\int_n^{+\infty}f(x)\,dx=\frac{\pi}{2}-\tan^{-1}(n)$, so

$$\pi/2-\tan^{-1}(11)<\sum_{k=1}^{\infty}\frac{1}{k^2+1}-s_{10}<\pi/2-\tan^{-1}(10)$$

(c) $f(x)=\frac{x}{e^x}$, $\int_n^{+\infty}f(x)\,dx=(n+1)e^{-n}$, so $12e^{-11}<\sum_{k=1}^{\infty}\frac{k}{e^k}-s_{10}<11e^{-10}$

34. **(a)** $\displaystyle\int_n^{+\infty} \frac{1}{x^3}\,dx = \frac{1}{2n^2}$; use Exercise 30(b)

(b) $\displaystyle\frac{1}{2n^2} - \frac{1}{2(n+1)^2} < 0.01$ for $n = 5$.

(c) From Part (a) with $n = 5$ obtain $1.200 < S < 1.206$, so $S \approx 1.203$.

35. **(a)** $\displaystyle\int_n^{+\infty} \frac{1}{x^4}\,dx = \frac{1}{3n^3}$; choose n so that $\displaystyle\frac{1}{3n^3} - \frac{1}{3(n+1)^3} < 0.005,\ n = 4;\ S \approx 1.08$

36. **(a)** Let $F(x) = \dfrac{1}{x}$, then $\displaystyle\int_1^n \frac{1}{x}dx = \ln n$ and $\displaystyle\int_1^{n+1} \frac{1}{x}dx = \ln(n+1)$, $u_1 = 1$ so $\ln(n+1) < s_n < 1 + \ln n$.

(b) $\ln(1,000,001) < s_{1,000,000} < 1 + \ln(1,000,000)$, $13 < s_{1,000,000} < 15$

(c) $s_{10^9} < 1 + \ln 10^9 = 1 + 9\ln 10 < 22$

(d) $s_n > \ln(n+1) \geq 100$, $n \geq e^{100} - 1 \approx 2.688 \times 10^{43}$; $n = 2.69 \times 10^{43}$

37. p-series with $p = \ln a$; convergence for $p > 1, a > e$

38. x^2e^{-x} is decreasing and positive for $x > 2$ so the Integral Test applies:
$\displaystyle\int_1^{\infty} x^2e^{-x}\,dx = -(x^2 + 2x + 2)e^{-x}\Big]_1^{\infty} = 5e^{-1}$ so the series converges.

39. **(a)** $f(x) = 1/(x^3 + 1)$ is decreasing and continuous on the interval $[1, +\infty]$, so the Integral Test applies.

(c)

n	10	20	30	40	50
s_n	0.681980	0.685314	0.685966	0.686199	0.686307

n	60	70	80	90	100
s_n	0.686367	0.686403	0.686426	0.686442	0.686454

(e) Set $\displaystyle g(n) = \int_n^{+\infty} \frac{1}{x^3+1}\,dx = \frac{\sqrt{3}}{6}\pi + \frac{1}{6}\ln\frac{n^3+1}{(n+1)^3} - \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{2n-1}{\sqrt{3}}\right)$; for $n \geq 13$, $g(n) - g(n+1) \leq 0.0005$; $s_{13} + (g(13) + g(14))/2 \approx 0.6865$, so the sum ≈ 0.6865 to three decimal places.

EXERCISE SET 10.6

1. **(a)** $\displaystyle\frac{1}{5k^2 - k} \leq \frac{1}{5k^2 - k^2} = \frac{1}{4k^2}, \sum_{k=1}^{\infty} \frac{1}{4k^2}$ converges

(b) $\displaystyle\frac{3}{k - 1/4} > \frac{3}{k}, \sum_{k=1}^{\infty} 3/k$ diverges

2. **(a)** $\displaystyle\frac{k+1}{k^2 - k} > \frac{k}{k^2} = \frac{1}{k}, \sum_{k=2}^{\infty} 1/k$ diverges

(b) $\displaystyle\frac{2}{k^4 + k} < \frac{2}{k^4}, \sum_{k=1}^{\infty} \frac{2}{k^4}$ converges

3. **(a)** $\dfrac{1}{3^k+5} < \dfrac{1}{3^k}, \displaystyle\sum_{k=1}^{\infty} \frac{1}{3^k}$ converges **(b)** $\dfrac{5\sin^2 k}{k!} < \dfrac{5}{k!}, \displaystyle\sum_{k=1}^{\infty} \frac{5}{k!}$ converges

4. **(a)** $\dfrac{\ln k}{k} > \dfrac{1}{k}$ for $k \geq 3$, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ diverges

(b) $\dfrac{k}{k^{3/2}-1/2} > \dfrac{k}{k^{3/2}} = \dfrac{1}{\sqrt{k}}; \displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges

5. compare with the convergent series $\displaystyle\sum_{k=1}^{\infty} 1/k^5$, $\rho = \displaystyle\lim_{k\to+\infty} \frac{4k^7-2k^6+6k^5}{8k^7+k-8} = 1/2$, converges

6. compare with the divergent series $\displaystyle\sum_{k=1}^{\infty} 1/k$, $\rho = \displaystyle\lim_{k\to+\infty} \frac{k}{9k+6} = 1/9$, diverges

7. compare with the convergent series $\displaystyle\sum_{k=1}^{\infty} 5/3^k$, $\rho = \displaystyle\lim_{k\to+\infty} \frac{3^k}{3^k+1} = 1$, converges

8. compare with the divergent series $\displaystyle\sum_{k=1}^{\infty} 1/k$, $\rho = \displaystyle\lim_{k\to+\infty} \frac{k^2(k+3)}{(k+1)(k+2)(k+5)} = 1$, diverges

9. compare with the divergent series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{2/3}}$,

$$\rho = \lim_{k\to+\infty} \frac{k^{2/3}}{(8k^2-3k)^{1/3}} = \lim_{k\to+\infty} \frac{1}{(8-3/k)^{1/3}} = 1/2, \text{ diverges}$$

10. compare with the convergent series $\displaystyle\sum_{k=1}^{\infty} 1/k^{17}$,

$$\rho = \lim_{k\to+\infty} \frac{k^{17}}{(2k+3)^{17}} = \lim_{k\to+\infty} \frac{1}{(2+3/k)^{17}} = 1/2^{17}, \text{ converges}$$

11. $\rho = \displaystyle\lim_{k\to+\infty} \frac{3^{k+1}/(k+1)!}{3^k/k!} = \lim_{k\to+\infty} \frac{3}{k+1} = 0$, the series converges

12. $\rho = \displaystyle\lim_{k\to+\infty} \frac{4^{k+1}/(k+1)^2}{4^k/k^2} = \lim_{k\to+\infty} \frac{4k^2}{(k+1)^2} = 4$, the series diverges

13. $\rho = \displaystyle\lim_{k\to+\infty} \frac{k}{k+1} = 1$, the result is inconclusive

14. $\rho = \displaystyle\lim_{k\to+\infty} \frac{(k+1)(1/2)^{k+1}}{k(1/2)^k} = \lim_{k\to+\infty} \frac{k+1}{2k} = 1/2$, the series converges

15. $\rho = \displaystyle\lim_{k\to+\infty} \frac{(k+1)!/(k+1)^3}{k!/k^3} = \lim_{k\to+\infty} \frac{k^3}{(k+1)^2} = +\infty$, the series diverges

16. $\rho = \displaystyle\lim_{k\to+\infty} \frac{(k+1)/[(k+1)^2+1]}{k/(k^2+1)} = \lim_{k\to+\infty} \frac{(k+1)(k^2+1)}{k(k^2+2k+2)} = 1$, the result is inconclusive.

17. $\rho = \lim_{k\to+\infty} \dfrac{3k+2}{2k-1} = 3/2$, the series diverges

18. $\rho = \lim_{k\to+\infty} k/100 = +\infty$, the series diverges

19. $\rho = \lim_{k\to+\infty} \dfrac{k^{1/k}}{5} = 1/5$, the series converges

20. $\rho = \lim_{k\to+\infty} (1 - e^{-k}) = 1$, the result is inconclusive

21. Ratio Test, $\rho = \lim_{k\to+\infty} 7/(k+1) = 0$, converges

22. Limit Comparison Test, compare with the divergent series $\sum_{k=1}^{\infty} 1/k$

23. Ratio Test, $\rho = \lim_{k\to+\infty} \dfrac{(k+1)^2}{5k^2} = 1/5$, converges

24. Ratio Test, $\rho = \lim_{k\to+\infty} (10/3)(k+1) = +\infty$, diverges

25. Ratio Test, $\rho = \lim_{k\to+\infty} e^{-1}(k+1)^{50}/k^{50} = e^{-1} < 1$, converges

26. Limit Comparison Test, compare with the divergent series $\sum_{k=1}^{\infty} 1/k$

27. Limit Comparison Test, compare with the convergent series $\sum_{k=1}^{\infty} 1/k^{5/2}$, $\rho = \lim_{k\to+\infty} \dfrac{k^3}{k^3+1} = 1$, converges

28. $\dfrac{4}{2+3^k k} < \dfrac{4}{3^k k}$, $\sum_{k=1}^{\infty} \dfrac{4}{3^k k}$ converges (Ratio Test) so $\sum_{k=1}^{\infty} \dfrac{4}{2+k3^k}$ converges by the Comparison Test

29. Limit Comparison Test, compare with the divergent series $\sum_{k=1}^{\infty} 1/k$, $\rho = \lim_{k\to+\infty} \dfrac{k}{\sqrt{k^2+k}} = 1$, diverges

30. $\dfrac{2+(-1)^k}{5^k} \le \dfrac{3}{5^k}$, $\sum_{k=1}^{\infty} 3/5^k$ converges so $\sum_{k=1}^{\infty} \dfrac{2+(-1)^k}{5^k}$ converges

31. Limit Comparison Test, compare with the convergent series $\sum_{k=1}^{\infty} 1/k^{5/2}$,

$\rho = \lim_{k\to+\infty} \dfrac{k^3+2k^{5/2}}{k^3+3k^2+3k} = 1$, converges

32. $\dfrac{4+|\cos\ k|}{k^3} < \dfrac{5}{k^3}$, $\sum_{k=1}^{\infty} 5/k^3$ converges so $\sum_{k=1}^{\infty} \dfrac{4+|\cos k|}{k^3}$ converges

33. Limit Comparison Test, compare with the divergent series $\sum_{k=1}^{\infty} 1/\sqrt{k}$

34. Ratio Test, $\rho = \lim_{k\to+\infty}(1+1/k)^{-k} = 1/e < 1$, converges

35. Ratio Test, $\rho = \lim_{k\to+\infty}\dfrac{\ln(k+1)}{e\ln k} = \lim_{k\to+\infty}\dfrac{k}{e(k+1)} = 1/e < 1$, converges

36. Ratio Test, $\rho = \lim_{k\to+\infty}\dfrac{k+1}{e^{2k+1}} = \lim_{k\to+\infty}\dfrac{1}{2e^{2k+1}} = 0$, converges

37. Ratio Test, $\rho = \lim_{k\to+\infty}\dfrac{k+5}{4(k+1)} = 1/4$, converges

38. Root Test, $\rho = \lim_{k\to+\infty}(\dfrac{k}{k+1})^k = \lim_{k\to+\infty}\dfrac{1}{(1+1/k)^k} = 1/e$, converges

39. diverges because $\lim_{k\to+\infty}\dfrac{1}{4+2^{-k}} = 1/4 \neq 0$

40. $\sum_{k=1}^{\infty}\dfrac{\sqrt{k}\ln k}{k^3+1} = \sum_{k=2}^{\infty}\dfrac{\sqrt{k}\ln k}{k^3+1}$ because $\ln 1 = 0$, $\dfrac{\sqrt{k}\ln k}{k^3+1} < \dfrac{k\ln k}{k^3} = \dfrac{\ln k}{k^2}$,

$\int_2^{+\infty}\dfrac{\ln x}{x^2}dx = \lim_{\ell\to+\infty}\left(-\dfrac{\ln x}{x} - \dfrac{1}{x}\right)\Big]_2^{\ell} = \dfrac{1}{2}(\ln 2+1)$ so $\sum_{k=2}^{\infty}\dfrac{\ln k}{k^2}$ converges and so does $\sum_{k=1}^{\infty}\dfrac{\sqrt{k}\ln k}{k^3+1}$.

41. $\dfrac{\tan^{-1}k}{k^2} < \dfrac{\pi/2}{k^2}$, $\sum_{k=1}^{\infty}\dfrac{\pi/2}{k^2}$ converges so $\sum_{k=1}^{\infty}\dfrac{\tan^{-1}k}{k^2}$ converges

42. $\dfrac{5^k+k}{k!+3} < \dfrac{5^k+5^k}{k!} = \dfrac{2\left(5^k\right)}{k!}$, $\sum_{k=1}^{\infty}2\left(\dfrac{5^k}{k!}\right)$ converges (Ratio Test) so $\sum_{k=1}^{\infty}\dfrac{5^k+k}{k!+3}$ converges

43. Ratio Test, $\rho = \lim_{k\to+\infty}\dfrac{(k+1)^2}{(2k+2)(2k+1)} = 1/4$, converges

44. Ratio Test, $\rho = \lim_{k\to+\infty}\dfrac{2(k+1)^2}{(2k+4)(2k+3)} = 1/2$, converges

45. $u_k = \dfrac{k!}{1\cdot 3\cdot 5\cdots(2k-1)}$, by the Ratio Test $\rho = \lim_{k\to+\infty}\dfrac{k+1}{2k+1} = 1/2$; converges

46. $u_k = \dfrac{1\cdot 3\cdot 5\cdots(2k-1)}{(2k-1)!}$, by the Ratio Test $\rho = \lim_{k\to+\infty}\dfrac{1}{2k} = 0$; converges

47. Root Test: $\rho = \lim_{k\to+\infty}\dfrac{1}{3}(\ln k)^{1/k} = 1/3$, converges

48. Root Test: $\rho = \lim_{k\to+\infty}\dfrac{\pi(k+1)}{k^{1+1/k}} = \lim_{k\to+\infty}\pi\dfrac{k+1}{k} = \pi$, diverges

49. **(b)** $\rho = \lim_{k\to+\infty}\dfrac{\sin(\pi/k)}{\pi/k} = 1$ and $\sum_{k=1}^{\infty}\pi/k$ diverges

50. **(a)** $\cos x \approx 1 - x^2/2, 1-\cos\left(\dfrac{1}{k}\right) \approx \dfrac{1}{2k^2}$ **(b)** $\rho = \lim_{k\to+\infty}\dfrac{1-\cos(1/k)}{1/k^2} = 2$, converges

51. Set $g(x) = \sqrt{x} - \ln x$; $\dfrac{d}{dx}g(x) = \dfrac{1}{2\sqrt{x}} - \dfrac{1}{x} = 0$ when $x = 4$. Since $\lim\limits_{x\to 0+} g(x) = \lim\limits_{x\to+\infty} g(x) = +\infty$ it follows that $g(x)$ has its minimum at $x = 4$, $g(4) = \sqrt{4} - \ln 4 > 0$, and thus $\sqrt{x} - \ln x > 0$ for $x > 0$.

(a) $\dfrac{\ln k}{k^2} < \dfrac{\sqrt{k}}{k^2} = \dfrac{1}{k^{3/2}}$, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ converges so $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k^2}$ converges.

(b) $\dfrac{1}{(\ln k)^2} > \dfrac{1}{k}$, $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k}$ diverges so $\displaystyle\sum_{k=2}^{\infty} \frac{1}{(\ln k)^2}$ diverges.

52. By the Root Test, $\rho = \lim\limits_{k\to+\infty} \dfrac{\alpha}{(k^{1/k})^{\alpha}} = \dfrac{\alpha}{1^{\alpha}} = \alpha$, the series converges if $\alpha < 1$ and diverges if $\alpha > 1$. If $\alpha = 1$ then the series is $\displaystyle\sum_{k=1}^{\infty} 1/k$ which diverges.

53. **(a)** If $\sum b_k$ converges, then set $M = \sum b_k$. Then $a_1 + a_2 + \cdots + a_n \le b_1 + b_2 + \cdots + b_n \le M$; apply Theorem 10.5.6 to get convergence of $\sum a_k$.

(b) Assume the contrary, that $\sum b_k$ converges; then use Part (a) of the Theorem to show that $\sum a_k$ converges, a contradiction.

54. **(a)** If $\lim\limits_{k\to+\infty} (a_k/b_k) = 0$ then for $k \ge K$, $a_k/b_k < 1$, $a_k < b_k$ so $\sum a_k$ converges by the Comparison Test.

(b) If $\lim\limits_{k\to+\infty} (a_k/b_k) = +\infty$ then for $k \ge K$, $a_k/b_k > 1$, $a_k > b_k$ so $\sum a_k$ diverges by the Comparison Test.

EXERCISE SET 10.7

1. $a_{k+1} < a_k$, $\lim\limits_{k\to+\infty} a_k = 0$, $a_k > 0$

2. $\dfrac{a_{k+1}}{a_k} = \dfrac{k+1}{3k} \le \dfrac{2k}{3k} = \dfrac{2}{3}$ for $k \ge 1$, so $\{a_k\}$ is decreasing and tends to zero.

3. diverges because $\lim\limits_{k\to+\infty} a_k = \lim\limits_{k\to+\infty} \dfrac{k+1}{3k+1} = 1/3 \ne 0$

4. diverges because $\lim\limits_{k\to+\infty} a_k = \lim\limits_{k\to+\infty} \dfrac{k+1}{\sqrt{k}+1} = +\infty \ne 0$

5. $\{e^{-k}\}$ is decreasing and $\lim\limits_{k\to+\infty} e^{-k} = 0$, converges

6. $\left\{\dfrac{\ln k}{k}\right\}$ is decreasing and $\lim\limits_{k\to+\infty} \dfrac{\ln k}{k} = 0$, converges

7. $\rho = \lim\limits_{k\to+\infty} \dfrac{(3/5)^{k+1}}{(3/5)^k} = 3/5$, converges absolutely

8. $\rho = \lim\limits_{k\to+\infty} \dfrac{2}{k+1} = 0$, converges absolutely

9. $\rho = \lim_{k\to+\infty} \dfrac{3k^2}{(k+1)^2} = 3$, diverges

10. $\rho = \lim_{k\to+\infty} \dfrac{k+1}{5k} = 1/5$, converges absolutely

11. $\rho = \lim_{k\to+\infty} \dfrac{(k+1)^3}{ek^3} = 1/e$, converges absolutely

12. $\rho = \lim_{k\to+\infty} \dfrac{(k+1)^{k+1}k!}{(k+1)!k^k} = \lim_{k\to+\infty}(1+1/k)^k = e$, diverges

13. conditionally convergent, $\sum_{k=1}^{\infty} \dfrac{(-1)^{k+1}}{3k}$ converges by the Alternating Series Test but $\sum_{k=1}^{\infty} \dfrac{1}{3k}$ diverges

14. absolutely convergent, $\sum_{k=1}^{\infty} \dfrac{1}{k^{4/3}}$ converges

15. divergent, $\lim_{k\to+\infty} a_k \neq 0$

16. absolutely convergent, Ratio Test for absolute convergence

17. $\sum_{k=1}^{\infty} \dfrac{\cos k\pi}{k} = \sum_{k=1}^{\infty} \dfrac{(-1)^k}{k}$ is conditionally convergent, $\sum_{k=1}^{\infty} \dfrac{(-1)^k}{k}$ converges by the Alternating Series Test but $\sum_{k=1}^{\infty} 1/k$ diverges.

18. conditionally convergent, $\sum_{k=3}^{\infty} \dfrac{(-1)^k \ln k}{k}$ converges by the Alternating Series Test but $\sum_{k=3}^{\infty} \dfrac{\ln k}{k}$ diverges (Limit Comparison Test with $\sum 1/k$).

19. conditionally convergent, $\sum_{k=1}^{\infty}(-1)^{k+1}\dfrac{k+2}{k(k+3)}$ converges by the Alternating Series Test but $\sum_{k=1}^{\infty} \dfrac{k+2}{k(k+3)}$ diverges (Limit Comparison Test with $\sum 1/k$)

20. conditionally convergent, $\sum_{k=1}^{\infty} \dfrac{(-1)^{k+1}k^2}{k^3+1}$ converges by the Alternating Series Test but $\sum_{k=1}^{\infty} \dfrac{k^2}{k^3+1}$ diverges (Limit Comparison Test with $\sum(1/k)$)

21. $\sum_{k=1}^{\infty} \sin(k\pi/2) = 1+0-1+0+1+0-1+0+\cdots$, divergent ($\lim_{k\to+\infty} \sin(k\pi/2)$ does not exist)

22. absolutely convergent, $\sum_{k=1}^{\infty} \dfrac{|\sin k|}{k^3}$ converges (compare with $\sum 1/k^3$)

23. conditionally convergent, $\sum_{k=2}^{\infty} \frac{(-1)^k}{k \ln k}$ converges by the Alternating Series Test but $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ diverges (Integral Test)

24. conditionally convergent, $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k(k+1)}}$ converges by the Alternating Series Test but $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k(k+1)}}$ diverges (Limit Comparison Test with $\sum 1/k$)

25. absolutely convergent, $\sum_{k=2}^{\infty} (1/\ln k)^k$ converges by the Root Test

26. conditionally convergent, $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k+1}+\sqrt{k}}$ converges by the Alternating Series Test but $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k+1}+\sqrt{k}}$ diverges (Limit Comparison Test with $\sum 1/\sqrt{k}$)

27. conditionally convergent, let $f(x) = \frac{x^2+1}{x^3+2}$ then $f'(x) = \frac{x(4-3x-x^3)}{(x^3+2)^2} \le 0$ for $x \ge 1$ so $\{a_k\}_{k=2}^{+\infty} = \left\{\frac{k^2+1}{k^3+2}\right\}_{k=2}^{+\infty}$ is decreasing, $\lim_{k\to+\infty} a_k = 0$; the series converges by the Alternating Series Test but $\sum_{k=2}^{\infty} \frac{k^2+1}{k^3+2}$ diverges (Limit Comparison Test with $\sum 1/k$)

28. $\sum_{k=1}^{\infty} \frac{k \cos k\pi}{k^2+1} = \sum_{k=1}^{\infty} \frac{(-1)^k k}{k^2+1}$ is conditionally convergent, $\sum_{k=1}^{\infty} \frac{(-1)^k k}{k^2+1}$ converges by the Alternating Series Test but $\sum_{k=1}^{\infty} \frac{k}{k^2+1}$ diverges

29. absolutely convergent by the Ratio Test, $\rho = \lim_{k\to+\infty} \frac{k+1}{(2k+1)(2k)} = 0$

30. divergent, $\lim_{k\to+\infty} a_k = +\infty$

31. $|\text{error}| < a_8 = 1/8 = 0.125$

32. $|\text{error}| < a_6 = 1/6! < 0.0014$

33. $|\text{error}| < a_{100} = 1/\sqrt{100} = 0.1$

34. $|\text{error}| < a_4 = 1/(5\ln 5) < 0.125$

35. $|\text{error}| < 0.0001$ if $a_{n+1} \le 0.0001$, $1/(n+1) \le 0.0001$, $n+1 \ge 10,000$, $n \ge 9,999$, $n = 9,999$

36. $|\text{error}| < 0.00001$ if $a_{n+1} \le 0.00001$, $1/(n+1)! \le 0.00001$, $(n+1)! \ge 100,000$. But $8! = 40,320$, $9! = 362,880$ so $(n+1)! \ge 100,000$ if $n+1 \ge 9$, $n \ge 8$, $n = 8$

37. $|\text{error}| < 0.005$ if $a_{n+1} \le 0.005$, $1/\sqrt{n+1} \le 0.005$, $\sqrt{n+1} \ge 200$, $n+1 \ge 40,000$, $n \ge 39,999$, $n = 39,999$

38. $|\text{error}| < 0.05$ if $a_{n+1} \le 0.05$, $1/[(n+2)\ln(n+2)] \le 0.05$, $(n+2)\ln(n+2) \ge 20$. But $9\ln 9 \approx 19.8$ and $10\ln 10 \approx 23.0$ so $(n+2)\ln(n+2) \ge 20$ if $n+2 \ge 10$, $n \ge 8$, $n = 8$

39. $a_k = \dfrac{3}{2^{k+1}}$, $|\text{error}| < a_{11} = \dfrac{3}{2^{12}} < 0.00074$; $s_{10} \approx 0.4995$; $S = \dfrac{3/4}{1-(-1/2)} = 0.5$

40. $a_k = \left(\dfrac{2}{3}\right)^{k-1}$, $|\text{error}| < a_{11} = \left(\dfrac{2}{3}\right)^{10} < 0.01735$; $s_{10} \approx 0.5896$; $S = \dfrac{1}{1-(-2/3)} = 0.6$

41. $a_k = \dfrac{1}{(2k-1)!}$, $a_{n+1} = \dfrac{1}{(2n+1)!} \le 0.005$, $(2n+1)! \ge 200$, $2n+1 \ge 6$, $n \ge 2.5$; $n = 3$, $s_3 = 1 - 1/6 + 1/120 \approx 0.84$

42. $a_k = \dfrac{1}{(2k-2)!}$, $a_{n+1} = \dfrac{1}{(2n)!} \le 0.005$, $(2n)! \ge 200$, $2n \ge 6$, $n \ge 3$; $n = 3$, $s_3 \approx 0.54$

43. $a_k = \dfrac{1}{k2^k}$, $a_{n+1} = \dfrac{1}{(n+1)2^{n+1}} \le 0.005$, $(n+1)2^{n+1} \ge 200$, $n+1 \ge 6$, $n \ge 5$; $n = 5$, $s_5 \approx 0.41$

44. $a_k = \dfrac{1}{(2k-1)^5 + 4(2k-1)}$, $a_{n+1} = \dfrac{1}{(2n+1)^5 + 4(2n+1)} \le 0.005$, $(2n+1)^5 + 4(2n+1) \ge 200$, $2n+1 \ge 3$, $n \ge 1$; $n = 1$, $s_1 = 0.20$

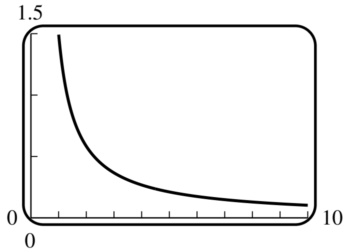
45. **(c)** $a_k = \dfrac{1}{2k-1}$, $a_{n+1} = \dfrac{1}{2n+1} \le 10^{-2}$, $2n+1 \ge 100$, $n \ge 49.5$; $n = 50$

46. $\sum(1/k^p)$ converges if $p > 1$ and diverges if $p \le 1$, so $\displaystyle\sum_{k=1}^{\infty}(-1)^k \frac{1}{k^p}$ converges absolutely if $p > 1$, and converges conditionally if $0 < p \le 1$ since it satisfies the Alternating Series Test; it diverges for $p \le 0$ since $\displaystyle\lim_{k\to+\infty} a_k \ne 0$.

47.
$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right] - \left[\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots\right]$$
$$= \frac{\pi^2}{6} - \frac{1}{2^2}\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right] = \frac{\pi^2}{6} - \frac{1}{4}\frac{\pi^2}{6} = \frac{\pi^2}{8}$$

48.
$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \left[1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\right] - \left[\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \cdots\right]$$
$$= \frac{\pi^4}{90} - \frac{1}{2^4}\left[1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\right] = \frac{\pi^4}{90} - \frac{1}{16}\frac{\pi^4}{90} = \frac{\pi^4}{96}$$

49. Every positive integer can be written in exactly one of the three forms $2k-1$ or $4k-2$ or $4k$, so a rearrangement is
$$\left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{10} - \frac{1}{12}\right) + \cdots + \left(\frac{1}{2k-1} - \frac{1}{4k-2} - \frac{1}{4k}\right) + \cdots$$
$$= \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{10} - \frac{1}{12}\right) + \cdots + \left(\frac{1}{4k-2} - \frac{1}{4k}\right) + \cdots = \frac{1}{2}\ln 2$$

50. **(a)** 

(b) Yes; since $f(x)$ is decreasing for $x \geq 1$ and $\lim_{x\to+\infty} f(x) = 0$, the series satisfies the Alternating Series Test.

51. **(a)** The distance d from the starting point is
$d = 180 - \frac{180}{2} + \frac{180}{3} - \cdots - \frac{180}{1000} = 180\left[1 - \frac{1}{2} + \frac{1}{3} - \cdots - \frac{1}{1000}\right]$.
From Theorem 10.7.2, $1 - \frac{1}{2} + \frac{1}{3} - \cdots - \frac{1}{1000}$ differs from $\ln 2$ by less than $1/1001$ so $180(\ln 2 - 1/1001) < d < 180 \ln 2$, $124.58 < d < 124.77$.

(b) The total distance traveled is $s = 180 + \frac{180}{2} + \frac{180}{3} + \cdots + \frac{180}{1000}$, and from inequality (2) in Section 10.5,

$$\int_1^{1001} \frac{180}{x}dx < s < 180 + \int_1^{1000} \frac{180}{x}dx$$

$$180 \ln 1001 < s < 180(1 + \ln 1000)$$

$$1243 < s < 1424$$

52. **(a)** Suppose $\Sigma|a_k|$ converges, then $\lim_{k\to+\infty} |a_k| = 0$ so $|a_k| < 1$ for $k \geq K$ and thus $|a_k|^2 < |a_k|$, $a_k^2 < |a_k|$ hence Σa_k^2 converges by the Comparison Test.

(b) Let $a_k = \frac{1}{k}$, then $\sum a_k^2$ converges but $\sum a_k$ diverges.

EXERCISE SET 10.8

1. $f^{(k)}(x) = (-1)^k e^{-x}$, $f^{(k)}(0) = (-1)^k$; $\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k$

2. $f^{(k)}(x) = a^k e^{ax}$, $f^{(k)}(0) = a^k$; $\sum_{k=0}^{\infty} \frac{a^k}{k!} x^k$

3. $f^{(k)}(0) = 0$ if k is odd, $f^{(k)}(0)$ is alternately π^k and $-\pi^k$ if k is even; $\sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k}}{(2k)!} x^{2k}$

4. $f^{(k)}(0) = 0$ if k is even, $f^{(k)}(0)$ is alternately π^k and $-\pi^k$ if k is odd; $\sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k+1}}{(2k+1)!} x^{2k+1}$

5. $f^{(0)}(0) = 0$; for $k \geq 1$, $f^{(k)}(x) = \frac{(-1)^{k+1}(k-1)!}{(1+x)^k}$, $f^{(k)}(0) = (-1)^{k+1}(k-1)!$; $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k$

6. $f^{(k)}(x) = (-1)^k \dfrac{k!}{(1+x)^{k+1}}$; $f^{(k)}(0) = (-1)^k k!$; $\displaystyle\sum_{k=0}^{\infty}(-1)^k x^k$

7. $f^{(k)}(0) = 0$ if k is odd, $f^{(k)}(0) = 1$ if k is even; $\displaystyle\sum_{k=0}^{\infty}\frac{1}{(2k)!}x^{2k}$

8. $f^{(k)}(0) = 0$ if k is even, $f^{(k)}(0) = 1$ if k is odd; $\displaystyle\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}x^{2k+1}$

9. $f^{(k)}(x) = \begin{cases} (-1)^{k/2}(x\sin x - k\cos x) & k \text{ even} \\ (-1)^{(k-1)/2}(x\cos x + k\sin x) & k \text{ odd} \end{cases}$, $\quad f^{(k)}(0) = \begin{cases} (-1)^{1+k/2}k & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$

$\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+2}$

10. $f^{(k)}(x) = (k+x)e^x$, $f^{(k)}(0) = k$; $\displaystyle\sum_{k=1}^{\infty}\frac{1}{(k-1)!}x^k$

11. $f^{(k)}(x_0) = e$; $\displaystyle\sum_{k=0}^{\infty}\frac{e}{k!}(x-1)^k$

12. $f^{(k)}(x) = (-1)^k e^{-x}$, $f^{(k)}(\ln 2) = (-1)^k \dfrac{1}{2}$; $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{2\cdot k!}(x-\ln 2)^k$

13. $f^{(k)}(x) = \dfrac{(-1)^k k!}{x^{k+1}}$, $f^{(k)}(-1) = -k!$; $\displaystyle\sum_{k=0}^{\infty}(-1)(x+1)^k$

14. $f^{(k)}(x) = \dfrac{(-1)^k k!}{(x+2)^{k+1}}$, $f^{(k)}(3) = \dfrac{(-1)^k k!}{5^{k+1}}$; $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{5^{k+1}}(x-3)^k$

15. $f^{(k)}(1/2) = 0$ if k is odd, $f^{(k)}(1/2)$ is alternately π^k and $-\pi^k$ if k is even;

$\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k \pi^{2k}}{(2k)!}(x-1/2)^{2k}$

16. $f^{(k)}(\pi/2) = 0$ if k is even, $f^{(k)}(\pi/2)$ is alternately -1 and 1 if k is odd; $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{(2k+1)!}(x-\pi/2)^{2k+1}$

17. $f(1) = 0$, for $k \geq 1$, $f^{(k)}(x) = \dfrac{(-1)^{k-1}(k-1)!}{x^k}$; $f^{(k)}(1) = (-1)^{k-1}(k-1)!$;

$\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}(x-1)^k$

18. $f(e) = 1$, for $k \geq 1$, $f^{(k)}(x) = \dfrac{(-1)^{k-1}(k-1)!}{x^k}$; $f^{(k)}(e) = \dfrac{(-1)^{k-1}(k-1)!}{e^k}$;

$\displaystyle 1 + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{ke^k}(x-e)^k$

19. geometric series, $\rho = \lim_{k\to+\infty} \left|\frac{u_{k+1}}{u_k}\right| = |x|$, so the interval of convergence is $-1 < x < 1$, converges there to $\frac{1}{1+x}$ (the series diverges for $x = \pm 1$)

20. geometric series, $\rho = \lim_{k\to+\infty} \left|\frac{u_{k+1}}{u_k}\right| = |x|^2$, so the interval of convergence is $-1 < x < 1$, converges there to $\frac{1}{1-x^2}$ (the series diverges for $x = \pm 1$)

21. geometric series, $\rho = \lim_{k\to+\infty} \left|\frac{u_{k+1}}{u_k}\right| = |x-2|$, so the interval of convergence is $1 < x < 3$, converges there to $\frac{1}{1-(x-2)} = \frac{1}{3-x}$ (the series diverges for $x = 1, 3$)

22. geometric series, $\rho = \lim_{k\to+\infty} \left|\frac{u_{k+1}}{u_k}\right| = |x+3|$, so the interval of convergence is $-4 < x < -2$, converges there to $\frac{1}{1+(x+3)} = \frac{1}{4+x}$ (the series diverges for $x = -4, -2$)

23. **(a)** geometric series, $\rho = \lim_{k\to+\infty} \left|\frac{u_{k+1}}{u_k}\right| = |x/2|$, so the interval of convergence is $-2 < x < 2$, converges there to $\frac{1}{1+x/2} = \frac{2}{2+x}$; (the series diverges for $x = -2, 2$)

(b) $f(0) = 1;\ f(1) = 2/3$

24. **(a)** geometric series, $\rho = \lim_{k\to+\infty} \left|\frac{u_{k+1}}{u_k}\right| = \left|\frac{x-5}{3}\right|$, so the interval of convergence is $2 < x < 8$, converges to $\frac{1}{1+(x-5)/3} = \frac{3}{x-2}$ (the series diverges for $x = 2, 8$)

(b) $f(3) = 3,\ f(6) = 3/4$

25. $\rho = \lim_{k\to+\infty} \frac{k+1}{k+2}|x| = |x|$, the series converges if $|x| < 1$ and diverges if $|x| > 1$. If $x = -1$, $\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}$ converges by the Alternating Series Test; if $x = 1$, $\sum_{k=0}^{\infty} \frac{1}{k+1}$ diverges. The radius of convergence is 1, the interval of convergence is $[-1, 1)$.

26. $\rho = \lim_{k\to+\infty} 3|x| = 3|x|$, the series converges if $3|x| < 1$ or $|x| < 1/3$ and diverges if $|x| > 1/3$. If $x = -1/3$, $\sum_{k=0}^{\infty} (-1)^k$ diverges, if $x = 1/3$, $\sum_{k=0}^{\infty} (1)$ diverges. The radius of convergence is $1/3$, the interval of convergence is $(-1/3,\ 1/3)$.

27. $\rho = \lim_{k\to+\infty} \frac{|x|}{k+1} = 0$, the radius of convergence is $+\infty$, the interval is $(-\infty, +\infty)$.

28. $\rho = \lim_{k\to+\infty} \frac{k+1}{2}|x| = +\infty$, the radius of convergence is 0, the series converges only if $x = 0$.

29. $\rho = \lim_{k\to+\infty} \frac{5k^2|x|}{(k+1)^2} = 5|x|$, converges if $|x| < 1/5$ and diverges if $|x| > 1/5$. If $x = -1/5$, $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ converges; if $x = 1/5$, $\sum_{k=1}^{\infty} 1/k^2$ converges. Radius of convergence is 1/5, interval of convergence is $[-1/5, 1/5]$.

30. $\rho = \lim_{k\to+\infty} \frac{\ln k}{\ln(k+1)}|x| = |x|$, the series converges if $|x| < 1$ and diverges if $|x| > 1$. If $x = -1$, $\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln k}$ converges; if $x = 1$, $\sum_{k=2}^{\infty} 1/(\ln k)$ diverges (compare to $\sum(1/k)$). Radius of convergence is 1, interval of convergence is $[-1, 1)$.

31. $\rho = \lim_{k\to+\infty} \frac{k|x|}{k+2} = |x|$, converges if $|x| < 1$, diverges if $|x| > 1$. If $x = -1$, $\sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)}$ converges; if $x = 1$, $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ converges. Radius of convergence is 1, interval of convergence is $[-1, 1]$.

32. $\rho = \lim_{k\to+\infty} 2\frac{k+1}{k+2}|x| = 2|x|$, converges if $|x| < 1/2$, diverges if $|x| > 1/2$. If $x = -1/2$, $\sum_{k=0}^{\infty} \frac{-1}{2(k+1)}$ diverges; if $x = 1/2$, $\sum_{k=0}^{\infty} \frac{(-1)^k}{2(k+1)}$ converges. Radius of convergence is 1/2, interval of convergence is $(-1/2, 1/2]$.

33. $\rho = \lim_{k\to+\infty} \frac{\sqrt{k}}{\sqrt{k+1}}|x| = |x|$, converges if $|x| < 1$, diverges if $|x| > 1$. If $x = -1$, $\sum_{k=1}^{\infty} \frac{-1}{\sqrt{k}}$ diverges; if $x = 1$, $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\sqrt{k}}$ converges. Radius of convergence is 1, interval of convergence is $(-1, 1]$.

34. $\rho = \lim_{k\to+\infty} \frac{|x|^2}{(2k+2)(2k+1)} = 0$, radius of convergence is $+\infty$, interval of convergence is $(-\infty, +\infty)$.

35. $\rho = \lim_{k\to+\infty} \frac{|x|^2}{(2k+3)(2k+2)} = 0$, radius of convergence is $+\infty$, interval of convergence is $(-\infty, +\infty)$.

36. $\rho = \lim_{k\to+\infty} \frac{k^{3/2}|x|^3}{(k+1)^{3/2}} = |x|^3$, converges if $|x| < 1$, diverges if $|x| > 1$. If $x = -1$, $\sum_{k=0}^{\infty} \frac{1}{k^{3/2}}$ converges; if $x = 1$, $\sum_{k=0}^{\infty} \frac{(-1)^k}{k^{3/2}}$ converges. Radius of convergence is 1, interval of convergence is $[-1, 1]$.

37. $\rho = \lim_{k\to+\infty} \frac{3|x|}{k+1} = 0$, radius of convergence is $+\infty$, interval of convergence is $(-\infty, +\infty)$.

38. $\rho = \lim_{k\to+\infty} \dfrac{k(\ln k)^2|x|}{(k+1)[\ln(k+1)]^2} = |x|$, converges if $|x| < 1$, diverges if $|x| > 1$. If $x = -1$, then, by Exercise 10.5.25, $\sum_{k=2}^{\infty} \dfrac{-1}{k(\ln k)^2}$ converges; if $x = 1$, $\sum_{k=2}^{\infty} \dfrac{(-1)^{k+1}}{k(\ln k)^2}$ converges. Radius of convergence is 1, interval of convergence is $[-1, 1]$.

39. $\rho = \lim_{k\to+\infty} \dfrac{1+k^2}{1+(k+1)^2}|x| = |x|$, converges if $|x| < 1$, diverges if $|x| > 1$. If $x = -1$, $\sum_{k=0}^{\infty} \dfrac{(-1)^k}{1+k^2}$ converges; if $x = 1$, $\sum_{k=0}^{\infty} \dfrac{1}{1+k^2}$ converges. Radius of convergence is 1, interval of convergence is $[-1, 1]$.

40. $\rho = \lim_{k\to+\infty} \dfrac{1}{2}|x-3| = \dfrac{1}{2}|x-3|$, converges if $|x-3| < 2$, diverges if $|x-3| > 2$. If $x = 1$, $\sum_{k=0}^{\infty}(-1)^k$ diverges; if $x = 5$, $\sum_{k=0}^{\infty} 1$ diverges. Radius of convergence is 2, interval of convergence is $(1, 5)$.

41. $\rho = \lim_{k\to+\infty} \dfrac{k|x+1|}{k+1} = |x+1|$, converges if $|x+1| < 1$, diverges if $|x+1| > 1$. If $x = -2$, $\sum_{k=1}^{\infty} \dfrac{-1}{k}$ diverges; if $x = 0$, $\sum_{k=1}^{\infty} \dfrac{(-1)^{k+1}}{k}$ converges. Radius of convergence is 1, interval of convergence is $(-2, 0]$.

42. $\rho = \lim_{k\to+\infty} \dfrac{(k+1)^2}{(k+2)^2}|x-4| = |x-4|$, converges if $|x-4| < 1$, diverges if $|x-4| > 1$. If $x = 3$, $\sum_{k=0}^{\infty} 1/(k+1)^2$ converges; if $x = 5$, $\sum_{k=0}^{\infty}(-1)^k/(k+1)^2$ converges. Radius of convergence is 1, interval of convergence is $[3, 5]$.

43. $\rho = \lim_{k\to+\infty} (3/4)|x+5| = \dfrac{3}{4}|x+5|$, converges if $|x+5| < 4/3$, diverges if $|x+5| > 4/3$. If $x = -19/3$, $\sum_{k=0}^{\infty}(-1)^k$ diverges; if $x = -11/3$, $\sum_{k=0}^{\infty} 1$ diverges. Radius of convergence is $4/3$, interval of convergence is $(-19/3, -11/3)$.

44. $\rho = \lim_{k\to+\infty} \dfrac{(2k+3)(2k+2)k^3}{(k+1)^3}|x-2| = +\infty$, radius of convergence is 0, series converges only at $x = 2$.

45. $\rho = \lim_{k\to+\infty} \dfrac{k^2+4}{(k+1)^2+4}|x+1|^2 = |x+1|^2$, converges if $|x+1| < 1$, diverges if $|x+1| > 1$. If $x = -2$, $\sum_{k=1}^{\infty} \dfrac{(-1)^{3k+1}}{k^2+4}$ converges; if $x = 0$, $\sum_{k=1}^{\infty} \dfrac{(-1)^k}{k^2+4}$ converges. Radius of convergence is 1, interval of convergence is $[-2, 0]$.

46. $\rho = \lim_{k\to+\infty} \frac{k\ln(k+1)}{(k+1)\ln k}|x-3| = |x-3|$, converges if $|x-3|<1$, diverges if $|x-3|>1$. If $x=2$, $\sum_{k=1}^{\infty}\frac{(-1)^k \ln k}{k}$ converges; if $x=4$, $\sum_{k=1}^{\infty}\frac{\ln k}{k}$ diverges. Radius of convergence is 1, interval of convergence is $[2,4)$.

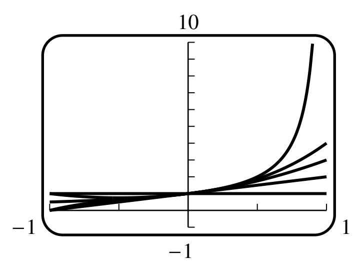
47. $\rho = \lim_{k\to+\infty} \frac{\pi|x-1|^2}{(2k+3)(2k+2)} = 0$, radius of convergence $+\infty$, interval of convergence $(-\infty,+\infty)$.

48. $\rho = \lim_{k\to+\infty} \frac{1}{16}|2x-3| = \frac{1}{16}|2x-3|$, converges if $\frac{1}{16}|2x-3|<1$ or $|x-3/2|<8$, diverges if $|x-3/2|>8$. If $x=-13/2$, $\sum_{k=0}^{\infty}(-1)^k$ diverges; if $x=19/2$, $\sum_{k=0}^{\infty}1$ diverges. Radius of convergence is 8, interval of convergence is $(-13/2, 19/2)$.

49. $\rho = \lim_{k\to+\infty}\sqrt[k]{|u_k|} = \lim_{k\to+\infty}\frac{|x|}{\ln k} = 0$, the series converges absolutely for all x so the interval of convergence is $(-\infty,+\infty)$.

50. $\rho = \lim_{k\to+\infty}\frac{2k+1}{(2k)(2k-1)}|x| = 0$
so $R=+\infty$.

51. **(a)**



52. Ratio Test: $\rho = \lim_{k\to+\infty}\frac{|x|^2}{4(k+1)(k+2)} = 0,\ R=+\infty$

53. By the Ratio Test for absolute convergence,

$$\rho = \lim_{k\to+\infty}\frac{(pk+p)!(k!)^p}{(pk)![(k+1)!]^p}|x| = \lim_{k\to+\infty}\frac{(pk+p)(pk+p-1)(pk+p-2)\cdots(pk+p-[p-1])}{(k+1)^p}|x|$$

$$= \lim_{k\to+\infty} p\left(p-\frac{1}{k+1}\right)\left(p-\frac{2}{k+1}\right)\cdots\left(p-\frac{p-1}{k+1}\right)|x| = p^p|x|,$$

converges if $|x|<1/p^p$, diverges if $|x|>1/p^p$. Radius of convergence is $1/p^p$.

54. By the Ratio Test for absolute convergence,

$$\rho = \lim_{k\to+\infty}\frac{(k+1+p)!k!(k+q)!}{(k+p)!(k+1)!(k+1+q)!}|x| = \lim_{k\to+\infty}\frac{k+1+p}{(k+1)(k+1+q)}|x| = 0,$$

radius of convergence is $+\infty$.

55. **(a)** By Theorem 10.5.3(b) both series converge or diverge together, so they have the same radius of convergence.

(b) By Theorem 10.5.3(a) the series $\sum(c_k+d_k)(x-x_0)^k$ converges if $|x-x_0|<R$; if $|x-x_0|>R$ then $\sum(c_k+d_k)(x-x_0)^k$ cannot converge, as otherwise $\sum c_k(x-x_0)^k$ would converge by the same Theorem. Hence the radius of convergence of $\sum(c_k+d_k)(x-x_0)^k$ is R.

(c) Let r be the radius of convergence of $\sum(c_k+d_k)(x-x_0)^k$. If $|x-x_0|<\min(R_1,R_2)$ then $\sum c_k(x-x_0)^k$ and $\sum d_k(x-x_0)^k$ converge, so $\sum(c_k+d_k)(x-x_0)^k$ converges. Hence $r\geq\min(R_1,R_2)$ (to see that $r>\min(R_1,R_2)$ is possible consider the case $c_k=-d_k=1$). If in addition $R_1\neq R_2$, and $R_1<|x-x_0|<R_2$ (or $R_2<|x-x_0|<R_1$) then $\sum(c_k+d_k)(x-x_0)^k$ cannot converge, as otherwise all three series would converge. Thus in this case $r=\min(R_1,R_2)$.

56. By the Root Test for absolute convergence,
$\rho=\lim_{k\to+\infty}|c_k|^{1/k}|x|=L|x|, L|x|<1$ if $|x|<1/L$ so the radius of convergence is $1/L$.

57. By assumption $\sum_{k=0}^{\infty}c_kx^k$ converges if $|x|<R$ so $\sum_{k=0}^{\infty}c_kx^{2k}=\sum_{k=0}^{\infty}c_k(x^2)^k$ converges if $|x^2|<R$, $|x|<\sqrt{R}$. Moreover, $\sum_{k=0}^{\infty}c_kx^{2k}=\sum_{k=0}^{\infty}c_k(x^2)^k$ diverges if $|x^2|>R$, $|x|>\sqrt{R}$. Thus $\sum_{k=0}^{\infty}c_kx^{2k}$ has radius of convergence $\sqrt{R}$.

58. The assumption is that $\sum_{k=0}^{\infty}c_kR^k$ is convergent and $\sum_{k=0}^{\infty}c_k(-R)^k$ is divergent. Suppose that $\sum_{k=0}^{\infty}c_kR^k$ is absolutely convergent then $\sum_{k=0}^{\infty}c_k(-R)^k$ is also absolutely convergent and hence convergent because $|c_kR^k|=|c_k(-R)^k|$, which contradicts the assumption that $\sum_{k=0}^{\infty}c_k(-R)^k$ is divergent so $\sum_{k=0}^{\infty}c_kR^k$ must be conditionally convergent.

EXERCISE SET 10.9

1. $\sin 4^\circ=\sin\left(\dfrac{\pi}{45}\right)=\dfrac{\pi}{45}-\dfrac{(\pi/45)^3}{3!}+\dfrac{(\pi/45)^5}{5!}-\cdots$

(a) Method 1: $|R_n(\pi/45)|\leq\dfrac{(\pi/45)^{n+1}}{(n+1)!}<0.000005$ for $n+1=4, n=3$;
$\sin 4^\circ\approx\dfrac{\pi}{45}-\dfrac{(\pi/45)^3}{3!}\approx 0.069756$

(b) Method 2: The first term in the alternating series that is less than 0.000005 is $\dfrac{(\pi/45)^5}{5!}$, so the result is the same as in Part (a).

2. $\cos 3^\circ=\cos\left(\dfrac{\pi}{60}\right)=1-\dfrac{(\pi/60)^2}{2}+\dfrac{(\pi/60)^4}{4!}-\cdots$

(a) Method 1: $|R_n(\pi/60)|\leq\dfrac{(\pi/60)^{n+1}}{(n+1)!}<0.0005$ for $n=2$; $\cos 3^\circ\approx 1-\dfrac{(\pi/60)^2}{2}\approx 0.9986$.

(b) Method 2: The first term in the alternating series that is less than 0.0005 is $\dfrac{(\pi/60)^4}{4!}$, so the result is the same as in Part (a).

3. $|R_n(0.1)| \le \dfrac{(0.1)^{n+1}}{(n+1)!} \le 0.000005$ for $n = 3$; $\cos 0.1 \approx 1 - (0.1)^2/2 = 0.99500$, calculator value $0.995004\ldots$

4. $(0.1)^3/3 < 0.5 \times 10^{-3}$ so $\tan^{-1}(0.1) \approx 0.100$, calculator value ≈ 0.0997

5. Expand about $\pi/2$ to get $\sin x = 1 - \dfrac{1}{2!}(x-\pi/2)^2 + \dfrac{1}{4!}(x-\pi/2)^4 - \cdots$, $85° = 17\pi/36$ radians,

$|R_n(x)| \le \dfrac{|x-\pi/2|^{n+1}}{(n+1)!}$, $|R_n(17\pi/36)| \le \dfrac{|17\pi/36 - \pi/2|^{n+1}}{(n+1)!} = \dfrac{(\pi/36)^{n+1}}{(n+1)!} < 0.5 \times 10^{-4}$

if $n = 3$, $\sin 85° \approx 1 - \dfrac{1}{2}(-\pi/36)^2 \approx 0.99619$, calculator value $0.99619\ldots$

6. $-175° = -\pi + \pi/36$ rad; $x_0 = -\pi, x = -\pi + \pi/36$, $\cos x = -1 + \dfrac{(x+\pi)^2}{2} - \dfrac{(x+\pi)^4}{4!} - \cdots$;

$|R_n| \le \dfrac{(\pi/36)^{n+1}}{(n+1)!} \le 0.00005$ for $n = 3$; $\cos(-\pi + \pi/36) = -1 + \dfrac{(\pi/36)^2}{2} \approx -0.99619$, calculator value $-0.99619\ldots$

7. $f^{(k)}(x) = \cosh x$ or $\sinh x, |f^{(k)}(x)| \le \cosh x \le \cosh 0.5 = \dfrac{1}{2}\left(e^{0.5} + e^{-0.5}\right) < \dfrac{1}{2}(2+1) = 1.5$

so $|R_n(x)| < \dfrac{1.5(0.5)^{n+1}}{(n+1)!} \le 0.5 \times 10^{-3}$ if $n = 4$, $\sinh 0.5 \approx 0.5 + \dfrac{(0.5)^3}{3!} \approx 0.5208$, calculator value $0.52109\ldots$

8. $f^{(k)}(x) = \cosh x$ or $\sinh x, |f^{(k)}(x)| \le \cosh x \le \cosh 0.1 = \dfrac{1}{2}\left(e^{0.1} + e^{-0.1}\right) < 1.06$ so

$|R_n(x)| < \dfrac{1.06(0.1)^{n+1}}{(n+1)!} \le 0.5 \times 10^{-3}$ for $n = 2$, $\cosh 0.1 \approx 1 + \dfrac{(0.1)^2}{2!} = 1.005$, calculator value $1.0050\ldots$

9. $f(x) = \sin x$, $f^{(n+1)}(x) = \pm\sin x$ or $\pm\cos x$, $|f^{(n+1)}(x)| \le 1$, $|R_n(x)| \le \dfrac{|x-\pi/4|^{n+1}}{(n+1)!}$,

$\lim\limits_{n\to+\infty} \dfrac{|x-\pi/4|^{n+1}}{(n+1)!} = 0$; by the Squeezing Theorem, $\lim\limits_{n\to+\infty} |R_n(x)| = 0$

so $\lim\limits_{n\to+\infty} R_n(x) = 0$ for all x.

10. $f(x) = e^x$, $f^{(n+1)}(x) = e^x$; if $x > 1$ then $|R_n(x)| \le \dfrac{e^x}{(n+1)!}|x-1|^{n+1}$; if $x < 1$ then

$|R_n(x)| \le \dfrac{e}{(n+1)!}|x-1|^{n+1}$. But $\lim\limits_{n\to+\infty} \dfrac{|x-1|^{n+1}}{(n+1)!} = 0$ so $\lim\limits_{n\to+\infty} R_n(x) = 0$.

11. **(a)** Let $x = 1/9$ in series (13).

(b) $\ln 1.25 \approx 2\left(1/9 + \dfrac{(1/9)^3}{3}\right) = 2(1/9 + 1/3^7) \approx 0.223$, which agrees with the calculator value $0.22314\ldots$ to three decimal places.

12. **(a)** Let $x = 1/2$ in series (13).

(b) $\ln 3 \approx 2\left(1/2 + \frac{(1/2)^3}{3}\right) = 2(1/2 + 1/24) = 13/12 \approx 1.083$; the calculator value is 1.099 to three decimal places.

13. **(a)** $(1/2)^9/9 < 0.5 \times 10^{-3}$ and $(1/3)^7/7 < 0.5 \times 10^{-3}$ so

$$\tan^{-1}(1/2) \approx 1/2 - \frac{(1/2)^3}{3} + \frac{(1/2)^5}{5} - \frac{(1/2)^7}{7} \approx 0.4635$$

$$\tan^{-1}(1/3) \approx 1/3 - \frac{(1/3)^3}{3} + \frac{(1/3)^5}{5} \approx 0.3218$$

(b) From Formula (17), $\pi \approx 4(0.4635 + 0.3218) = 3.1412$

(c) Let $a = \tan^{-1}\frac{1}{2}, b = \tan^{-1}\frac{1}{3}$; then $|a - 0.4635| < 0.0005$ and $|b - 0.3218| < 0.0005$, so $|4(a + b) - 3.1412| \le 4|a - 0.4635| + 4|b - 0.3218| < 0.004$, so two decimal-place accuracy is guaranteed, but not three.

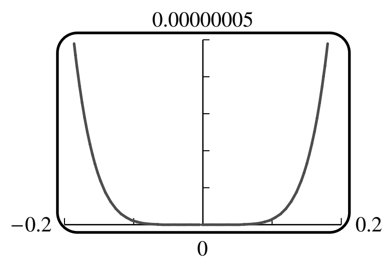
14. $(27+x)^{1/3} = 3(1+x/3^3)^{1/3} = 3\left(1 + \frac{1}{3^4}x - \frac{1 \cdot 2}{3^8 2}x^2 + \frac{1 \cdot 2 \cdot 5}{3^{12} 3!}x^3 + \ldots\right)$, alternates after first term,

$\frac{3 \cdot 2}{3^8 2} < 0.0005$, $\sqrt{28} \approx 3\left(1 + \frac{1}{3^4}\right) \approx 3.0370$

15. **(a)** $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + (0)x^5 + R_5(x)$,

$|R_5(x)| \le \frac{|x|^6}{6!} \le \frac{(0.2)^6}{6!} < 9 \times 10^{-8}$

(b)



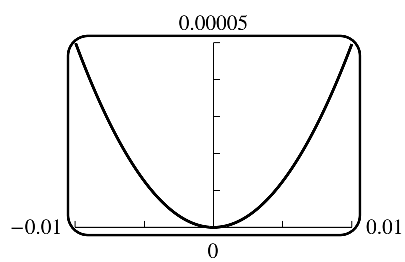
16. **(a)** $f''(x) = -1/(1+x)^2$,

$|f''(x)| < 1/(0.99)^2 \le 1.03$,

$|R_1(x)| \le \frac{1.03|x|^2}{2} \le \frac{1.03(0.01)^2}{2}$

$\le 5.15 \times 10^{-5}$ for $-0.01 \le x \le 0.01$

(b)



17. **(a)** $(1+x)^{-1} = 1 - x + \frac{-1(-2)}{2!}x^2 + \frac{-1(-2)(-3)}{3!}x^3 + \cdots + \frac{-1(-2)(-3)\cdots(-k)}{k!}x^k + \cdots$

$$= \sum_{k=0}^{\infty}(-1)^k x^k$$

(b) $(1+x)^{1/3} = 1 + (1/3)x + \dfrac{(1/3)(-2/3)}{2!}x^2 + \dfrac{(1/3)(-2/3)(-5/3)}{3!}x^3 + \cdots$

$$+ \frac{(1/3)(-2/3)\cdots(4-3k)/3}{k!}x^k + \cdots = 1 + x/3 + \sum_{k=2}^{\infty}(-1)^{k-1}\frac{2\cdot 5\cdots(3k-4)}{3^k k!}x^k$$

(c) $(1+x)^{-3} = 1 - 3x + \dfrac{(-3)(-4)}{2!}x^2 + \dfrac{(-3)(-4)(-5)}{3!}x^3 + \cdots + \dfrac{(-3)(-4)\cdots(-2-k)}{k!}x^k + \cdots$

$$= \sum_{k=0}^{\infty}(-1)^k\frac{(k+2)!}{2\cdot k!}x^k = \sum_{k=0}^{\infty}(-1)^k\frac{(k+2)(k+1)}{2}x^k$$

18. $(1+x)^m = \binom{m}{0} + \sum_{k=1}^{\infty}\binom{m}{k}x^k = \sum_{k=0}^{\infty}\binom{m}{k}x^k$

19. **(a)** $\dfrac{d}{dx}\ln(1+x) = \dfrac{1}{1+x}, \dfrac{d^k}{dx^k}\ln(1+x) = (-1)^{k-1}\dfrac{(k-1)!}{(1+x)^k}$; similarly $\dfrac{d}{dx}\ln(1-x) = -\dfrac{(k-1)!}{(1-x)^k}$,

so $f^{(n+1)}(x) = n!\left[\dfrac{(-1)^n}{(1+x)^{n+1}} + \dfrac{1}{(1-x)^{n+1}}\right]$.

(b) $\left|f^{(n+1)}(x)\right| \le n!\left|\dfrac{(-1)^n}{(1+x)^{n+1}}\right| + n!\left|\dfrac{1}{(1-x)^{n+1}}\right| = n!\left[\dfrac{1}{(1+x)^{n+1}} + \dfrac{1}{(1-x)^{n+1}}\right]$

(c) If $\left|f^{(n+1)}(x)\right| \le M$ on the interval $[0, 1/3]$ then $|R_n(1/3)| \le \dfrac{M}{(n+1)!}\left(\dfrac{1}{3}\right)^{n+1}$.

(d) If $0 \le x \le 1/3$ then $1+x \ge 1, 1-x \ge 2/3$, $\left|f^{(n+1)}(x)\right| \le M = n!\left[1 + \dfrac{1}{(2/3)^{n+1}}\right]$.

(e) $0.000005 \ge \dfrac{M}{(n+1)!}\left(\dfrac{1}{3}\right)^{n+1} = \dfrac{1}{n+1}\left[\left(\dfrac{1}{3}\right)^{n+1} + \dfrac{(1/3)^{n+1}}{(2/3)^{n+1}}\right] = \dfrac{1}{n+1}\left[\left(\dfrac{1}{3}\right)^{n+1} + \left(\dfrac{1}{2}\right)^{n+1}\right]$

20. Set $x = 1/4$ in Formula (13). Follow the argument of Exercise 19: Parts (a) and (b) remain unchanged; in Part (c) replace (1/3) with (1/4):
$\left|R_n\left(\dfrac{1}{4}\right)\right| \le \dfrac{M}{(n+1)!}\left(\dfrac{1}{4}\right)^{n+1} \le 0.000005$ for x in the interval $[0, 1/4]$. From Part (b), together with $0 \le x \le 1/4, 1+x \ge 1, 1-x \ge 3/4$, follows Part (d): $M = n!\left[1 + \dfrac{1}{(3/4)^{n+1}}\right]$. Part (e) now becomes $0.000005 \ge \dfrac{M}{(n+1)!}\left(\dfrac{1}{4}\right)^{n+1} = \dfrac{1}{n+1}\left[\left(\dfrac{1}{4}\right)^{n+1} + \left(\dfrac{1}{3}\right)^{n+1}\right]$, which is true for $n = 9$.

21. $f(x) = \cos x, f^{(n+1)}(x) = \pm\sin x$ or $\pm\cos x, |f^{(n+1)}(x)| \le 1,$ set $M = 1$,
$|R_n(x)| \le \dfrac{1}{(n+1)!}|x-a|^{n+1}, \lim_{n\to+\infty}\dfrac{|x-a|^{n+1}}{(n+1)!} = 0$ so $\lim_{n\to+\infty}R_n(x) = 0$ for all x.

22. $f(x) = \sin x, f^{(n+1)}(x) = \pm\sin x$ or $\pm\cos x, |f^{(n+1)}(x)| \le 1$, follow Exercise 21.

23. **(a)** From Machin's formula and a CAS, $\dfrac{\pi}{4} \approx 0.7853981633974483096156608$, accurate to the 25th decimal place.

(b)

n	s_n
0	0.3183098 78 . . .
1	0.3183098 861837906 067 . . .
2	0.3183098 861837906 7153776 695 . . .
3	0.3183098 861837906 7153776 752674502 34 . . .
$1/\pi$	0.3183098 861837906 7153776 752674502 87 . . .

24. (a) $f'(0) = \lim_{h\to 0} \dfrac{f(h)-f(0)}{h} = \lim_{h\to 0} \dfrac{e^{-1/h^2}}{h}$, let $t = 1/h$ then $h = 1/t$ and

$\lim_{h\to 0^+} \dfrac{e^{-1/h^2}}{h} = \lim_{t\to +\infty} te^{-t^2} = \lim_{t\to +\infty} \dfrac{t}{e^{t^2}} = \lim_{t\to +\infty} \dfrac{1}{2te^{t^2}} = 0$, similarly $\lim_{h\to 0^-} \dfrac{e^{-1/h^2}}{h} = 0$ so $f'(0) = 0$.

(b) The Maclaurin series is $0 + 0\cdot x + 0\cdot x^2 + \cdots = 0$, but $f(0) = 0$ and $f(x) > 0$ if $x \neq 0$ so the series converges to $f(x)$ only at the point $x = 0$.

EXERCISE SET 10.10

1. (a) Replace x with $-x$: $\dfrac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^k x^k + \cdots;\ R = 1$.

(b) Replace x with x^2 : $\dfrac{1}{1-x^2} = 1 + x^2 + x^4 + \cdots + x^{2k} + \cdots;\ R = 1$.

(c) Replace x with $2x$: $\dfrac{1}{1-2x} = 1 + 2x + 4x^2 + \cdots + 2^k x^k + \cdots;\ R = 1/2$.

(d) $\dfrac{1}{2-x} = \dfrac{1/2}{1-x/2}$; replace x with $x/2$: $\dfrac{1}{2-x} = \dfrac{1}{2} + \dfrac{1}{2^2}x + \dfrac{1}{2^3}x^2 + \cdots + \dfrac{1}{2^{k+1}}x^k + \cdots;\ R = 2$.

2. (a) Replace x with $-x$: $\ln(1-x) = -x - x^2/2 - x^3/3 - \cdots - x^k/k - \cdots;\ R = 1$.

(b) Replace x with x^2 : $\ln(1+x^2) = x^2 - x^4/2 + x^6/3 - \cdots + (-1)^{k-1}x^{2k}/k + \cdots;\ R = 1$.

(c) Replace x with $2x$: $\ln(1+2x) = 2x - (2x)^2/2 + (2x)^3/3 - \cdots + (-1)^{k-1}(2x)^k/k + \cdots;\ R = 1/2$.

(d) $\ln(2+x) = \ln 2 + \ln(1+x/2)$; replace x with $x/2$:
$\ln(2+x) = \ln 2 + x/2 - (x/2)^2/2 + (x/2)^3/3 + \cdots + (-1)^{k-1}(x/2)^k/k + \cdots;\ R = 2$.

3. (a) From Section 10.9, Example 5(b), $\dfrac{1}{\sqrt{1+x}} = 1 - \dfrac{1}{2}x + \dfrac{1\cdot 3}{2^2\cdot 2!}x^2 - \dfrac{1\cdot 3\cdot 5}{2^3\cdot 3!}x^3 + \cdots$, so

$$(2+x)^{-1/2} = \frac{1}{\sqrt{2}\sqrt{1+x/2}} = \frac{1}{2^{1/2}} - \frac{1}{2^{5/2}}x + \frac{1\cdot 3}{2^{9/2}\cdot 2!}x^2 - \frac{1\cdot 3\cdot 5}{2^{13/2}\cdot 3!}x^3 + \cdots$$

(b) Example 5(a): $\dfrac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$, so $\dfrac{1}{(1-x^2)^2} = 1 + 2x^2 + 3x^4 + 4x^6 + \cdots$

4. (a) $\dfrac{1}{a-x} = \dfrac{1/a}{1-x/a} = 1/a + x/a^2 + x^2/a^3 + \cdots + x^k/a^{k+1} + \cdots;\ R = |a|$.

(b) $1/(a+x)^2 = \dfrac{1}{a^2}\dfrac{1}{(1+x/a)^2} = \dfrac{1}{a^2}\left(1 - 2(x/a) + 3(x/a)^2 - 4(x/a)^3 + \cdots\right)$

$$= \frac{1}{a^2} - \frac{2}{a^3}x + \frac{3}{a^4}x^2 - \frac{4}{a^5}x^3 + \cdots; \quad R = |a|$$

5. **(a)** $2x - \frac{2^3}{3!}x^3 + \frac{2^5}{5!}x^5 - \frac{2^7}{7!}x^7 + \cdots$; $R = +\infty$

(b) $1 - 2x + 2x^2 - \frac{4}{3}x^3 + \cdots$; $R = +\infty$

(c) $1 + x^2 + \frac{1}{2!}x^4 + \frac{1}{3!}x^6 + \cdots$; $R = +\infty$

(d) $x^2 - \frac{\pi^2}{2}x^4 + \frac{\pi^4}{4!}x^6 - \frac{\pi^6}{6!}x^8 + \cdots$; $R = +\infty$

6. **(a)** $1 - \frac{2^2}{2!}x^2 + \frac{2^4}{4!}x^4 - \frac{2^6}{6!}x^6 + \cdots$; $R = +\infty$

(b) $x^2\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots\right) = x^2 + x^3 + \frac{1}{2!}x^4 + \frac{1}{3!}x^5 + \cdots$; $R = +\infty$

(c) $x\left(1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \cdots\right) = x - x^2 + \frac{1}{2!}x^3 - \frac{1}{3!}x^4 + \cdots$; $R = +\infty$

(d) $x^2 - \frac{1}{3!}x^6 + \frac{1}{5!}x^{10} - \frac{1}{7!}x^{14} + \cdots$; $R = +\infty$

7. **(a)** $x^2\left(1 - 3x + 9x^2 - 27x^3 + \cdots\right) = x^2 - 3x^3 + 9x^4 - 27x^5 + \cdots$; $R = 1/3$

(b) $x\left(2x + \frac{2^3}{3!}x^3 + \frac{2^5}{5!}x^5 + \frac{2^7}{7!}x^7 + \cdots\right) = 2x^2 + \frac{2^3}{3!}x^4 + \frac{2^5}{5!}x^6 + \frac{2^7}{7!}x^8 + \cdots$; $R = +\infty$

(c) Substitute $3/2$ for m and $-x^2$ for x in Equation (18) of Section 10.9, then multiply by x: $x - \frac{3}{2}x^3 + \frac{3}{8}x^5 + \frac{1}{16}x^7 + \cdots$; $R = 1$

8. **(a)** $\frac{x}{x-1} = \frac{-x}{1-x} = -x\left(1 + x + x^2 + x^3 + \cdots\right) = -x - x^2 - x^3 - x^4 - \cdots$; $R = 1$.

(b) $3 + \frac{3}{2!}x^4 + \frac{3}{4!}x^8 + \frac{3}{6!}x^{12} + \cdots$; $R = +\infty$

(c) From Table 10.9.1 with $m = -3$, $(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \cdots$, so
$x(1+2x)^{-3} = x - 6x^2 + 24x^3 - 80x^4 + \cdots$; $R = 1/2$

9. **(a)** $\sin^2 x = \frac{1}{2}(1 - \cos 2x) = \frac{1}{2}\left[1 - \left(1 - \frac{2^2}{2!}x^2 + \frac{2^4}{4!}x^4 - \frac{2^6}{6!}x^6 + \cdots\right)\right]$

$$= x^2 - \frac{2^3}{4!}x^4 + \frac{2^5}{6!}x^6 - \frac{2^7}{8!}x^8 + \cdots$$

(b) $\ln\left[(1+x^3)^{12}\right] = 12\ln(1+x^3) = 12x^3 - 6x^6 + 4x^9 - 3x^{12} + \cdots$

10. **(a)** $\cos^2 x = \frac{1}{2}(1 + \cos 2x) = \frac{1}{2}\left[1 + \left(1 - \frac{2^2}{2!}x^2 + \frac{2^4}{4!}x^4 - \frac{2^6}{6!}x^6 + \cdots\right)\right]$

$$= 1 - x^2 + \frac{2^3}{4!}x^4 - \frac{2^5}{6!}x^6 + \cdots$$

(b) In Equation (13) of Section 10.9 replace x with $-x$: $\ln\left(\frac{1-x}{1+x}\right) = -2\left(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots\right)$

11. **(a)** $\dfrac{1}{x} = \dfrac{1}{1-(1-x)} = 1 + (1-x) + (1-x)^2 + \cdots + (1-x)^k + \cdots$
$= 1 - (x-1) + (x-1)^2 - \cdots + (-1)^k (x-1)^k + \cdots$

(b) $(0, 2)$

12. **(a)** $\dfrac{1}{x} = \dfrac{1/x_0}{1 + (x-x_0)/x_0} = 1/x_0 - (x-x_0)/x_0^2 + (x-x_0)^2/x_0^3 - \cdots + (-1)^k (x-x_0)^k / x_0^{k+1} + \cdots$

(b) $(0, 2x_0)$

13. **(a)** $(1 + x + x^2/2 + x^3/3! + x^4/4! + \cdots)(x - x^3/3! + x^5/5! - \cdots) = x + x^2 + x^3/3 - x^5/30 + \cdots$

(b) $(1 + x/2 - x^2/8 + x^3/16 - (5/128)x^4 + \cdots)(x - x^2/2 + x^3/3 - x^4/4 + x^5/5 - \cdots)$
$= x - x^3/24 + x^4/24 - (71/1920)x^5 + \cdots$

14. **(a)** $(1 - x^2 + x^4/2 - x^6/6 + \cdots)\left(1 - \dfrac{1}{2}x^2 + \dfrac{1}{24}x^4 - \dfrac{1}{720}x^6 \cdots\right) = 1 - \dfrac{3}{2}x^2 + \dfrac{25}{24}x^4 - \dfrac{331}{720}x^6 + \cdots$

(b) $\left(1 + \dfrac{4}{3}x^2 + \cdots\right)\left(1 + \dfrac{1}{3}x - \dfrac{1}{9}x^2 + \dfrac{5}{81}x^3 - \cdots\right) = 1 + \dfrac{1}{3}x + \dfrac{11}{9}x^2 + \dfrac{41}{81}x^3 + \cdots$

15. **(a)** $\dfrac{1}{\cos x} = 1 \Big/ \left(1 - \dfrac{1}{2!}x^2 + \dfrac{1}{4!}x^4 - \dfrac{1}{6!}x^6 + \cdots\right) = 1 + \dfrac{1}{2}x^2 + \dfrac{5}{24}x^4 + \dfrac{61}{720}x^6 + \cdots$

(b) $\dfrac{\sin x}{e^x} = \left(x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots\right) \Big/ \left(1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots\right) = x - x^2 + \dfrac{1}{3}x^3 - \dfrac{1}{30}x^5 + \cdots$

16. **(a)** $\dfrac{\tan^{-1} x}{1+x} = \left(x - x^3/3 + x^5/5 - \cdots\right) / (1+x) = x - x^2 + \dfrac{2}{3}x^3 - \dfrac{2}{3}x^4 \cdots$

(b) $\dfrac{\ln(1+x)}{1-x} = \left(x - x^2/2 + x^3/3 - x^4/4 + \cdots\right) / (1-x) = x + \dfrac{1}{2}x^2 + \dfrac{5}{6}x^3 + \dfrac{7}{12}x^4 + \cdots$

17. $e^x = 1 + x + x^2/2 + x^3/3! + \cdots + x^k/k! + \cdots,\ e^{-x} = 1 - x + x^2/2 - x^3/3! + \cdots + (-1)^k x^k/k! + \cdots;$

$\sinh x = \dfrac{1}{2}\left(e^x - e^{-x}\right) = x + x^3/3! + x^5/5! + \cdots + x^{2k+1}/(2k+1)! + \cdots, R = +\infty$

$\cosh x = \dfrac{1}{2}\left(e^x + e^{-x}\right) = 1 + x^2/2 + x^4/4! + \cdots + x^{2k}/(2k)! + \cdots, R = +\infty$

18. $\tanh x = \dfrac{x + x^3/3! + x^5/5! + x^7/7! + \cdots}{1 + x^2/2 + x^4/4! + x^6/6! \cdots} = x - \dfrac{1}{3}x^3 + \dfrac{2}{15}x^5 - \dfrac{17}{315}x^7 + \cdots$

19. $\dfrac{4x-2}{x^2-1} = \dfrac{-1}{1-x} + \dfrac{3}{1+x} = -\left(1 + x + x^2 + x^3 + x^4 + \cdots\right) + 3\left(1 - x + x^2 - x^3 + x^4 + \cdots\right)$
$= 2 - 4x + 2x^2 - 4x^3 + 2x^4 + \cdots$

20. $\dfrac{x^3 + x^2 + 2x - 2}{x^2 - 1} = x + 1 - \dfrac{1}{1-x} + \dfrac{2}{1+x}$
$= x + 1 - \left(1 + x + x^2 + x^3 + x^4 + \cdots\right) + 2\left(1 - x + x^2 - x^3 + x^4 + \cdots\right)$
$= 2 - 2x + x^2 - 3x^3 + x^4 - \cdots$

21. **(a)** $\dfrac{d}{dx}\left(1 - x^2/2! + x^4/4! - x^6/6! + \cdots\right) = -x + x^3/3! - x^5/5! + \cdots = -\sin x$

(b) $\dfrac{d}{dx}\left(x - x^2/2 + x^3/3 - \cdots\right) = 1 - x + x^2 - \cdots = 1/(1+x)$

22. **(a)** $\dfrac{d}{dx}\left(x + x^3/3! + x^5/5! + \cdots\right) = 1 + x^2/2! + x^4/4! + \cdots = \cosh x$

(b) $\dfrac{d}{dx}\left(x - x^3/3 + x^5/5 - x^7/7 + \cdots\right) = 1 - x^2 + x^4 - x^6 + \cdots = \dfrac{1}{1+x^2}$

23. **(a)** $$\int \left(1 + x + x^2/2! + \cdots\right) dx = (x + x^2/2! + x^3/3! + \cdots) + C_1 = \left(1 + x + x^2/2! + x^3/3! + \cdots\right) + C_1 - 1 = e^x + C$$

(b) $$\int \left(x + x^3/3! + x^5/5! + \cdots\right) = x^2/2! + x^4/4! + \cdots + C_1 = 1 + x^2/2! + x^4/4! + \cdots + C_1 - 1 = \cosh x + C$$

24. **(a)** $$\int \left(x - x^3/3! + x^5/5! - \cdots\right) dx = \left(x^2/2! - x^4/4! + x^6/6! - \cdots\right) + C_1 = -\left(1 - x^2/2! + x^4/4! - x^6/6! + \cdots\right) + C_1 + 1 = -\cos x + C$$

(b) $$\int \left(1 - x + x^2 - \cdots\right) dx = \left(x - x^2/2 + x^3/3 - \cdots\right) + C = \ln(1+x) + C$$

(*Note:* $-1 < x < 1$, so $|1+x| = 1 + x$)

25. **(a)** Substitute x^2 for x in the Maclaurin Series for $1/(1-x)$ (Table 10.9.1) and then multiply by x: $\dfrac{x}{1-x^2} = x\displaystyle\sum_{k=0}^{\infty}(x^2)^k = \sum_{k=0}^{\infty} x^{2k+1}$

(b) $f^{(5)}(0) = 5!c_5 = 5!,\ f^{(6)}(0) = 6!c_6 = 0$ **(c)** $f^{(n)}(0) = n!c_n = \begin{cases} n! & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$

26. $x^2 \cos 2x = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k}}{(2k)!} x^{2k+2}$; $f^{(99)}(0) = 0$ because $c_{99} = 0$.

27. **(a)** $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0}\left(1 - x^2/3! + x^4/5! - \cdots\right) = 1$

(b) $\displaystyle\lim_{x\to 0} \frac{\tan^{-1} x - x}{x^3} = \lim_{x\to 0} \frac{\left(x - x^3/3 + x^5/5 - x^7/7 + \cdots\right) - x}{x^3} = -1/3$

28. **(a)** $$\frac{1-\cos x}{\sin x} = \frac{1 - \left(1 - x^2/2! + x^4/4! - x^6/6! + \cdots\right)}{x - x^3/3! + x^5/5! - \cdots} = \frac{x^2/2! - x^4/4! + x^6/6! - \cdots}{x - x^3/3! + x^5/5! - \cdots}$$
$$= \frac{x/2! - x^3/4! + x^5/6! - \cdots}{1 - x^2/3! + x^4/5! - \cdots}, x \neq 0;\ \lim_{x\to 0} \frac{1-\cos x}{\sin x} = \frac{0}{1} = 0$$

(b) $\lim_{x\to 0}\frac{1}{x}\left[\ln\sqrt{1+x}-\sin 2x\right]=\lim_{x\to 0}\frac{1}{x}\left[\frac{1}{2}\ln(1+x)-\sin 2x\right]$

$$=\lim_{x\to 0}\frac{1}{x}\left[\frac{1}{2}\left(x-\frac{1}{2}x^2+\frac{1}{3}x^3-\cdots\right)-\left(2x-\frac{4}{3}x^3+\frac{4}{15}x^5-\cdots\right)\right]$$

$$=\lim_{x\to 0}\left(-\frac{3}{2}-\frac{1}{4}x+\frac{3}{2}x^2+\cdots\right)=-3/2$$

29. $\int_0^1 \sin\left(x^2\right)dx=\int_0^1\left(x^2-\frac{1}{3!}x^6+\frac{1}{5!}x^{10}-\frac{1}{7!}x^{14}+\cdots\right)dx$

$$=\frac{1}{3}x^3-\frac{1}{7\cdot 3!}x^7+\frac{1}{11\cdot 5!}x^{11}-\frac{1}{15\cdot 7!}x^{15}+\cdots\Big]_0^1$$

$$=\frac{1}{3}-\frac{1}{7\cdot 3!}+\frac{1}{11\cdot 5!}-\frac{1}{15\cdot 7!}+\cdots,$$

but $\frac{1}{15\cdot 7!}<0.5\times 10^{-3}$ so $\int_0^1\sin(x^2)dx\approx\frac{1}{3}-\frac{1}{7\cdot 3!}+\frac{1}{11\cdot 5!}\approx 0.3103$

30. $\int_0^{1/2}\tan^{-1}\left(2x^2\right)dx=\int_0^{1/2}\left(2x^2-\frac{8}{3}x^6+\frac{32}{5}x^{10}-\frac{128}{7}x^{14}+\cdots\right)dx$

$$=\frac{2}{3}x^3-\frac{8}{21}x^7+\frac{32}{55}x^{11}-\frac{128}{105}x^{15}+\cdots\Big]_0^{1/2}$$

$$=\frac{2}{3}\frac{1}{2^3}-\frac{8}{21}\frac{1}{2^7}+\frac{32}{55}\frac{1}{2^{11}}-\frac{128}{105}\frac{1}{2^{15}}-\cdots,$$

but $\frac{32}{55\cdot 2^{11}}<0.5\times 10^{-3}$ so $\int_0^{1/2}\tan^{-1}(2x^2)dx\approx\frac{2}{3\cdot 2^3}-\frac{8}{21\cdot 2^7}\approx 0.0804$

31. $\int_0^{0.2}\left(1+x^4\right)^{1/3}dx=\int_0^{0.2}\left(1+\frac{1}{3}x^4-\frac{1}{9}x^8+\cdots\right)dx$

$$=x+\frac{1}{15}x^5-\frac{1}{81}x^9+\cdots\Big]_0^{0.2}=0.2+\frac{1}{15}(0.2)^5-\frac{1}{81}(0.2)^9+\cdots,$$

but $\frac{1}{15}(0.2)^5<0.5\times 10^{-3}$ so $\int_0^{0.2}(1+x^4)^{1/3}dx\approx 0.200$

32. $\int_0^{1/2}(1+x^2)^{-1/4}dx=\int_0^{1/2}\left(1-\frac{1}{4}x^2+\frac{5}{32}x^4-\frac{15}{128}x^6+\cdots\right)dx$

$$=x-\frac{1}{12}x^3+\frac{1}{32}x^5-\frac{15}{896}x^7+\cdots\Big]_0^{1/2}$$

$$=1/2-\frac{1}{12}(1/2)^3+\frac{1}{32}(1/2)^5-\frac{15}{896}(1/2)^7+\cdots,$$

but $\frac{15}{896}(1/2)^7<0.5\times 10^{-3}$ so $\int_0^{1/2}(1+x^2)^{-1/4}dx\approx 1/2-\frac{1}{12}(1/2)^3+\frac{1}{32}(1/2)^5\approx 0.4906$

33. **(a)** $\dfrac{x}{(1-x)^2}=x\dfrac{d}{dx}\left[\dfrac{1}{1-x}\right]=x\dfrac{d}{dx}\left[\sum_{k=0}^{\infty}x^k\right]=x\left[\sum_{k=1}^{\infty}kx^{k-1}\right]=\sum_{k=1}^{\infty}kx^k$

(b) $$-\ln(1-x)=\int\frac{1}{1-x}dx-C=\int\left[\sum_{k=0}^{\infty}x^k\right]dx-C$$
$$=\sum_{k=0}^{\infty}\frac{x^{k+1}}{k+1}-C=\sum_{k=1}^{\infty}\frac{x^k}{k}-C,-\ln(1-0)=0 \text{ so } C=0.$$

(c) Replace x with $-x$ in Part (b): $\ln(1+x)=-\sum_{k=1}^{+\infty}\dfrac{(-1)^k}{k}x^k=\sum_{k=1}^{+\infty}\dfrac{(-1)^{k+1}}{k}x^k$

(d) $\sum_{k=1}^{+\infty}\dfrac{(-1)^{k+1}}{k}$ converges by the Alternating Series Test.

(e) By Parts (c) and (d) and the remark, $\sum_{k=1}^{+\infty}\dfrac{(-1)^{k+1}}{k}x^k$ converges to $\ln(1+x)$ for $-1<x\le 1$.

34. **(a)** In Exercise 33(a), set $x=\dfrac{1}{3}$, $S=\dfrac{1/3}{(1-1/3)^2}=\dfrac{3}{4}$

(b) In Part (b) set $x=1/4, S=\ln(4/3)$

(c) In Part (e) set $x=1, S=\ln 2$

35. **(a)** $$\sinh^{-1}x=\int\left(1+x^2\right)^{-1/2}dx-C=\int\left(1-\frac{1}{2}x^2+\frac{3}{8}x^4-\frac{5}{16}x^6+\cdots\right)dx-C$$
$$=\left(x-\frac{1}{6}x^3+\frac{3}{40}x^5-\frac{5}{112}x^7+\cdots\right)-C;\ \sinh^{-1}0=0 \text{ so } C=0.$$

(b) $$\left(1+x^2\right)^{-1/2}=1+\sum_{k=1}^{\infty}\frac{(-1/2)(-3/2)(-5/2)\cdots(-1/2-k+1)}{k!}(x^2)^k$$
$$=1+\sum_{k=1}^{\infty}(-1)^k\frac{1\cdot 3\cdot 5\cdots(2k-1)}{2^k k!}x^{2k},$$
$$\sinh^{-1}x=x+\sum_{k=1}^{\infty}(-1)^k\frac{1\cdot 3\cdot 5\cdots(2k-1)}{2^k k!(2k+1)}x^{2k+1}$$

(c) $R=1$

36. **(a)** $$\sin^{-1}x=\int(1-x^2)^{-1/2}dx-C=\int\left(1+\frac{1}{2}x^2+\frac{3}{8}x^4+\frac{5}{16}x^6+\cdots\right)dx-C$$
$$=\left(x+\frac{1}{6}x^3+\frac{3}{40}x^5+\frac{5}{112}x^7+\cdots\right)-C,\sin^{-1}0=0 \text{ so } C=0$$

(b) $$\begin{aligned}(1-x^2)^{-1/2} &= 1+\sum_{k=1}^{\infty}\frac{(-1/2)(-3/2)(-5/2)\cdots(-1/2-k+1)}{k!}(-x^2)^k \\ &= 1+\sum_{k=1}^{\infty}\frac{(-1)^k(1/2)^k(1)(3)(5)\cdots(2k-1)}{k!}(-1)^k x^{2k} \\ &= 1+\sum_{k=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2k-1)}{2^k k!}x^{2k} \\ \sin^{-1}x &= x+\sum_{k=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2k-1)}{2^k k!(2k+1)}x^{2k+1}\end{aligned}$$

(c) $R=1$

37. **(a)** $y(t)=y_0\sum_{k=0}^{\infty}\frac{(-1)^k(0.000121)^k t^k}{k!}$

(b) $y(1)\approx y_0(1-0.000121t)\Big]_{t=1}=0.999879y_0$

(c) $y_0e^{-0.000121}\approx 0.9998790073y_0$

38. **(a)** If $\frac{ct}{m}\approx 0$ then $e^{-ct/m}\approx 1-\frac{ct}{m}$, and $v(t)\approx\left(1-\frac{ct}{m}\right)\left(v_0+\frac{mg}{c}\right)-\frac{mg}{c}=v_0-\left(\frac{cv_0}{m}+g\right)t$.

(b) The quadratic approximation is

$$v_0\approx\left(1-\frac{ct}{m}+\frac{(ct)^2}{2m^2}\right)\left(v_0+\frac{mg}{c}\right)-\frac{mg}{c}=v_0-\left(\frac{cv_0}{m}+g\right)t+\frac{c^2}{2m^2}\left(v_0+\frac{mg}{c}\right)t^2.$$

39. $\theta_0=5^\circ=\pi/36$ rad, $k=\sin(\pi/72)$

(a) $T\approx 2\pi\sqrt{\frac{L}{g}}=2\pi\sqrt{1/9.8}\approx 2.00709$

(b) $T\approx 2\pi\sqrt{\frac{L}{g}}\left(1+\frac{k^2}{4}\right)\approx 2.008044621$

(c) 2.008045644

40. The third order model gives the same result as the second, because there is no term of degree three in (5). By the Wallis sine formula, $\int_0^{\pi/2}\sin^4\phi\,d\phi=\frac{1\cdot 3}{2\cdot 4}\frac{\pi}{2}$, and

$$\begin{aligned}T\approx 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left(1+\frac{1}{2}k^2\sin^2\phi+\frac{1\cdot 3}{2^2 2!}k^4\sin^4\phi\right)d\phi &= 4\sqrt{\frac{L}{g}}\left(\frac{\pi}{2}+\frac{k^2}{2}\frac{\pi}{4}+\frac{3k^4}{8}\frac{3\pi}{16}\right) \\ &= 2\pi\sqrt{\frac{L}{g}}\left(1+\frac{k^2}{4}+\frac{9k^4}{64}\right)\end{aligned}$$

41. **(a)** $F = \dfrac{mgR^2}{(R+h)^2} = \dfrac{mg}{(1+h/R)^2} = mg\left(1 - 2h/R + 3h^2/R^2 - 4h^3/R^3 + \cdots\right)$

(b) If $h = 0$, then the binomial series converges to 1 and $F = mg$.

(c) Sum the series to the linear term, $F \approx mg - 2mgh/R$.

(d) $\dfrac{mg - 2mgh/R}{mg} = 1 - \dfrac{2h}{R} = 1 - \dfrac{2 \cdot 29{,}028}{4000 \cdot 5280} \approx 0.9973$, so about 0.27% less.

42. **(a)** We can differentiate term-by-term:

$$y' = \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k-1}}{2^{2k-1}k!(k-1)!} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+1}}{2^{2k+1}(k+1)!k!},\ y'' = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}(2k+1)x^{2k}}{2^{2k+1}(k+1)!k!}, \text{ and}$$

$$xy'' + y' + xy = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}(2k+1)x^{2k+1}}{2^{2k+1}(k+1)!k!} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}x^{2k+1}}{2^{2k+1}(k+1)!k!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k}(k!)^2},$$

$$xy'' + y' + xy = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}x^{2k+1}}{2^{2k}(k!)^2}\left[\frac{2k+1}{2(k+1)} + \frac{1}{2(k+1)} - 1\right] = 0$$

(b) $y' = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k(2k+1)x^{2k}}{2^{2k+1}k!(k+1)!},\ y'' = \sum_{k=1}^{\infty} \frac{(-1)^k(2k+1)x^{2k-1}}{2^{2k}(k-1)!(k+1)!}.$

Since $J_1(x) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1}k!(k+1)!}$ and $x^2 J_1(x) = \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}x^{2k+1}}{2^{2k-1}(k-1)!k!}$, it follows that

$$x^2y'' + xy' + (x^2-1)y$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k(2k+1)x^{2k+1}}{2^{2k}(k-1)!(k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k(2k+1)x^{2k+1}}{2^{2k+1}(k!)(k+1)!} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}x^{2k+1}}{2^{2k-1}(k-1)!k!} - \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1}k!(k+1)!}$$

$$= \frac{x}{2} - \frac{x}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k-1}(k-1)!k!}\left(\frac{2k+1}{2(k+1)} + \frac{2k+1}{4k(k+1)} - 1 - \frac{1}{4k(k+1)}\right) = 0.$$

(c) From Part (a), $J_0'(x) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^{k+1}x^{2k+1}}{2^{2k+1}(k+1)!k!} = -J_1(x).$

43. Let $f(x) = \displaystyle\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} b_k x^k$ for $-r < x < r$. Then $a_k = f^{(k)}(0)/k! = b_k$ for all k.

CHAPTER 10 SUPPLEMENTARY EXERCISES

4. **(a)** $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$ **(b)** $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$

9. **(a)** always true by Theorem 10.5.2

(b) sometimes false, for example the harmonic series diverges but $\sum(1/k^2)$ converges

(c) sometimes false, for example $f(x) = \sin \pi x, a_k = 0, L = 0$
(d) always true by the comments which follow Example 3(d) of Section 10.2
(e) sometimes false, for example $a_n = \frac{1}{2} + (-1)^n \frac{1}{4}$
(f) sometimes false, for example $u_k = 1/2$
(g) always false by Theorem 10.5.3
(h) sometimes false, for example $u_k = 1/k, v_k = 2/k$
(i) always true by the Comparison Test
(j) always true by the Comparison Test
(k) sometimes false, for example $\sum (-1)^k/k$
(l) sometimes false, for example $\sum (-1)^k/k$

10. **(a)** false, $f(x)$ is not differentiable at $x = 0$, Definition 10.8.1
(b) true: $s_n = 1$ if n is odd and $s_{2n} = 1 + 1/(n+1)$; $\lim_{n\to+\infty} s_n = 1$
(c) false, $\lim a_k \neq 0$

11. **(a)** geometric, $r = 1/5$, converges **(b)** $1/(5^k + 1) < 1/5^k$, converges
(c) $\dfrac{9}{\sqrt{k}+1} \geq \dfrac{9}{\sqrt{k}+\sqrt{k}} = \dfrac{9}{2\sqrt{k}}, \sum_{k=1}^{\infty} \dfrac{9}{2\sqrt{k}}$ diverges

12. **(a)** converges by Alternating Series Test
(b) absolutely convergent, $\sum_{k=1}^{\infty} \left[\dfrac{k+2}{3k-1}\right]^k$ converges by the Root Test.
(c) $\dfrac{k^{-1/2}}{2+\sin^2 k} > \dfrac{k^{-1}}{2+1} = \dfrac{1}{3k}, \quad \sum_{k=1}^{\infty} \dfrac{1}{3k}$ diverges

13. **(a)** $\dfrac{1}{k^3+2k+1} < \dfrac{1}{k^3}, \sum_{k=1}^{\infty} 1/k^3$ converges, so $\sum_{k=1}^{\infty} \dfrac{1}{k^3+2k+1}$ converges by the Comparison Test
(b) Limit Comparison Test, compare with the divergent series $\sum_{k=1}^{\infty} \dfrac{1}{k^{2/5}}$, diverges
(c) $\left|\dfrac{\cos(1/k)}{k^2}\right| < \dfrac{1}{k^2}, \sum_{k=1}^{\infty} \dfrac{1}{k^2}$ converges, so $\sum_{k=1}^{\infty} \dfrac{\cos(1/k)}{k^2}$ converges absolutely

14. **(a)** $\sum_{k=1}^{\infty} \dfrac{\ln k}{k\sqrt{k}} = \sum_{k=2}^{\infty} \dfrac{\ln k}{k\sqrt{k}}$ because $\ln 1 = 0$,

$$\int_2^{+\infty} \frac{\ln x}{x^{3/2}} dx = \lim_{\ell\to+\infty} \left[-\frac{2\ln x}{x^{1/2}} - \frac{4}{x^{1/2}}\right]_2^{\ell} = \sqrt{2}(\ln 2 + 2) \text{ so } \sum_{k=2}^{\infty} \frac{\ln k}{k^{3/2}} \text{ converges}$$

(b) $\dfrac{k^{4/3}}{8k^2+5k+1} \geq \dfrac{k^{4/3}}{8k^2+5k^2+k^2} = \dfrac{1}{14k^{2/3}}, \sum_{k=1}^{\infty} \dfrac{1}{14k^{2/3}}$ diverges
(c) absolutely convergent, $\sum_{k=1}^{\infty} \dfrac{1}{k^2+1}$ converges (compare with $\sum 1/k^2$)

15. $\sum_{k=0}^{\infty}\frac{1}{5^k}-\sum_{k=0}^{99}\frac{1}{5^k}=\sum_{k=100}^{\infty}\frac{1}{5^k}=\frac{1}{5^{100}}\sum_{k=0}^{\infty}\frac{1}{5^k}=\frac{1}{4\cdot 5^{99}}$

16. no, $\lim_{k\to+\infty} a_k=\frac{1}{2}\neq 0$ (Divergence Test)

17. **(a)** $p_0(x)=1, p_1(x)=1-7x, p_2(x)=1-7x+5x^2, p_3(x)=1-7x+5x^2+4x^3$, $p_4(x)=1-7x+5x^2+4x^3$

(b) If $f(x)$ is a polynomial of degree n and $k\geq n$ then the Maclaurin polynomial of degree k is the polynomial itself; if $k<n$ then it is the truncated polynomial.

18. $\ln(1+x)=x-x^2/2+\cdots$; so $|\ln(1+x)-x|\leq x^2/2$ by Theorem 10.7.2.

19. $\sin x=x-x^3/3!+x^5/5!-x^7/7!+\cdots$ is an alternating series, so
$|\sin x-x+x^3/3!-x^5/5!|\leq x^7/7!\leq \pi^7/(4^7 7!)\leq 0.00005$

20. $\int_0^1\frac{1-\cos x}{x}\,dx=\left[\frac{x^2}{2\cdot 2!}-\frac{x^4}{4\cdot 4!}+\frac{x^6}{6\cdot 6!}-\cdots\right]_0^1=\frac{1}{2\cdot 2!}-\frac{1}{4\cdot 4!}+\frac{1}{6\cdot 6!}-\cdots$, and $\frac{1}{6\cdot 6!}<0.0005$,

so $\int_0^1\frac{1-\cos x}{x}\,dx=\frac{1}{2\cdot 2!}-\frac{1}{4\cdot 4!}=0.2396$ to three decimal-place accuracy.

21. **(a)** $\rho=\lim_{k\to+\infty}\left(\frac{2^k}{k!}\right)^{1/k}=\lim_{k\to+\infty}\frac{2}{\sqrt[k]{k!}}=0$, converges

(b) $\rho=\lim_{k\to+\infty}u_k^{1/k}=\lim_{k\to+\infty}\frac{k}{\sqrt[k]{k!}}=e$, diverges

22. **(a)** $1\leq k, 2\leq k, 3\leq k,\ldots,k\leq k$, therefore $1\cdot 2\cdot 3\cdots k\leq k\cdot k\cdot k\cdots k$, or $k!\leq k^k$.

(b) $\sum\frac{1}{k^k}\leq\sum\frac{1}{k!}$, converges

(c) $\lim_{k\to+\infty}\left(\frac{1}{k^k}\right)^{1/k}=\lim_{k\to+\infty}\frac{1}{k}=0$, converges

23. **(a)** $u_{100}=\sum_{k=1}^{100}u_k-\sum_{k=1}^{99}u_k=\left(2-\frac{1}{100}\right)-\left(2-\frac{1}{99}\right)=\frac{1}{9900}$

(b) $u_1=1$; for $k\geq 2, u_k=\left(2-\frac{1}{k}\right)-\left(2-\frac{1}{k-1}\right)=\frac{1}{k(k-1)}, \lim_{k\to+\infty}u_k=0$

(c) $\sum_{k=1}^{\infty}u_k=\lim_{n\to+\infty}\sum_{k=1}^{n}u_k=\lim_{n\to+\infty}\left(2-\frac{1}{n}\right)=2$

24. **(a)** $\sum_{k=1}^{\infty}\left(\frac{3}{2^k}-\frac{2}{3^k}\right)=\sum_{k=1}^{\infty}\frac{3}{2^k}-\sum_{k=1}^{\infty}\frac{2}{3^k}=\left(\frac{3}{2}\right)\frac{1}{1-(1/2)}-\left(\frac{2}{3}\right)\frac{1}{1-(1/3)}=2$ (geometric series)

(b) $\sum_{k=1}^{n}[\ln(k+1)-\ln k]=\ln(n+1)$, so $\sum_{k=1}^{\infty}[\ln(k+1)-\ln k]=\lim_{n\to+\infty}\ln(n+1)=+\infty$, diverges

(c) $\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{1}{2}\left(\frac{1}{k}-\frac{1}{k+2}\right)=\lim_{n\to+\infty}\frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right)=\frac{3}{4}$

(d) $\lim_{n\to+\infty}\sum_{k=1}^{n}\left[\tan^{-1}(k+1)-\tan^{-1}k\right]=\lim_{n\to+\infty}\left[\tan^{-1}(n+1)-\tan^{-1}(1)\right]=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$

25. **(a)** e^2-1 **(b)** $\sin\pi=0$ **(c)** $\cos e$ **(d)** $e^{-\ln 3}=1/3$

26. $a_k=\sqrt{a_{k-1}}=a_{k-1}^{1/2}=a_{k-2}^{1/4}=\cdots=a_1^{1/2^{k-1}}=c^{1/2^k}$

(a) If $c=1/2$ then $\lim_{k\to+\infty}a_k=1$. **(b)** if $c=3/2$ then $\lim_{k\to+\infty}a_k=1$.

27. $e^{-x}=1-x+x^2/2!+\cdots$. Replace x with $-(\frac{x-100}{16})^2/2$ to obtain

$e^{-(\frac{x-100}{16})^2/2}=1-\frac{(x-100)^2}{2\cdot 16^2}+\frac{(x-100)^4}{8\cdot 16^4}+\cdots$, thus

$p\approx\frac{1}{16\sqrt{2\pi}}\int_{100}^{110}\left[1-\frac{(x-100)^2}{2\cdot 16^2}+\frac{(x-100)^4}{8\cdot 16^4}\right]dx\approx 0.23406$ or 23.406%.

28. $f(x)=xe^x=x+x^2+\frac{x^3}{2!}+\frac{x^4}{3!}+\cdots=\sum_{k=0}^{\infty}\frac{x^{k+1}}{k!}$,

$f'(x)=(x+1)e^x=1+2x+\frac{3x^2}{2!}+\frac{4x^3}{3!}+\cdots=\sum_{k=0}^{\infty}\frac{k+1}{k!}x^k$; $\sum_{k=0}^{\infty}\frac{k+1}{k!}=f'(1)=2e$.

29. Let $A=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots$; since the series all converge absolutely,

$\frac{\pi^2}{6}-A=2\frac{1}{2^2}+2\frac{1}{4^2}+2\frac{1}{6^2}+\cdots=\frac{1}{2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right)=\frac{1}{2}\frac{\pi^2}{6}$, so $A=\frac{1}{2}\frac{\pi^2}{6}=\frac{\pi^2}{12}$.

30. Compare with $1/k^p$: converges if $p>1$, diverges otherwise.

31. **(a)** $x+\frac{1}{2}x^2+\frac{3}{14}x^3+\frac{3}{35}x^4+\cdots$; $\rho=\lim_{k\to+\infty}\frac{k+1}{3k+1}|x|=\frac{1}{3}|x|$,

converges if $\frac{1}{3}|x|<1$, $|x|<3$ so $R=3$.

(b) $-x^3+\frac{2}{3}x^5-\frac{2}{5}x^7+\frac{8}{35}x^9-\cdots$; $\rho=\lim_{k\to+\infty}\frac{k+1}{2k+1}|x|^2=\frac{1}{2}|x|^2$,

converges if $\frac{1}{2}|x|^2<1$, $|x|^2<2$, $|x|<\sqrt{2}$ so $R=\sqrt{2}$.

32. By the Ratio Test for absolute convergence, $\rho=\lim_{k\to+\infty}\frac{|x-x_0|}{b}=\frac{|x-x_0|}{b}$; converges if $|x-x_0|<b$, diverges if $|x-x_0|>b$. If $x=x_0-b$, $\sum_{k=0}^{\infty}(-1)^k$ diverges; if $x=x_0+b$, $\sum_{k=0}^{\infty}1$ diverges. The interval of convergence is (x_0-b,x_0+b).

33. If $x \geq 0$, then $\cos\sqrt{x} = 1 - \dfrac{(\sqrt{x})^2}{2!} + \dfrac{(\sqrt{x})^4}{4!} - \dfrac{(\sqrt{x})^6}{6!} + \cdots = 1 - \dfrac{x}{2!} + \dfrac{x^2}{4!} - \dfrac{x^3}{6!} + \cdots$; if $x \leq 0$, then

$$\cosh(\sqrt{-x}) = 1 + \frac{(\sqrt{-x})^2}{2!} + \frac{(\sqrt{-x})^4}{4!} + \frac{(\sqrt{-x})^6}{6!} + \cdots = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots.$$

34. By Exercise 74 of Section 3.5, the derivative of an odd (even) function is even (odd); hence all odd-numbered derivatives of an odd function are even, all even-numbered derivatives of an odd function are odd; a similar statement holds for an even function.

(a) If $f(x)$ is an even function, then $f^{(2k-1)}(x)$ is an odd function, so $f^{(2k-1)}(0) = 0$, and thus the MacLaurin series coefficients $a_{2k-1} = 0, k = 1, 2, \cdots$.

(b) If $f(x)$ is an odd function, then $f^{(2k)}(x)$ is an odd function, so $f^{(2k)}(0) = 0$, and thus the MacLaurin series coefficients $a_{2k} = 0, k = 1, 2, \cdots$.

35. $\left(1 - \dfrac{v^2}{c^2}\right)^{-1/2} \approx 1 + \dfrac{v^2}{2c^2}$, so $K = m_0c^2\left[\dfrac{1}{\sqrt{1 - v^2/c^2}} - 1\right] \approx m_0c^2(v^2/(2c^2) = m_0v^2/2$

36. **(a)** $\displaystyle\int_n^{+\infty} \frac{1}{x^{3.7}}dx < 0.005$ if $n > 4.93$; let $n = 5$.

(b) $s_n \approx 1.1062$; CAS: 1.10628824