CHAPTER 8
Principles of Integral Evaluation

EXERCISE SET 8.1

1. $u = 3 - 2x, du = -2dx, \quad -\frac{1}{2}\int u^3\,du = -\frac{1}{8}u^4 + C = -\frac{1}{8}(3-2x)^4 + C$

2. $u = 4 + 9x, du = 9dx, \quad \frac{1}{9}\int u^{1/2}\,du = \frac{2}{3\cdot 9}u^{3/2} + C = \frac{2}{27}(4+9x)^{3/2} + C$

3. $u = x^2, du = 2xdx, \quad \frac{1}{2}\int \sec^2 u\,du = \frac{1}{2}\tan u + C = \frac{1}{2}\tan(x^2) + C$

4. $u = x^2, du = 2xdx, \quad 2\int \tan u\,du = -2\ln|\cos u| + C = -2\ln|\cos(x^2)| + C$

5. $u = 2 + \cos 3x, du = -3\sin 3x dx, \quad -\frac{1}{3}\int \frac{du}{u} = -\frac{1}{3}\ln|u| + C = -\frac{1}{3}\ln(2+\cos 3x) + C$

6. $u = \frac{3x}{2}, du = \frac{3}{2}dx, \quad \frac{2}{3}\int \frac{du}{4+4u^2} = \frac{1}{6}\int \frac{du}{1+u^2} = \frac{1}{6}\tan^{-1}u + C = \frac{1}{6}\tan^{-1}(3x/2) + C$

7. $u = e^x, du = e^x dx, \quad \int \sinh u\,du = \cosh u + C = \cosh e^x + C$

8. $u = \ln x, du = \frac{1}{x}dx, \quad \int \sec u \tan u\,du = \sec u + C = \sec(\ln x) + C$

9. $u = \cot x, du = -\csc^2 x dx, \quad -\int e^u\,du = -e^u + C = -e^{\cot x} + C$

10. $u = x^2, du = 2xdx, \quad \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1}u + C = \frac{1}{2}\sin^{-1}(x^2) + C$

11. $u = \cos 7x, du = -7\sin 7x dx, \quad -\frac{1}{7}\int u^5\,du = -\frac{1}{42}u^6 + C = -\frac{1}{42}\cos^6 7x + C$

12. $u = \sin x, du = \cos x\,dx, \quad \int \frac{du}{u\sqrt{u^2+1}} = -\ln\left|\frac{1+\sqrt{1+u^2}}{u}\right| + C = -\ln\left|\frac{1+\sqrt{1+\sin^2 x}}{\sin x}\right| + C$

13. $u = e^x, du = e^x dx, \quad \int \frac{du}{\sqrt{4+u^2}} = \ln\left(u + \sqrt{u^2+4}\right) + C = \ln\left(e^x + \sqrt{e^{2x}+4}\right) + C$

14. $u = \tan^{-1}x, du = \frac{1}{1+x^2}\,dx, \quad \int e^u\,du = e^u + C = e^{\tan^{-1}x} + C$

15. $u = \sqrt{x-2}, du = \frac{1}{2\sqrt{x-2}}\,dx, \quad 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{x-2}} + C$

16. $u = 3x^2 + 2x, du = (6x+2)dx, \quad \frac{1}{2}\int \cot u\,du = \frac{1}{2}\ln|\sin u| + C = \frac{1}{2}\ln\sin|3x^2+2x| + C$

17. $u = \sqrt{x}, du = \frac{1}{2\sqrt{x}}\,dx, \quad \int 2\cosh u\,du = 2\sinh u + C = 2\sinh\sqrt{x} + C$

18. $u = \ln x, du = \dfrac{dx}{x}, \quad \displaystyle\int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C$

19. $u = \sqrt{x}, du = \dfrac{1}{2\sqrt{x}}\,dx, \quad \displaystyle\int \frac{2\,du}{3^u} = 2\int e^{-u\ln 3}\,du = -\frac{2}{\ln 3}e^{-u\ln 3} + C = -\frac{2}{\ln 3}3^{-\sqrt{x}} + C$

20. $u = \sin\theta, du = \cos\theta d\theta, \quad \displaystyle\int \sec u \tan u\,du = \sec u + C = \sec(\sin\theta) + C$

21. $u = \dfrac{2}{x}, du = -\dfrac{2}{x^2}\,dx, \quad \displaystyle -\frac{1}{2}\int \operatorname{csch}^2 u\,du = \frac{1}{2}\coth u + C = \frac{1}{2}\coth\frac{2}{x} + C$

22. $\displaystyle\int \frac{dx}{\sqrt{x^2-3}} = \ln\left|x + \sqrt{x^2-3}\right| + C$

23. $u = e^{-x}, du = -e^{-x}dx, \quad \displaystyle -\int \frac{du}{4-u^2} = -\frac{1}{4}\ln\left|\frac{2+u}{2-u}\right| + C = -\frac{1}{4}\ln\left|\frac{2+e^{-x}}{2-e^{-x}}\right| + C$

24. $u = \ln x, du = \dfrac{1}{x}dx, \quad \displaystyle\int \cos u\,du = \sin u + C = \sin(\ln x) + C$

25. $u = e^x, du = e^x dx, \quad \displaystyle\int \frac{e^x\,dx}{\sqrt{1-e^{2x}}} = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1} e^x + C$

26. $u = x^{-1/2}, du = -\dfrac{1}{2x^{3/2}}\,dx, \quad \displaystyle -\int 2\sinh u\,du = -2\cosh u + C = -2\cosh(x^{-1/2}) + C$

27. $u = x^2, du = 2xdx, \quad \displaystyle \frac{1}{2}\int \frac{du}{\sec u} = \frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u + C = \frac{1}{2}\sin(x^2) + C$

28. $2u = e^x, 2du = e^x dx, \quad \displaystyle\int \frac{2du}{\sqrt{4-4u^2}} = \sin^{-1} u + C = \sin^{-1}(e^x/2) + C$

29. $4^{-x^2} = e^{-x^2\ln 4}, u = -x^2\ln 4, du = -2x\ln 4\,dx = -x\ln 16\,dx,$

$\displaystyle -\frac{1}{\ln 16}\int e^u\,du = -\frac{1}{\ln 16}e^u + C = -\frac{1}{\ln 16}e^{-x^2\ln 4} + C = -\frac{1}{\ln 16}4^{-x^2} + C$

30. $2^{\pi x} = e^{\pi x\ln 2}, \quad \displaystyle\int 2^{\pi x}\,dx = \frac{1}{\pi\ln 2}e^{\pi x\ln 2} + C = \frac{1}{\pi\ln 2}2^{\pi x} + C$

EXERCISE SET 8.2

1. $u = x,\ dv = e^{-x}dx,\ du = dx,\ v = -e^{-x};\ \displaystyle\int xe^{-x}dx = -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x} + C$

2. $u = x,\ dv = e^{3x}dx,\ du = dx,\ v = \dfrac{1}{3}e^{3x};\ \displaystyle\int xe^{3x}dx = \frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$

3. $u = x^2,\ dv = e^x dx,\ du = 2x\,dx,\ v = e^x;\ \displaystyle\int x^2e^x dx = x^2e^x - 2\int xe^x dx.$

For $\displaystyle\int xe^x dx$ use $u = x,\ dv = e^x dx,\ du = dx,\ v = e^x$ to get

$\displaystyle\int xe^x dx = xe^x - e^x + C_1$ so $\displaystyle\int x^2e^x dx = x^2e^x - 2xe^x + 2e^x + C$

4. $u = x^2$, $dv = e^{-2x}dx$, $du = 2x\,dx$, $v = -\frac{1}{2}e^{-2x}$; $\int x^2e^{-2x}dx = -\frac{1}{2}x^2e^{-2x} + \int xe^{-2x}dx$

For $\int xe^{-2x}dx$ use $u = x$, $dv = e^{-2x}dx$ to get

$\int xe^{-2x}dx = -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}dx = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$

so $\int x^2e^{-2x}dx = -\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$

5. $u = x$, $dv = \sin 2x\,dx$, $du = dx$, $v = -\frac{1}{2}\cos 2x$;

$\int x\sin 2x\,dx = -\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x\,dx = -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C$

6. $u = x$, $dv = \cos 3x\,dx$, $du = dx$, $v = \frac{1}{3}\sin 3x$;

$\int x\cos 3x\,dx = \frac{1}{3}x\sin 3x - \frac{1}{3}\int \sin 3x\,dx = \frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x + C$

7. $u = x^2$, $dv = \cos x\,dx$, $du = 2x\,dx$, $v = \sin x$; $\int x^2\cos x\,dx = x^2\sin x - 2\int x\sin x\,dx$

For $\int x\sin x\,dx$ use $u = x$, $dv = \sin x\,dx$ to get

$\int x\sin x\,dx = -x\cos x + \sin x + C_1$ so $\int x^2\cos x\,dx = x^2\sin x + 2x\cos x - 2\sin x + C$

8. $u = x^2$, $dv = \sin x\,dx$, $du = 2x\,dx$, $v = -\cos x$;

$\int x^2\sin x\,dx = -x^2\cos x + 2\int x\cos x\,dx$; for $\int x\cos x\,dx$ use $u = x$, $dv = \cos x\,dx$ to get

$\int x\cos x\,dx = x\sin x + \cos x + C_1$ so $\int x^2\sin x\,dx = -x^2\cos x + 2x\sin x + 2\cos x + C$

9. $u = \ln x$, $dv = \sqrt{x}\,dx$, $du = \frac{1}{x}dx$, $v = \frac{2}{3}x^{3/2}$;

$\int \sqrt{x}\ln x\,dx = \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int x^{1/2}dx = \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + C$

10. $u = \ln x$, $dv = x\,dx$, $du = \frac{1}{x}dx$, $v = \frac{1}{2}x^2$; $\int x\ln x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$

11. $u = (\ln x)^2$, $dv = dx$, $du = 2\frac{\ln x}{x}dx$, $v = x$; $\int (\ln x)^2dx = x(\ln x)^2 - 2\int \ln x\,dx$.

Use $u = \ln x$, $dv = dx$ to get $\int \ln x\,dx = x\ln x - \int dx = x\ln x - x + C_1$ so

$\int (\ln x)^2dx = x(\ln x)^2 - 2x\ln x + 2x + C$

12. $u = \ln x$, $dv = \frac{1}{\sqrt{x}}dx$, $du = \frac{1}{x}dx$, $v = 2\sqrt{x}$; $\int \frac{\ln x}{\sqrt{x}}dx = 2\sqrt{x}\ln x - 2\int \frac{1}{\sqrt{x}}dx = 2\sqrt{x}\ln x - 4\sqrt{x} + C$

13. $u = \ln(2x+3)$, $dv = dx$, $du = \dfrac{2}{2x+3}dx$, $v = x$; $\displaystyle\int \ln(2x+3)dx = x\ln(2x+3) - \int \frac{2x}{2x+3}dx$

but $\displaystyle\int \frac{2x}{2x+3}dx = \int \left(1 - \frac{3}{2x+3}\right)dx = x - \frac{3}{2}\ln(2x+3) + C_1$ so

$$\int \ln(2x+3)dx = x\ln(2x+3) - x + \frac{3}{2}\ln(2x+3) + C$$

14. $u = \ln(x^2+4)$, $dv = dx$, $du = \dfrac{2x}{x^2+4}dx$, $v = x$; $\displaystyle\int \ln(x^2+4)dx = x\ln(x^2+4) - 2\int \frac{x^2}{x^2+4}dx$

but $\displaystyle\int \frac{x^2}{x^2+4}dx = \int \left(1 - \frac{4}{x^2+4}\right)dx = x - 2\tan^{-1}\frac{x}{2} + C_1$ so

$$\int \ln(x^2+4)dx = x\ln(x^2+4) - 2x + 4\tan^{-1}\frac{x}{2} + C$$

15. $u = \sin^{-1} x$, $dv = dx$, $du = 1/\sqrt{1-x^2}dx$, $v = x$;

$$\int \sin^{-1} x\,dx = x\sin^{-1} x - \int x/\sqrt{1-x^2}dx = x\sin^{-1} x + \sqrt{1-x^2} + C$$

16. $u = \cos^{-1}(2x)$, $dv = dx$, $du = -\dfrac{2}{\sqrt{1-4x^2}}dx$, $v = x$;

$$\int \cos^{-1}(2x)dx = x\cos^{-1}(2x) + \int \frac{2x}{\sqrt{1-4x^2}}dx = x\cos^{-1}(2x) - \frac{1}{2}\sqrt{1-4x^2} + C$$

17. $u = \tan^{-1}(2x)$, $dv = dx$, $du = \dfrac{2}{1+4x^2}dx$, $v = x$;

$$\int \tan^{-1}(2x)dx = x\tan^{-1}(2x) - \int \frac{2x}{1+4x^2}dx = x\tan^{-1}(2x) - \frac{1}{4}\ln(1+4x^2) + C$$

18. $u = \tan^{-1} x$, $dv = x\,dx$, $du = \dfrac{1}{1+x^2}dx$, $v = \dfrac{1}{2}x^2$; $\displaystyle\int x\tan^{-1} x\,dx = \frac{1}{2}x^2\tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2}dx$

but $\displaystyle\int \frac{x^2}{1+x^2}dx = \int \left(1 - \frac{1}{1+x^2}\right)dx = x - \tan^{-1} x + C_1$ so

$$\int x\tan^{-1} x\,dx = \frac{1}{2}x^2\tan^{-1} x - \frac{1}{2}x + \frac{1}{2}\tan^{-1} x + C$$

19. $u = e^x$, $dv = \sin x\,dx$, $du = e^x dx$, $v = -\cos x$; $\displaystyle\int e^x \sin x\,dx = -e^x\cos x + \int e^x\cos x\,dx.$

For $\displaystyle\int e^x\cos x\,dx$ use $u = e^x$, $dv = \cos x\,dx$ to get $\displaystyle\int e^x\cos x = e^x\sin x - \int e^x\sin x\,dx$ so

$$\int e^x\sin x\,dx = -e^x\cos x + e^x\sin x - \int e^x\sin x\,dx,$$

$$2\int e^x\sin x\,dx = e^x(\sin x - \cos x) + C_1, \quad \int e^x\sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + C$$

20. $u = e^{2x}$, $dv = \cos 3x\,dx$, $du = 2e^{2x}dx$, $v = \dfrac{1}{3}\sin 3x$;

$\displaystyle\int e^{2x}\cos 3x\,dx = \frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\int e^{2x}\sin 3x\,dx.$ Use $u = e^{2x}$, $dv = \sin 3x\,dx$ to get

$$\int e^{2x}\sin 3x\,dx = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\int e^{2x}\cos 3x\,dx \text{ so}$$

$$\int e^{2x}\cos 3x\,dx = \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x - \frac{4}{9}\int e^{2x}\cos 3x\,dx,$$

$$\frac{13}{9}\int e^{2x}\cos 3x\,dx = \frac{1}{9}e^{2x}(3\sin 3x + 2\cos 3x) + C_1, \int e^{2x}\cos 3x\,dx = \frac{1}{13}e^{2x}(3\sin 3x + 2\cos 3x) + C$$

21. $u = e^{ax},\ dv = \sin bx\,dx,\ du = ae^{ax}dx,\ v = -\frac{1}{b}\cos bx \quad (b \neq 0);$

$\int e^{ax}\sin bx\,dx = -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\,dx$. Use $u = e^{ax},\ dv = \cos bx\,dx$ to get

$$\int e^{ax}\cos bx\,dx = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\int e^{ax}\sin bx\,dx \text{ so}$$

$$\int e^{ax}\sin bx\,dx = -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}\int e^{ax}\sin bx\,dx,$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C$$

22. From Exercise 21 with $a = -3, b = 5, x = \theta$, answer $= \frac{e^{-3\theta}}{\sqrt{34}}(-3\sin 5\theta - 5\cos 5\theta) + C$

23. $u = \sin(\ln x),\ dv = dx,\ du = \frac{\cos(\ln x)}{x}dx,\ v = x;$

$\int \sin(\ln x)dx = x\sin(\ln x) - \int \cos(\ln x)dx$. Use $u = \cos(\ln x),\ dv = dx$ to get

$$\int \cos(\ln x)dx = x\cos(\ln x) + \int \sin(\ln x)dx \text{ so}$$

$$\int \sin(\ln x)dx = x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)dx,$$

$$\int \sin(\ln x)dx = \frac{1}{2}x[\sin(\ln x) - \cos(\ln x)] + C$$

24. $u = \cos(\ln x),\ dv = dx,\ du = -\frac{1}{x}\sin(\ln x)dx,\ v = x;$

$\int \cos(\ln x)dx = x\cos(\ln x) + \int \sin(\ln x)dx$. Use $u = \sin(\ln x),\ dv = dx$ to get

$$\int \sin(\ln x)dx = x\sin(\ln x) - \int \cos(\ln x)dx \text{ so}$$

$$\int \cos(\ln x)dx = x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)dx,$$

$$\int \cos(\ln x)dx = \frac{1}{2}x[\cos(\ln x) + \sin(\ln x)] + C$$

25. $u = x,\ dv = \sec^2 x\,dx,\ du = dx,\ v = \tan x;$

$$\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx = x\tan x - \int \frac{\sin x}{\cos x}dx = x\tan x + \ln|\cos x| + C$$

26. $u = x,\ dv = \tan^2 x\,dx = (\sec^2 x - 1)dx,\ du = dx,\ v = \tan x - x;$

$$\int x\tan^2 x\,dx = x\tan x - x^2 - \int(\tan x - x)dx$$
$$= x\tan x - x^2 + \ln|\cos x| + \frac{1}{2}x^2 + C = x\tan x - \frac{1}{2}x^2 + \ln|\cos x| + C$$

27. $u = x^2,\ dv = xe^{x^2}dx,\ du = 2x\,dx,\ v = \frac{1}{2}e^{x^2};$

$$\int x^3e^{x^2}dx = \frac{1}{2}x^2e^{x^2} - \int xe^{x^2}dx = \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2} + C$$

28. $u = xe^x,\ dv = \frac{1}{(x+1)^2}dx,\ du = (x+1)e^x\,dx,\ v = -\frac{1}{x+1};$

$$\int \frac{xe^x}{(x+1)^2}dx = -\frac{xe^x}{x+1} + \int e^x dx = -\frac{xe^x}{x+1} + e^x + C = \frac{e^x}{x+1} + C$$

29. $u = x,\ dv = e^{-5x}dx,\ du = dx,\ v = -\frac{1}{5}e^{-5x};$

$$\int_0^1 xe^{-5x}dx = -\frac{1}{5}xe^{-5x}\Big]_0^1 + \frac{1}{5}\int_0^1 e^{-5x}dx$$
$$= -\frac{1}{5}e^{-5} - \frac{1}{25}e^{-5x}\Big]_0^1 = -\frac{1}{5}e^{-5} - \frac{1}{25}(e^{-5} - 1) = (1 - 6e^{-5})/25$$

30. $u = x,\ dv = e^{2x}dx,\ du = dx,\ v = \frac{1}{2}e^{2x};$

$$\int_0^2 xe^{2x}dx = \frac{1}{2}xe^{2x}\Big]_0^2 - \frac{1}{2}\int_0^2 e^{2x}dx = e^4 - \frac{1}{4}e^{2x}\Big]_0^2 = e^4 - \frac{1}{4}(e^4 - 1) = (3e^4 + 1)/4$$

31. $u = \ln x,\ dv = x^2dx,\ du = \frac{1}{x}dx,\ v = \frac{1}{3}x^3;$

$$\int_1^e x^2\ln x\,dx = \frac{1}{3}x^3\ln x\Big]_1^e - \frac{1}{3}\int_1^e x^2dx = \frac{1}{3}e^3 - \frac{1}{9}x^3\Big]_1^e = \frac{1}{3}e^3 - \frac{1}{9}(e^3 - 1) = (2e^3 + 1)/9$$

32. $u = \ln x,\ dv = \frac{1}{x^2}dx,\ du = \frac{1}{x}dx,\ v = -\frac{1}{x};$

$$\int_{\sqrt{e}}^e \frac{\ln x}{x^2}dx = -\frac{1}{x}\ln x\Big]_{\sqrt{e}}^e + \int_{\sqrt{e}}^e \frac{1}{x^2}dx$$
$$= -\frac{1}{e} + \frac{1}{\sqrt{e}}\ln\sqrt{e} - \frac{1}{x}\Big]_{\sqrt{e}}^e = -\frac{1}{e} + \frac{1}{2\sqrt{e}} - \frac{1}{e} + \frac{1}{\sqrt{e}} = \frac{3\sqrt{e} - 4}{2e}$$

33. $u = \ln(x+3),\ dv = dx,\ du = \dfrac{1}{x+3}dx,\ v = x;$

$$\int_{-2}^{2} \ln(x+3)dx = x\ln(x+3)\Big]_{-2}^{2} - \int_{-2}^{2}\frac{x}{x+3}dx = 2\ln 5 + 2\ln 1 - \int_{-2}^{2}\left[1 - \frac{3}{x+3}\right]dx$$

$$= 2\ln 5 - [x - 3\ln(x+3)]\Big]_{-2}^{2} = 2\ln 5 - (2 - 3\ln 5) + (-2 - 3\ln 1) = 5\ln 5 - 4$$

34. $u = \sin^{-1} x,\ dv = dx,\ du = \dfrac{1}{\sqrt{1-x^2}}dx,\ v = x;$

$$\int_{0}^{1/2} \sin^{-1} x\,dx = x\sin^{-1} x\Big]_{0}^{1/2} - \int_{0}^{1/2}\frac{x}{\sqrt{1-x^2}}dx = \frac{1}{2}\sin^{-1}\frac{1}{2} + \sqrt{1-x^2}\Big]_{0}^{1/2}$$

$$= \frac{1}{2}\left(\frac{\pi}{6}\right) + \sqrt{\frac{3}{4}} - 1 = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$

35. $u = \sec^{-1}\sqrt{\theta},\ dv = d\theta,\ du = \dfrac{1}{2\theta\sqrt{\theta-1}}d\theta,\ v = \theta;$

$$\int_{2}^{4} \sec^{-1}\sqrt{\theta}d\theta = \theta\sec^{-1}\sqrt{\theta}\Big]_{2}^{4} - \frac{1}{2}\int_{2}^{4}\frac{1}{\sqrt{\theta-1}}d\theta = 4\sec^{-1} 2 - 2\sec^{-1}\sqrt{2} - \sqrt{\theta-1}\Big]_{2}^{4}$$

$$= 4\left(\frac{\pi}{3}\right) - 2\left(\frac{\pi}{4}\right) - \sqrt{3} + 1 = \frac{5\pi}{6} - \sqrt{3} + 1$$

36. $u = \sec^{-1} x,\ dv = x\,dx,\ du = \dfrac{1}{x\sqrt{x^2-1}}dx,\ v = \dfrac{1}{2}x^2;$

$$\int_{1}^{2} x\sec^{-1} x\,dx = \frac{1}{2}x^2\sec^{-1} x\Big]_{1}^{2} - \frac{1}{2}\int_{1}^{2}\frac{x}{\sqrt{x^2-1}}dx$$

$$= \frac{1}{2}[(4)(\pi/3) - (1)(0)] - \frac{1}{2}\sqrt{x^2-1}\Big]_{1}^{2} = 2\pi/3 - \sqrt{3}/2$$

37. $u = x,\ dv = \sin 4x\,dx,\ du = dx,\ v = -\dfrac{1}{4}\cos 4x;$

$$\int_{0}^{\pi/2} x\sin 4x\,dx = -\frac{1}{4}x\cos 4x\Big]_{0}^{\pi/2} + \frac{1}{4}\int_{0}^{\pi/2}\cos 4x\,dx = -\pi/8 + \frac{1}{16}\sin 4x\Big]_{0}^{\pi/2} = -\pi/8$$

38. $\displaystyle\int_{0}^{\pi}(x + x\cos x)dx = \frac{1}{2}x^2\Big]_{0}^{\pi} + \int_{0}^{\pi} x\cos x\,dx = \frac{\pi^2}{2} + \int_{0}^{\pi} x\cos x\,dx;$

$u = x,\ dv = \cos x\,dx,\ du = dx,\ v = \sin x$

$$\int_{0}^{\pi} x\cos x\,dx = x\sin x\Big]_{0}^{\pi} - \int_{0}^{\pi}\sin x\,dx = \cos x\Big]_{0}^{\pi} = -2 \text{ so } \int_{0}^{\pi}(x + x\cos x)dx = \pi^2/2 - 2$$

39. $u = \tan^{-1}\sqrt{x}$, $dv = \sqrt{x}dx$, $du = \dfrac{1}{2\sqrt{x}(1+x)}dx$, $v = \dfrac{2}{3}x^{3/2}$;

$$\int_1^3 \sqrt{x}\tan^{-1}\sqrt{x}dx = \frac{2}{3}x^{3/2}\tan^{-1}\sqrt{x}\Big]_1^3 - \frac{1}{3}\int_1^3 \frac{x}{1+x}dx$$
$$= \frac{2}{3}x^{3/2}\tan^{-1}\sqrt{x}\Big]_1^3 - \frac{1}{3}\int_1^3 \left[1 - \frac{1}{1+x}\right]dx$$
$$= \left[\frac{2}{3}x^{3/2}\tan^{-1}\sqrt{x} - \frac{1}{3}x + \frac{1}{3}\ln|1+x|\right]_1^3 = (2\sqrt{3}\pi - \pi/2 - 2 + \ln 2)/3$$

40. $u = \ln(x^2+1)$, $dv = dx$, $du = \dfrac{2x}{x^2+1}dx$, $v = x$;

$$\int_0^2 \ln(x^2+1)dx = x\ln(x^2+1)\Big]_0^2 - \int_0^2 \frac{2x^2}{x^2+1}dx = 2\ln 5 - 2\int_0^2\left(1 - \frac{1}{x^2+1}\right)dx$$
$$= 2\ln 5 - 2(x - \tan^{-1}x)\Big]_0^2 = 2\ln 5 - 4 + 2\tan^{-1}2$$

41. $t = \sqrt{x},\ t^2 = x,\ dx = 2t\,dt$

(a) $\displaystyle\int e^{\sqrt{x}}dx = 2\int te^t\,dt$; $u = t, dv = e^t dt, du = dt, v = e^t$,

$$\int e^{\sqrt{x}}dx = 2te^t - 2\int e^t\,dt = 2(t-1)e^t + C = 2(\sqrt{x}-1)e^{\sqrt{x}} + C$$

(b) $\displaystyle\int \cos\sqrt{x}\,dx = 2\int t\cos t\,dt$; $u = t, dv = \cos t dt, du = dt, v = \sin t$,

$$\int \cos\sqrt{x}\,dx = 2t\sin t - 2\int \sin t dt = 2t\sin t + 2\cos t + C = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$$

42. Let $q_1(x), q_2(x), q_3(x)$ denote successive antiderivatives of $q(x)$,
so that $q_3'(x) = q_2(x), q_2'(x) = q_1(x), q_1'(x) = q(x)$. Let $p(x) = ax^2 + bx + c$.

Repeated Differentiation		Repeated Antidifferentiation
$ax^2 + bx + c$		$q(x)$
	$+$	
$2ax + b$		$q_1(x)$
	$-$	
$2a$		$q_2(x)$
	$+$	
0		$q_3(x)$

Then $\displaystyle\int p(x)q(x)\,dx = (ax^2+bx+c)q_1(x) - (2ax+b)q_2(x) + 2aq_3(x) + C$. Check:

$$\frac{d}{dx}[(ax^2+bx+c)q_1(x) - (2ax+b)q_2(x) + 2aq_3(x)]$$
$$= (2ax+b)q_1(x) + (ax^2+bx+c)q(x) - 2aq_2(x) - (2ax+b)q_1(x) + 2aq_2(x) = p(x)q(x)$$

43.

Repeated Differentiation		Repeated Antidifferentiation
$3x^2 - x + 2$		e^{-x}
	$+$	
$6x - 1$		$-e^{-x}$
	$-$	
6		e^{-x}
	$+$	
0		$-e^{-x}$

$$\int (3x^2 - x + 2)e^{-x} = -(3x^2 - x + 2)e^{-x} - (6x - 1)e^{-x} - 6e^{-x} + C = -e^{-x}[3x^2 + 5x + 7] + C$$

44.

Repeated Differentiation		Repeated Antidifferentiation
$x^2 + x + 1$		$\sin x$
	$+$	
$2x + 1$		$-\cos x$
	$-$	
2		$-\sin x$
	$+$	
0		$\cos x$

$$\int (x^2 + x + 1)\sin x\, dx = -(x^2 + x + 1)\cos x + (2x + 1)\sin x + 2\cos x + C$$
$$= -(x^2 + x - 1)\cos x + (2x + 1)\sin x + C$$

45.

Repeated Differentiation		Repeated Antidifferentiation
$8x^4$		$\cos 2x$
	$+$	
$32x^3$		$\frac{1}{2}\sin 2x$
	$-$	
$96x^2$		$-\frac{1}{4}\cos 2x$
	$+$	
$192x$		$-\frac{1}{8}\sin 2x$
	$-$	
192		$\frac{1}{16}\cos 2x$
	$+$	
0		$\frac{1}{32}\sin 2x$

$$\int 8x^4\cos 2x\, dx = (4x^4 - 12x^2 + 6)\sin 2x + (8x^3 - 12x)\cos 2x + C$$

46.

Repeated Differentiation		Repeated Antidifferentiation
x^3		$\sqrt{2x+1}$
	$+$	
$3x^2$		$\frac{1}{3}(2x+1)^{3/2}$
	$-$	
$6x$		$\frac{1}{15}(2x+1)^{5/2}$
	$+$	
6		$\frac{1}{105}(2x+1)^{7/2}$
	$-$	
0		$\frac{1}{945}(2x+1)^{9/2}$

$$\int x^3\sqrt{2x+1}\,dx = \frac{1}{3}x^3(2x+1)^{3/2} - \frac{1}{5}x^2(2x+1)^{5/2} + \frac{2}{35}x(2x+1)^{7/2} - \frac{2}{315}(2x+1)^{9/2} + C$$

47. **(a)** $A = \int_1^e \ln x\,dx = (x\ln x - x)\Big]_1^e = 1$

(b) $V = \pi\int_1^e (\ln x)^2 dx = \pi\Big[(x(\ln x)^2 - 2x\ln x + 2x)\Big]_1^e = \pi(e-2)$

48. $A = \int_0^{\pi/2}(x - x\sin x)dx = \frac{1}{2}x^2\Big]_0^{\pi/2} - \int_0^{\pi/2} x\sin x\,dx = \frac{\pi^2}{8} - (-x\cos x + \sin x)\Big]_0^{\pi/2} = \pi^2/8 - 1$

49. $V = 2\pi\int_0^{\pi} x\sin x\,dx = 2\pi(-x\cos x + \sin x)\Big]_0^{\pi} = 2\pi^2$

50. $V = 2\pi\int_0^{\pi/2} x\cos x\,dx = 2\pi(\cos x + x\sin x)\Big]_0^{\pi/2} = \pi(\pi - 2)$

51. distance $= \int_0^5 t^2e^{-t}dt; u = t^2, dv = e^{-t}dt, du = 2tdt, v = -e^{-t},$

distance $= -t^2e^{-t}\Big]_0^5 + 2\int_0^5 te^{-t}dt; u = 2t, dv = e^{-t}dt, du = 2dt, v = -e^{-t},$

distance $= -25e^{-5} - 2te^{-t}\Big]_0^5 + 2\int_0^5 e^{-t}dt = -25e^{-5} - 10e^{-5} - 2e^{-t}\Big]_0^5$

$= -25e^{-5} - 10e^{-5} - 2e^{-5} + 2 = -37e^{-5} + 2$

52. $u = 2t, dv = \sin(k\omega t)dt,\ du = 2dt,\ v = -\dfrac{1}{k\omega}\cos(k\omega t)$; the integrand is an even function of t so

$$\int_{-\pi/\omega}^{\pi/\omega} t\sin(k\omega t)\,dt = 2\int_0^{\pi/\omega} t\sin(k\omega t)\,dt = -\frac{2}{k\omega}t\cos(k\omega t)\bigg]_0^{\pi/\omega} + 2\int_0^{\pi/\omega}\frac{1}{k\omega}\cos(k\omega t)\,dt$$

$$= \frac{2\pi(-1)^{k+1}}{k\omega^2} + \frac{2}{k^2\omega^2}\sin(k\omega t)\bigg]_0^{\pi/\omega} = \frac{2\pi(-1)^{k+1}}{k\omega^2}$$

53. **(a)** $\displaystyle\int \sin^3 x\,dx = -\frac{1}{3}\sin^2 x\cos x + \frac{2}{3}\int \sin x\,dx = -\frac{1}{3}\sin^2 x\cos x - \frac{2}{3}\cos x + C$

(b) $\displaystyle\int \sin^4 x\,dx = -\frac{1}{4}\sin^3 x\cos x + \frac{3}{4}\int \sin^2 x\,dx,\ \int \sin^2 x\,dx = -\frac{1}{2}\sin x\cos x + \frac{1}{2}x + C_1$ so

$$\int_0^{\pi/4}\sin^4 x\,dx = \left[-\frac{1}{4}\sin^3 x\cos x - \frac{3}{8}\sin x\cos x + \frac{3}{8}x\right]_0^{\pi/4}$$

$$= -\frac{1}{4}(1/\sqrt{2})^3(1/\sqrt{2}) - \frac{3}{8}(1/\sqrt{2})(1/\sqrt{2}) + 3\pi/32 = 3\pi/32 - 1/4$$

54. **(a)** $\displaystyle\int \cos^5 x\,dx = \frac{1}{5}\cos^4 x\sin x + \frac{4}{5}\int \cos^3 x\,dx = \frac{1}{5}\cos^4 x\sin x + \frac{4}{5}\left[\frac{1}{3}\cos^2 x\sin x + \frac{2}{3}\sin x\right] + C$

$$= \frac{1}{5}\cos^4 x\sin x + \frac{4}{15}\cos^2 x\sin x + \frac{8}{15}\sin x + C$$

(b) $\displaystyle\int \cos^6 x\,dx = \frac{1}{6}\cos^5 x\sin x + \frac{5}{6}\int \cos^4 x\,dx$

$$= \frac{1}{6}\cos^5 x\sin x + \frac{5}{6}\left[\frac{1}{4}\cos^3 x\sin x + \frac{3}{4}\int \cos^2 x\,dx\right]$$

$$= \frac{1}{6}\cos^5 x\sin x + \frac{5}{24}\cos^3 x\sin x + \frac{5}{8}\left[\frac{1}{2}\cos x\sin x + \frac{1}{2}x\right] + C,$$

$$\left[\frac{1}{6}\cos^5 x\sin x + \frac{5}{24}\cos^3 x\sin x + \frac{5}{16}\cos x\sin x + \frac{5}{16}x\right]_0^{\pi/2} = 5\pi/32$$

55. $u = \sin^{n-1} x,\ dv = \sin x\,dx,\ du = (n-1)\sin^{n-2} x\cos x\,dx,\ v = -\cos x$;

$$\int \sin^n x\,dx = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\cos^2 x\,dx$$

$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,(1 - \sin^2 x)dx$$

$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx,$$

$$n\int \sin^n x\,dx = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx,$$

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

56. **(a)** $u = \sec^{n-2} x,\ dv = \sec^2 x\,dx,\ du = (n-2)\sec^{n-2} x \tan x\,dx,\ v = \tan x;$

$$\int \sec^n x\,dx = \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x\,dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x\,(\sec^2 x - 1)dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx,$$

$$(n-1)\int \sec^n x\,dx = \sec^{n-2} x \tan x + (n-2)\int \sec^{n-2} x\,dx,$$

$$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$$

(b) $$\int \tan^n x\,dx = \int \tan^{n-2} x\,(\sec^2 x - 1)\,dx = \int \tan^{n-1} x \sec^2 x\,dx - \int \tan^{n-2} x\,dx$$

$$= \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx$$

(c) $u = x^n,\ dv = e^x dx,\ du = nx^{n-1}dx,\ v = e^x;\ \int x^n e^x dx = x^n e^x - n\int x^{n-1}e^x dx$

57. **(a)** $$\int \tan^4 x\,dx = \frac{1}{3}\tan^3 x - \int \tan^2 x\,dx = \frac{1}{3}\tan^3 x - \tan x + \int dx = \frac{1}{3}\tan^3 x - \tan x + x + C$$

(b) $$\int \sec^4 x\,dx = \frac{1}{3}\sec^2 x \tan x + \frac{2}{3}\int \sec^2 x\,dx = \frac{1}{3}\sec^2 x \tan x + \frac{2}{3}\tan x + C$$

(c) $$\int x^3 e^x dx = x^3 e^x - 3\int x^2 e^x dx = x^3 e^x - 3\left[x^2 e^x - 2\int x e^x dx\right]$$

$$= x^3 e^x - 3x^2 e^x + 6\left[xe^x - \int e^x dx\right] = x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + C$$

58. **(a)** $u = 3x$,

$$\int x^2 e^{3x} dx = \frac{1}{27}\int u^2 e^u du = \frac{1}{27}\left[u^2 e^u - 2\int ue^u du\right] = \frac{1}{27}u^2 e^u - \frac{2}{27}\left[ue^u - \int e^u du\right]$$

$$= \frac{1}{27}u^2 e^u - \frac{2}{27}ue^u + \frac{2}{27}e^u + C = \frac{1}{3}x^2 e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C$$

(b) $u = -\sqrt{x}$,

$$\int_0^1 xe^{-\sqrt{x}}dx = 2\int_0^{-1} u^3 e^u du,$$

$$\int u^3 e^u du = u^3 e^u - 3\int u^2 e^u du = u^3 e^u - 3\left[u^2 e^u - 2\int ue^u du\right]$$

$$= u^3 e^u - 3u^2 e^u + 6\left[ue^u - \int e^u du\right] = u^3 e^u - 3u^2 e^u + 6ue^u - 6e^u + C,$$

$$2\int_0^{-1} u^3 e^u du = 2(u^3 - 3u^2 + 6u - 6)e^u\Big]_0^{-1} = 12 - 32e^{-1}$$

59. $u = x,\ dv = f''(x)dx,\ du = dx,\ v = f'(x);$

$$\int_{-1}^{1} x\, f''(x)dx = xf'(x)\Big]_{-1}^{1} - \int_{-1}^{1} f'(x)dx$$

$$= f'(1) + f'(-1) - f(x)\Big]_{-1}^{1} = f'(1) + f'(-1) - f(1) + f(-1)$$

60. **(a)** $u = f(x), dv = dx, du = f'(x), v = x;$

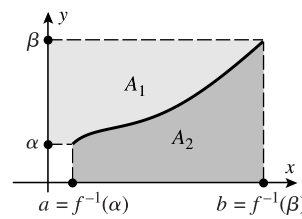
$$\int_a^b f(x)\,dx = xf(x)\Big]_a^b - \int_a^b xf'(x)\,dx = bf(b) - af(a) - \int_a^b xf'(x)\,dx$$

(b) Substitute $y = f(x), dy = f'(x)\,dx, x = a$ when $y = f(a)$, $x = b$ when $y = f(b)$,

$$\int_a^b xf'(x)\,dx = \int_{f(a)}^{f(b)} x\,dy = \int_{f(a)}^{f(b)} f^{-1}(y)\,dy$$

(c) From $a = f^{-1}(\alpha)$ and $b = f^{-1}(\beta)$ we get $bf(b) - af(a) = \beta f^{-1}(\beta) - \alpha f^{-1}(\alpha)$; then

$$\int_\alpha^\beta f^{-1}(x)\,dx = \int_\alpha^\beta f^{-1}(y)\,dy = \int_{f(a)}^{f(b)} f^{-1}(y)\,dy,$$



which, by Part (b), yields

$$\int_\alpha^\beta f^{-1}(x)\,dx = bf(b) - af(a) - \int_a^b f(x)\,dx$$

$$= \beta f^{-1}(\beta) - \alpha f^{-1}(\alpha) - \int_{f^{-1}(\alpha)}^{f^{-1}(\beta)} f(x)\,dx$$

Note from the figure that $A_1 = \displaystyle\int_\alpha^\beta f^{-1}(x)\,dx, A_2 = \int_{f^{-1}(\alpha)}^{f^{-1}(\beta)} f(x)\,dx$, and $A_1 + A_2 = \beta f^{-1}(\beta) - \alpha f^{-1}(\alpha)$, a "picture proof".

61. **(a)** Use Exercise 60(c);

$$\int_0^{1/2} \sin^{-1} x\,dx = \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right) - 0\cdot\sin^{-1} 0 - \int_{\sin^{-1}(0)}^{\sin^{-1}(1/2)} \sin x\,dx = \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right) - \int_0^{\pi/6} \sin x\,dx$$

(b) Use Exercise 60(b);

$$\int_e^{e^2} \ln x\,dx = e^2 \ln e^2 - e\ln e - \int_{\ln e}^{\ln e^2} f^{-1}(y)\,dy = 2e^2 - e - \int_1^2 e^y\,dy = 2e^2 - e - \int_1^2 e^x\,dx$$

62. **(a)** $\displaystyle\int u\,dv = uv - \int v\,du = x(\sin x + C_1) + \cos x - C_1 x + C_2 = x\sin x + \cos x + C_2;$

the constant C_1 cancels out and hence plays no role in the answer.

(b) $\displaystyle u(v + C_1) - \int (v + C_1)du = uv + C_1 u - \int v\,du - C_1 u = uv - \int v\,du$

63. $u = \ln(x+1), dv = dx, du = \dfrac{dx}{x+1}, v = x+1;$

$$\int \ln(x+1)\,dx = \int u\,dv = uv - \int v\,du = (x+1)\ln(x+1) - \int dx = (x+1)\ln(x+1) - x + C$$

64. $u = \ln(2x+3), dv = dx, du = \dfrac{2dx}{2x+3}, v = x + \dfrac{3}{2};$

$$\int \ln(2x+3)\,dx = \int u\,dv = uv - \int v\,du = (x+\frac{3}{2})\ln(2x+3) - \int dx$$

$$= \frac{1}{2}(2x+3)\ln(2x+3) - x + C$$

65. $u = \tan^{-1} x, dv = x\,dx, du = \dfrac{1}{1+x^2}\,dx, v = \dfrac{1}{2}(x^2+1)$

$$\int x\tan^{-1} x\,dx = \int u\,dv = uv - \int v\,du = \frac{1}{2}(x^2+1)\tan^{-1} x - \frac{1}{2}\int dx$$

$$= \frac{1}{2}(x^2+1)\tan^{-1} x - \frac{1}{2}x + C$$

66. $u = \dfrac{1}{\ln x}, \quad dv = \dfrac{1}{x}\,dx, du = -\dfrac{1}{x(\ln x)^2}\,dx, v = \ln x$

$$\int \frac{1}{x\ln x}\,dx = 1 + \int \frac{1}{x\ln x}\,dx.$$

This seems to imply that $1 = 0$, but recall that both sides represent a function *plus an arbitrary constant*; these two arbitrary constants will take care of the 1.

EXERCISE SET 8.3

1. $u = \cos x, -\displaystyle\int u^5 du = -\frac{1}{6}\cos^6 x + C$

2. $u = \sin 3x, \dfrac{1}{3}\displaystyle\int u^4 du = \frac{1}{15}\sin^5 3x + C$

3. $u = \sin ax, \dfrac{1}{a}\displaystyle\int u\,du = \frac{1}{2a}\sin^2 ax + C, \quad a \neq 0$

4. $\displaystyle\int \cos^2 3x\,dx = \frac{1}{2}\int(1+\cos 6x)dx = \frac{1}{2}x + \frac{1}{12}\sin 6x + C$

5. $\displaystyle\int \sin^2 5\theta\,d\theta = \frac{1}{2}\int(1-\cos 10\theta)d\theta = \frac{1}{2}\theta - \frac{1}{20}\sin 10\theta + C$

6. $\displaystyle\int \cos^3 at\,dt = \int(1-\sin^2 at)\cos at\,dt$

$$= \int \cos at\,dt - \int \sin^2 at\cos at\,dt = \frac{1}{a}\sin at - \frac{1}{3a}\sin^3 at + C \quad (a \neq 0)$$

7. $\displaystyle\int \cos^5 \theta d\theta = \int(1-\sin^2\theta)^2\cos\theta d\theta = \int(1-2\sin^2\theta+\sin^4\theta)\cos\theta d\theta$

$$= \sin\theta - \frac{2}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta + C$$

8. $\displaystyle\int \sin^3 x \cos^3 x\,dx = \int \sin^3 x(1-\sin^2 x)\cos x\,dx$

$\displaystyle = \int (\sin^3 x - \sin^5 x)\cos x\,dx = \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C$

9. $\displaystyle\int \sin^2 2t \cos^3 2t\,dt = \int \sin^2 2t(1-\sin^2 2t)\cos 2t\,dt = \int (\sin^2 2t - \sin^4 2t)\cos 2t\,dt$

$\displaystyle = \frac{1}{6}\sin^3 2t - \frac{1}{10}\sin^5 2t + C$

10. $\displaystyle\int \sin^3 2x \cos^2 2x\,dx = \int (1-\cos^2 2x)\cos^2 2x \sin 2x\,dx$

$\displaystyle = \int (\cos^2 2x - \cos^4 2x)\sin 2x\,dx = -\frac{1}{6}\cos^3 2x + \frac{1}{10}\cos^5 2x + C$

11. $\displaystyle\int \sin^2 x \cos^2 x\,dx = \frac{1}{4}\int \sin^2 2x\,dx = \frac{1}{8}\int (1-\cos 4x)dx = \frac{1}{8}x - \frac{1}{32}\sin 4x + C$

12. $\displaystyle\int \sin^2 x \cos^4 x\,dx = \frac{1}{8}\int (1-\cos 2x)(1+\cos 2x)^2 dx = \frac{1}{8}\int (1-\cos^2 2x)(1+\cos 2x)dx$

$\displaystyle = \frac{1}{8}\int \sin^2 2x\,dx + \frac{1}{8}\int \sin^2 2x\cos 2x\,dx = \frac{1}{16}\int (1-\cos 4x)dx + \frac{1}{48}\sin^3 2x$

$\displaystyle = \frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C$

13. $\displaystyle\int \sin x \cos 2x\,dx = \frac{1}{2}\int (\sin 3x - \sin x)dx = -\frac{1}{6}\cos 3x + \frac{1}{2}\cos x + C$

14. $\displaystyle\int \sin 3\theta \cos 2\theta d\theta = \frac{1}{2}\int (\sin 5\theta + \sin\theta)d\theta = -\frac{1}{10}\cos 5\theta - \frac{1}{2}\cos\theta + C$

15. $\displaystyle\int \sin x \cos(x/2)dx = \frac{1}{2}\int [\sin(3x/2) + \sin(x/2)]dx = -\frac{1}{3}\cos(3x/2) - \cos(x/2) + C$

16. $\displaystyle u = \cos x,\ -\int u^{1/5}du = -\frac{5}{6}\cos^{6/5} x + C$

17. $\displaystyle\int_0^{\pi/4} \cos^3 x\,dx = \int_0^{\pi/4} (1-\sin^2 x)\cos x\,dx$

$\displaystyle = \left[\sin x - \frac{1}{3}\sin^3 x\right]_0^{\pi/4} = (\sqrt{2}/2) - \frac{1}{3}(\sqrt{2}/2)^3 = 5\sqrt{2}/12$

18. $\displaystyle\int_0^{\pi/2} \sin^2(x/2)\cos^2(x/2)dx = \frac{1}{4}\int_0^{\pi/2} \sin^2 x\,dx = \frac{1}{8}\int_0^{\pi/2} (1-\cos 2x)dx$

$\displaystyle = \frac{1}{8}\left(x - \frac{1}{2}\sin 2x\right)\Bigg]_0^{\pi/2} = \pi/16$

19. $\displaystyle\int_0^{\pi/3} \sin^4 3x \cos^3 3x\,dx = \int_0^{\pi/3} \sin^4 3x(1-\sin^2 3x)\cos 3x\,dx = \left[\frac{1}{15}\sin^5 3x - \frac{1}{21}\sin^7 3x\right]_0^{\pi/3} = 0$

20. $\displaystyle\int_{-\pi}^{\pi} \cos^2 5\theta\, d\theta = \frac{1}{2}\int_{-\pi}^{\pi}(1+\cos 10\theta)d\theta = \frac{1}{2}\left(\theta + \frac{1}{10}\sin 10\theta\right)\Big]_{-\pi}^{\pi} = \pi$

21. $\displaystyle\int_0^{\pi/6} \sin 2x \cos 4x\, dx = \frac{1}{2}\int_0^{\pi/6}(\sin 6x - \sin 2x)dx = \left[-\frac{1}{12}\cos 6x + \frac{1}{4}\cos 2x\right]_0^{\pi/6}$

$= [(-1/12)(-1) + (1/4)(1/2)] - [-1/12 + 1/4] = 1/24$

22. $\displaystyle\int_0^{2\pi} \sin^2 kx\, dx = \frac{1}{2}\int_0^{2\pi}(1-\cos 2kx)dx = \frac{1}{2}\left(x - \frac{1}{2k}\sin 2kx\right)\Big]_0^{2\pi} = \pi - \frac{1}{4k}\sin 4\pi k \quad (k \neq 0)$

23. $\displaystyle\frac{1}{3}\tan(3x+1) + C$

24. $\displaystyle-\frac{1}{5}\ln|\cos 5x| + C$

25. $\displaystyle u = e^{-2x}, du = -2e^{-2x}\,dx;\ -\frac{1}{2}\int \tan u\, du = \frac{1}{2}\ln|\cos u| + C = \frac{1}{2}\ln|\cos(e^{-2x})| + C$

26. $\displaystyle\frac{1}{3}\ln|\sin 3x| + C$

27. $\displaystyle\frac{1}{2}\ln|\sec 2x + \tan 2x| + C$

28. $\displaystyle u = \sqrt{x}, du = \frac{1}{2\sqrt{x}}\,dx;\ \int 2\sec u\, du = 2\ln|\sec u + \tan u| + C = 2\ln\left|\sec\sqrt{x} + \tan\sqrt{x}\right| + C$

29. $\displaystyle u = \tan x,\ \int u^2 du = \frac{1}{3}\tan^3 x + C$

30. $\displaystyle\int \tan^5 x(1+\tan^2 x)\sec^2 x\, dx = \int(\tan^5 x + \tan^7 x)\sec^2 x\, dx = \frac{1}{6}\tan^6 x + \frac{1}{8}\tan^8 x + C$

31. $\displaystyle\int \tan^3 4x(1+\tan^2 4x)\sec^2 4x\, dx = \int(\tan^3 4x + \tan^5 4x)\sec^2 4x\, dx = \frac{1}{16}\tan^4 4x + \frac{1}{24}\tan^6 4x + C$

32. $\displaystyle\int \tan^4\theta(1+\tan^2\theta)\sec^2\theta\, d\theta = \frac{1}{5}\tan^5\theta + \frac{1}{7}\tan^7\theta + C$

33. $\displaystyle\int \sec^4 x(\sec^2 x - 1)\sec x\tan x\, dx = \int(\sec^6 x - \sec^4 x)\sec x \tan x\, dx = \frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + C$

34. $\displaystyle\int(\sec^2\theta - 1)^2\sec\theta\tan\theta d\theta = \int(\sec^4\theta - 2\sec^2\theta + 1)\sec\theta\tan\theta d\theta = \frac{1}{5}\sec^5\theta - \frac{2}{3}\sec^3\theta + \sec\theta + C$

35. $\displaystyle\int(\sec^2 x - 1)^2\sec x\, dx = \int(\sec^5 x - 2\sec^3 x + \sec x)dx = \int \sec^5 x\, dx - 2\int \sec^3 x\, dx + \int \sec x\, dx$

$\displaystyle= \frac{1}{4}\sec^3 x\tan x + \frac{3}{4}\int \sec^3 x\, dx - 2\int \sec^3 x\, dx + \ln|\sec x + \tan x|$

$\displaystyle= \frac{1}{4}\sec^3 x\tan x - \frac{5}{4}\left[\frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x + \tan x|\right] + \ln|\sec x + \tan x| + C$

$\displaystyle= \frac{1}{4}\sec^3 x\tan x - \frac{5}{8}\sec x\tan x + \frac{3}{8}\ln|\sec x + \tan x| + C$

36. $\int [\sec^2(x/2) - 1]\sec^3(x/2)dx = \int [\sec^5(x/2) - \sec^3(x/2)]dx$

$$= 2\left[\int \sec^5 u\,du - \int \sec^3 u\,du\right] \qquad (u = x/2)$$

$$= 2\left[\left(\frac{1}{4}\sec^3 u \tan u + \frac{3}{4}\int \sec^3 u\,du\right) - \int \sec^3 u\,du\right] \qquad \text{(equation (20))}$$

$$= \frac{1}{2}\sec^3 u \tan u - \frac{1}{2}\int \sec^3 u\,du$$

$$= \frac{1}{2}\sec^3 u \tan u - \frac{1}{4}\sec u \tan u - \frac{1}{4}\ln|\sec u + \tan u| + C \qquad \text{(equation (20), (22))}$$

$$= \frac{1}{2}\sec^3 \frac{x}{2}\tan\frac{x}{2} - \frac{1}{4}\sec\frac{x}{2}\tan\frac{x}{2} - \frac{1}{4}\ln\left|\sec\frac{x}{2} + \tan\frac{x}{2}\right| + C$$

37. $\int \sec^2 2t(\sec 2t \tan 2t)dt = \frac{1}{6}\sec^3 2t + C$

38. $\int \sec^4 x(\sec x \tan x)dx = \frac{1}{5}\sec^5 x + C$

39. $\int \sec^4 x\,dx = \int (1 + \tan^2 x)\sec^2 x\,dx = \int (\sec^2 x + \tan^2 x \sec^2 x)dx = \tan x + \frac{1}{3}\tan^3 x + C$

40. Using equation (20),

$$\int \sec^5 x\,dx = \frac{1}{4}\sec^3 x \tan x + \frac{3}{4}\int \sec^3 x\,dx$$

$$= \frac{1}{4}\sec^3 x \tan x + \frac{3}{8}\sec x \tan x + \frac{3}{8}\ln|\sec x + \tan x| + C$$

41. Use equation (19) to get $\int \tan^4 x\,dx = \frac{1}{3}\tan^3 x - \tan x + x + C$

42. $u = 4x$, use equation (19) to get

$$\frac{1}{4}\int \tan^3 u\,du = \frac{1}{4}\left[\frac{1}{2}\tan^2 u + \ln|\cos u|\right] + C = \frac{1}{8}\tan^2 4x + \frac{1}{4}\ln|\cos 4x| + C$$

43. $\int \sqrt{\tan x}(1 + \tan^2 x)\sec^2 x\,dx = \frac{2}{3}\tan^{3/2} x + \frac{2}{7}\tan^{7/2} x + C$

44. $\int \sec^{1/2} x(\sec x \tan x)dx = \frac{2}{3}\sec^{3/2} x + C$

45. $\int_0^{\pi/6} (\sec^2 2x - 1)dx = \left[\frac{1}{2}\tan 2x - x\right]_0^{\pi/6} = \sqrt{3}/2 - \pi/6$

46. $\int_0^{\pi/6} \sec^2\theta(\sec\theta\tan\theta)d\theta = \frac{1}{3}\sec^3\theta\Big]_0^{\pi/6} = (1/3)(2/\sqrt{3})^3 - 1/3 = 8\sqrt{3}/27 - 1/3$

47. $u = x/2$,

$$2\int_0^{\pi/4} \tan^5 u\,du = \left[\frac{1}{2}\tan^4 u - \tan^2 u - 2\ln|\cos u|\right]_0^{\pi/4} = 1/2 - 1 - 2\ln(1/\sqrt{2}) = -1/2 + \ln 2$$

48. $u = \pi x,\ \dfrac{1}{\pi}\displaystyle\int_0^{\pi/4} \sec u \tan u\, du = \dfrac{1}{\pi}\sec u\Big]_0^{\pi/4} = (\sqrt{2}-1)/\pi$

49. $\displaystyle\int (\csc^2 x - 1)\csc^2 x(\csc x \cot x)dx = \int (\csc^4 x - \csc^2 x)(\csc x \cot x)dx = -\frac{1}{5}\csc^5 x + \frac{1}{3}\csc^3 x + C$

50. $\displaystyle\int \frac{\cos^2 3t}{\sin^2 3t}\cdot\frac{1}{\cos 3t}dt = \int \csc 3t \cot 3t\, dt = -\frac{1}{3}\csc 3t + C$

51. $\displaystyle\int (\csc^2 x - 1)\cot x\, dx = \int \csc x(\csc x \cot x)dx - \int \frac{\cos x}{\sin x}dx = -\frac{1}{2}\csc^2 x - \ln|\sin x| + C$

52. $\displaystyle\int (\cot^2 x + 1)\csc^2 x\, dx = -\frac{1}{3}\cot^3 x - \cot x + C$

53. **(a)** $\displaystyle\int_0^{2\pi} \sin mx \cos nx\, dx = \frac{1}{2}\int_0^{2\pi}[\sin(m+n)x + \sin(m-n)x]dx = \left[-\frac{\cos(m+n)x}{2(m+n)} - \frac{\cos(m-n)x}{2(m-n)}\right]_0^{2\pi}$

but $\cos(m+n)x\Big]_0^{2\pi} = 0, \cos(m-n)x\Big]_0^{2\pi} = 0.$

(b) $\displaystyle\int_0^{2\pi} \cos mx \cos nx\, dx = \frac{1}{2}\int_0^{2\pi}[\cos(m+n)x + \cos(m-n)x]dx;$
since $m \neq n$, evaluate sin at integer multiples of 2π to get 0.

(c) $\displaystyle\int_0^{2\pi} \sin mx \sin nx\, dx = \frac{1}{2}\int_0^{2\pi}[\cos(m-n)x - \cos(m+n)x]\ dx;$
since $m \neq n$, evaluate sin at integer multiples of 2π to get 0.

54. **(a)** $\displaystyle\int_0^{2\pi} \sin mx \cos mx\, dx = \frac{1}{2}\int_0^{2\pi} \sin 2mx\, dx = -\frac{1}{4m}\cos 2mx\Big|_0^{2\pi} = 0$

(b) $\displaystyle\int_0^{2\pi} \cos^2 mx\, dx = \frac{1}{2}\int_0^{2\pi}(1+\cos 2mx)\, dx = \frac{1}{2}\left(x + \frac{1}{2m}\sin 2mx\right)\Big|_0^{2\pi} = \pi$

(c) $\displaystyle\int_0^{2\pi} \sin^2 mx\, dx = \frac{1}{2}\int_0^{2\pi}(1-\cos 2mx)\ dx = \frac{1}{2}\left(x - \frac{1}{2m}\sin 2mx\right)\Big|_0^{2\pi} = \pi$

55. $y' = \tan x,\ 1 + (y')^2 = 1 + \tan^2 x = \sec^2 x,$

$\displaystyle L = \int_0^{\pi/4}\sqrt{\sec^2 x}\, dx = \int_0^{\pi/4} \sec x\, dx = \ln|\sec x + \tan x|\Big]_0^{\pi/4} = \ln(\sqrt{2}+1)$

56. $\displaystyle V = \pi\int_0^{\pi/4}(1-\tan^2 x)dx = \pi\int_0^{\pi/4}(2-\sec^2 x)dx = \pi(2x - \tan x)\Big]_0^{\pi/4} = \frac{1}{2}\pi(\pi-2)$

57. $V = \pi \int_0^{\pi/4} (\cos^2 x - \sin^2 x)dx = \pi \int_0^{\pi/4} \cos 2x\, dx = \frac{1}{2}\pi \sin 2x \Big]_0^{\pi/4} = \pi/2$

58. $V = \pi \int_0^{\pi} \sin^2 x\, dx = \frac{\pi}{2} \int_0^{\pi} (1 - \cos 2x)dx = \frac{\pi}{2}\left(x - \frac{1}{2}\sin 2x\right)\Big]_0^{\pi} = \pi^2/2$

59. With $0 < \alpha < \beta, D = D_\beta - D_\alpha = \frac{L}{2\pi}\int_\alpha^\beta \sec x\, dx = \frac{L}{2\pi}\ln|\sec x + \tan x|\Big]_\alpha^\beta = \frac{L}{2\pi}\ln\left|\frac{\sec\beta + \tan\beta}{\sec\alpha + \tan\alpha}\right|$

60. **(a)** $D = \frac{100}{2\pi}\ln(\sec 25^\circ + \tan 25^\circ) = 7.18$ cm

(b) $D = \frac{100}{2\pi}\ln\left|\frac{\sec 50^\circ + \tan 50^\circ}{\sec 30^\circ + \tan 30^\circ}\right| = 7.34$ cm

61. **(a)** $\int \csc x\, dx = \int \sec(\pi/2 - x)dx = -\ln|\sec(\pi/2 - x) + \tan(\pi/2 - x)| + C$

$= -\ln|\csc x + \cot x| + C$

(b) $-\ln|\csc x + \cot x| = \ln\frac{1}{|\csc x + \cot x|} = \ln\frac{|\csc x - \cot x|}{|\csc^2 x - \cot^2 x|} = \ln|\csc x - \cot x|,$

$-\ln|\csc x + \cot x| = -\ln\left|\frac{1}{\sin x} + \frac{\cos x}{\sin x}\right| = \ln\left|\frac{\sin x}{1 + \cos x}\right|$

$= \ln\left|\frac{2\sin(x/2)\cos(x/2)}{2\cos^2(x/2)}\right| = \ln|\tan(x/2)|$

62. $\sin x + \cos x = \sqrt{2}\left[(1/\sqrt{2})\sin x + (1/\sqrt{2})\cos x\right]$

$= \sqrt{2}\,[\sin x \cos(\pi/4) + \cos x \sin(\pi/4)] = \sqrt{2}\sin(x + \pi/4),$

$\int \frac{dx}{\sin x + \cos x} = \frac{1}{\sqrt{2}}\int \csc(x + \pi/4)dx = -\frac{1}{\sqrt{2}}\ln|\csc(x + \pi/4) + \cot(x + \pi/4)| + C$

$= -\frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2} + \cos x - \sin x}{\sin x + \cos x}\right| + C$

63. $a\sin x + b\cos x = \sqrt{a^2 + b^2}\left[\frac{a}{\sqrt{a^2 + b^2}}\sin x + \frac{b}{\sqrt{a^2 + b^2}}\cos x\right] = \sqrt{a^2 + b^2}(\sin x\cos\theta + \cos x\sin\theta)$

where $\cos\theta = a/\sqrt{a^2 + b^2}$ and $\sin\theta = b/\sqrt{a^2 + b^2}$ so $a\sin x + b\cos x = \sqrt{a^2 + b^2}\sin(x + \theta)$

and $\int \frac{dx}{a\sin x + b\cos x} = \frac{1}{\sqrt{a^2 + b^2}}\int \csc(x + \theta)dx = -\frac{1}{\sqrt{a^2 + b^2}}\ln|\csc(x + \theta) + \cot(x + \theta)| + C$

$= -\frac{1}{\sqrt{a^2 + b^2}}\ln\left|\frac{\sqrt{a^2 + b^2} + a\cos x - b\sin x}{a\sin x + b\cos x}\right| + C$

64. **(a)** $\displaystyle\int_0^{\pi/2}\sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x\Big]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2}\sin^{n-2}x\,dx = \frac{n-1}{n}\int_0^{\pi/2}\sin^{n-2}x\,dx$

(b) By repeated application of the formula in Part (a)

$$\int_0^{\pi/2}\sin^n x\,dx = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\int_0^{\pi/2}\sin^{n-4}x\,dx$$

$$= \begin{cases} \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\cdots\left(\frac{1}{2}\right)\int_0^{\pi/2}dx, n \text{ even} \\ \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\cdots\left(\frac{2}{3}\right)\int_0^{\pi/2}\sin x\,dx, n \text{ odd} \end{cases}$$

$$= \begin{cases} \frac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n}\cdot\frac{\pi}{2}, n \text{ even} \\ \frac{2\cdot 4\cdot 6\cdots(n-1)}{3\cdot 5\cdot 7\cdots n}, n \text{ odd} \end{cases}$$

65. **(a)** $\displaystyle\int_0^{\pi/2}\sin^3 x\,dx = \frac{2}{3}$

(b) $\displaystyle\int_0^{\pi/2}\sin^4 x\,dx = \frac{1\cdot 3}{2\cdot 4}\cdot\frac{\pi}{2} = 3\pi/16$

(c) $\displaystyle\int_0^{\pi/2}\sin^5 x\,dx = \frac{2\cdot 4}{3\cdot 5} = 8/15$

(d) $\displaystyle\int_0^{\pi/2}\sin^6 x\,dx = \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{\pi}{2} = 5\pi/32$

66. Similar to proof in Exercise 64.

EXERCISE SET 8.4

1. $x = 2\sin\theta,\ dx = 2\cos\theta\,d\theta,$

$$4\int\cos^2\theta\,d\theta = 2\int(1+\cos 2\theta)d\theta = 2\theta + \sin 2\theta + C$$

$$= 2\theta + 2\sin\theta\cos\theta + C = 2\sin^{-1}(x/2) + \frac{1}{2}x\sqrt{4-x^2} + C$$

2. $x = \frac{1}{2}\sin\theta,\ dx = \frac{1}{2}\cos\theta\,d\theta,$

$$\frac{1}{2}\int\cos^2\theta\,d\theta = \frac{1}{4}\int(1+\cos 2\theta)d\theta = \frac{1}{4}\theta + \frac{1}{8}\sin 2\theta + C$$

$$= \frac{1}{4}\theta + \frac{1}{4}\sin\theta\cos\theta + C = \frac{1}{4}\sin^{-1}2x + \frac{1}{2}x\sqrt{1-4x^2} + C$$

3. $x = 3\sin\theta,\ dx = 3\cos\theta\,d\theta,$

$$9\int\sin^2\theta\,d\theta = \frac{9}{2}\int(1-\cos 2\theta)d\theta = \frac{9}{2}\theta - \frac{9}{4}\sin 2\theta + C = \frac{9}{2}\theta - \frac{9}{2}\sin\theta\cos\theta + C$$

$$= \frac{9}{2}\sin^{-1}(x/3) - \frac{1}{2}x\sqrt{9-x^2} + C$$

4. $x = 4\sin\theta,\ dx = 4\cos\theta\,d\theta,$

$$\frac{1}{16}\int\frac{1}{\sin^2\theta}d\theta = \frac{1}{16}\int\csc^2\theta\,d\theta = -\frac{1}{16}\cot\theta + C = -\frac{\sqrt{16-x^2}}{16x} + C$$

5. $x = 2\tan\theta,\ dx = 2\sec^2\theta\,d\theta,$

$$\frac{1}{8}\int\frac{1}{\sec^2\theta}d\theta = \frac{1}{8}\int\cos^2\theta\,d\theta = \frac{1}{16}\int(1+\cos 2\theta)d\theta = \frac{1}{16}\theta + \frac{1}{32}\sin 2\theta + C$$

$$= \frac{1}{16}\theta + \frac{1}{16}\sin\theta\cos\theta + C = \frac{1}{16}\tan^{-1}\frac{x}{2} + \frac{x}{8(4+x^2)} + C$$

6. $x = \sqrt{5}\tan\theta,\ dx = \sqrt{5}\sec^2\theta\,d\theta,$

$$5\int\tan^2\theta\sec\theta\,d\theta = 5\int(\sec^3\theta - \sec\theta)d\theta = 5\left(\frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta|\right) + C_1$$

$$= \frac{1}{2}x\sqrt{5+x^2} - \frac{5}{2}\ln\frac{\sqrt{5+x^2}+x}{\sqrt{5}} + C_1 = \frac{1}{2}x\sqrt{5+x^2} - \frac{5}{2}\ln(\sqrt{5+x^2}+x) + C$$

7. $x = 3\sec\theta,\ dx = 3\sec\theta\tan\theta\,d\theta,$

$$3\int\tan^2\theta\,d\theta = 3\int(\sec^2\theta - 1)d\theta = 3\tan\theta - 3\theta + C = \sqrt{x^2-9} - 3\sec^{-1}\frac{x}{3} + C$$

8. $x = 4\sec\theta,\ dx = 4\sec\theta\tan\theta\,d\theta,$

$$\frac{1}{16}\int\frac{1}{\sec\theta}d\theta = \frac{1}{16}\int\cos\theta\,d\theta = \frac{1}{16}\sin\theta + C = \frac{\sqrt{x^2-16}}{16x} + C$$

9. $x = \sqrt{2}\sin\theta,\ dx = \sqrt{2}\cos\theta\,d\theta,$

$$2\sqrt{2}\int\sin^3\theta\,d\theta = 2\sqrt{2}\int\left[1 - \cos^2\theta\right]\sin\theta\,d\theta$$

$$= 2\sqrt{2}\left(-\cos\theta + \frac{1}{3}\cos^3\theta\right) + C = -2\sqrt{2-x^2} + \frac{1}{3}(2-x^2)^{3/2} + C$$

10. $x = \sqrt{5}\sin\theta,\ dx = \sqrt{5}\cos\theta\,d\theta,$

$$25\sqrt{5}\int\sin^3\theta\cos^2\theta\,d\theta = 25\sqrt{5}\left(-\frac{1}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta\right) + C = -\frac{5}{3}(5-x^2)^{3/2} + \frac{1}{5}(5-x^2)^{5/2} + C$$

11. $x = \frac{3}{2}\sec\theta,\ dx = \frac{3}{2}\sec\theta\tan\theta\,d\theta,\ \frac{2}{9}\int\frac{1}{\sec\theta}d\theta = \frac{2}{9}\int\cos\theta\,d\theta = \frac{2}{9}\sin\theta + C = \frac{\sqrt{4x^2-9}}{9x} + C$

12. $t = \tan\theta,\ dt = \sec^2\theta\,d\theta,$

$$\int\frac{\sec^3\theta}{\tan\theta}d\theta = \int\frac{\tan^2\theta + 1}{\tan\theta}\sec\theta\,d\theta = \int(\sec\theta\tan\theta + \csc\theta)d\theta$$

$$= \sec\theta + \ln|\csc\theta - \cot\theta| + C = \sqrt{1+t^2} + \ln\frac{\sqrt{1+t^2}-1}{|t|} + C$$

13. $x = \sin\theta,\ dx = \cos\theta\,d\theta,\ \int\frac{1}{\cos^2\theta}d\theta = \int\sec^2\theta\,d\theta = \tan\theta + C = x/\sqrt{1-x^2} + C$

14. $x = 5\tan\theta,\ dx = 5\sec^2\theta\,d\theta,\ \frac{1}{25}\int\frac{\sec\theta}{\tan^2\theta}d\theta = \frac{1}{25}\int\csc\theta\cot\theta\,d\theta = -\frac{1}{25}\csc\theta + C = -\frac{\sqrt{x^2+25}}{25x} + C$

15. $x = \sec\theta,\ dx = \sec\theta\tan\theta\,d\theta,\ \int\sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|x + \sqrt{x^2-1}\right| + C$

16. $1+2x^2+x^4=(1+x^2)^2,\ x=\tan\theta,\ dx=\sec^2\theta\,d\theta,$

$$\int\frac{1}{\sec^2\theta}d\theta=\int\cos^2\theta\,d\theta=\frac{1}{2}\int(1+\cos 2\theta)d\theta=\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta+C$$
$$=\frac{1}{2}\theta+\frac{1}{2}\sin\theta\cos\theta+C=\frac{1}{2}\tan^{-1}x+\frac{x}{2(1+x^2)}+C$$

17. $x=\frac{1}{3}\sec\theta,\ dx=\frac{1}{3}\sec\theta\tan\theta\,d\theta,$

$$\frac{1}{3}\int\frac{\sec\theta}{\tan^2\theta}d\theta=\frac{1}{3}\int\csc\theta\cot\theta\,d\theta=-\frac{1}{3}\csc\theta+C=-x/\sqrt{9x^2-1}+C$$

18. $x=5\sec\theta,\ dx=5\sec\theta\tan\theta\,d\theta,$

$$25\int\sec^3\theta\,d\theta=\frac{25}{2}\sec\theta\tan\theta+\frac{25}{2}\ln|\sec\theta+\tan\theta|+C_1$$
$$=\frac{1}{2}x\sqrt{x^2-25}+\frac{25}{2}\ln|x+\sqrt{x^2-25}|+C$$

19. $e^x=\sin\theta,\ e^x dx=\cos\theta\,d\theta,$

$$\int\cos^2\theta\,d\theta=\frac{1}{2}\int(1+\cos 2\theta)d\theta=\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta+C=\frac{1}{2}\sin^{-1}(e^x)+\frac{1}{2}e^x\sqrt{1-e^{2x}}+C$$

20. $u=\sin\theta,\ \int\frac{1}{\sqrt{2-u^2}}du=\sin^{-1}\left(\frac{\sin\theta}{\sqrt{2}}\right)+C$

21. $x=4\sin\theta,\ dx=4\cos\theta\,d\theta,$

$$1024\int_0^{\pi/2}\sin^3\theta\cos^2\theta\,d\theta=1024\left[-\frac{1}{3}\cos^3\theta+\frac{1}{5}\cos^5\theta\right]_0^{\pi/2}=1024(1/3-1/5)=2048/15$$

22. $x=\frac{2}{3}\sin\theta,\ dx=\frac{2}{3}\cos\theta\,d\theta,$

$$\frac{1}{24}\int_0^{\pi/6}\frac{1}{\cos^3\theta}d\theta=\frac{1}{24}\int_0^{\pi/6}\sec^3\theta\,d\theta=\left[\frac{1}{48}\sec\theta\tan\theta+\frac{1}{48}\ln|\sec\theta+\tan\theta|\right]_0^{\pi/6}$$
$$=\frac{1}{48}[(2/\sqrt{3})(1/\sqrt{3})+\ln|2/\sqrt{3}+1/\sqrt{3}|]=\frac{1}{48}\left(\frac{2}{3}+\frac{1}{2}\ln 3\right)$$

23. $x=\sec\theta,\ dx=\sec\theta\tan\theta\,d\theta,\ \int_{\pi/4}^{\pi/3}\frac{1}{\sec\theta}d\theta=\int_{\pi/4}^{\pi/3}\cos\theta\,d\theta=\sin\theta\Big]_{\pi/4}^{\pi/3}=(\sqrt{3}-\sqrt{2})/2$

24. $x=\sqrt{2}\sec\theta,\ dx=\sqrt{2}\sec\theta\tan\theta\,d\theta,\ 2\int_0^{\pi/4}\tan^2\theta\,d\theta=\Big[2\tan\theta-2\theta\Big]_0^{\pi/4}=2-\pi/2$

25. $x=\sqrt{3}\tan\theta,\ dx=\sqrt{3}\sec^2\theta\,d\theta,$

$$\frac{1}{9}\int_{\pi/6}^{\pi/3}\frac{\sec\theta}{\tan^4\theta}d\theta=\frac{1}{9}\int_{\pi/6}^{\pi/3}\frac{\cos^3\theta}{\sin^4\theta}d\theta=\frac{1}{9}\int_{\pi/6}^{\pi/3}\frac{1-\sin^2\theta}{\sin^4\theta}\cos\theta\,d\theta=\frac{1}{9}\int_{1/2}^{\sqrt{3}/2}\frac{1-u^2}{u^4}du\ (u=\sin\theta)$$
$$=\frac{1}{9}\int_{1/2}^{\sqrt{3}/2}(u^{-4}-u^{-2})du=\frac{1}{9}\left[-\frac{1}{3u^3}+\frac{1}{u}\right]_{1/2}^{\sqrt{3}/2}=\frac{10\sqrt{3}+18}{243}$$

26. $x = \sqrt{3}\tan\theta,\ dx = \sqrt{3}\sec^2\theta\,d\theta,$

$$\frac{\sqrt{3}}{3}\int_0^{\pi/3}\frac{\tan^3\theta}{\sec^3\theta}d\theta = \frac{\sqrt{3}}{3}\int_0^{\pi/3}\sin^3\theta\,d\theta = \frac{\sqrt{3}}{3}\int_0^{\pi/3}\left[1-\cos^2\theta\right]\sin\theta\,d\theta$$
$$= \frac{\sqrt{3}}{3}\left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^{\pi/3} = \frac{\sqrt{3}}{3}\left[\left(-\frac{1}{2}+\frac{1}{24}\right)-\left(-1+\frac{1}{3}\right)\right] = 5\sqrt{3}/72$$

27. $u = x^2+4,\ du = 2x\,dx,$

$$\frac{1}{2}\int\frac{1}{u}du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+4)+C;\ \text{or } x = 2\tan\theta,\ dx = 2\sec^2\theta\,d\theta,$$

$$\int\tan\theta\,d\theta = \ln|\sec\theta| + C_1 = \ln\frac{\sqrt{x^2+4}}{2} + C_1 = \ln(x^2+4)^{1/2} - \ln 2 + C_1$$
$$= \frac{1}{2}\ln(x^2+4) + C \text{ with } C = C_1 - \ln 2$$

28. $x = 2\tan\theta, dx = 2\sec^2\theta\,d\theta,\ \displaystyle\int 2\tan^2\theta\,d\theta = 2\tan\theta - 2\theta + C = x - 2\tan^{-1}\frac{x}{2} + C$; alternatively

$$\int\frac{x^2}{x^2+4}\,dx = \int dx - 4\int\frac{dx}{x^2+4} = x - 2\tan^{-1}\frac{x}{2} + C$$

29. $y' = \dfrac{1}{x},\ 1+(y')^2 = 1+\dfrac{1}{x^2} = \dfrac{x^2+1}{x^2},$

$$L = \int_1^2\sqrt{\frac{x^2+1}{x^2}}dx = \int_1^2\frac{\sqrt{x^2+1}}{x}dx;\ x = \tan\theta,\ dx = \sec^2\theta\,d\theta,$$

$$L = \int_{\pi/4}^{\tan^{-1}(2)}\frac{\sec^3\theta}{\tan\theta}d\theta = \int_{\pi/4}^{\tan^{-1}(2)}\frac{\tan^2\theta+1}{\tan\theta}\sec\theta\,d\theta = \int_{\pi/4}^{\tan^{-1}(2)}(\sec\theta\tan\theta + \csc\theta)d\theta$$
$$= \Big[\sec\theta + \ln|\csc\theta - \cot\theta|\Big]_{\pi/4}^{\tan^{-1}(2)} = \sqrt{5} + \ln\left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right) - \left[\sqrt{2} + \ln|\sqrt{2}-1|\right]$$
$$= \sqrt{5} - \sqrt{2} + \ln\frac{2+2\sqrt{2}}{1+\sqrt{5}}$$

30. $y' = 2x,\ 1+(y')^2 = 1+4x^2,$

$$L = \int_0^1\sqrt{1+4x^2}dx;\ x = \frac{1}{2}\tan\theta,\ dx = \frac{1}{2}\sec^2\theta\,d\theta,$$

$$L = \frac{1}{2}\int_0^{\tan^{-1}2}\sec^3\theta\,d\theta = \frac{1}{2}\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|\right)\Big]_0^{\tan^{-1}2}$$
$$= \frac{1}{4}(\sqrt{5})(2) + \frac{1}{4}\ln|\sqrt{5}+2| = \frac{1}{2}\sqrt{5} + \frac{1}{4}\ln(2+\sqrt{5})$$

31. $y' = 2x,\ 1+(y')^2 = 1+4x^2,$

$$S = 2\pi\int_0^1 x^2\sqrt{1+4x^2}dx;\ x = \frac{1}{2}\tan\theta,\ dx = \frac{1}{2}\sec^2\theta\,d\theta,$$

$$S = \frac{\pi}{4}\int_0^{\tan^{-1}2} \tan^2\theta\sec^3\theta\,d\theta = \frac{\pi}{4}\int_0^{\tan^{-1}2}(\sec^2\theta - 1)\sec^3\theta\,d\theta = \frac{\pi}{4}\int_0^{\tan^{-1}2}(\sec^5\theta - \sec^3\theta)d\theta$$

$$= \frac{\pi}{4}\left[\frac{1}{4}\sec^3\theta\tan\theta - \frac{1}{8}\sec\theta\tan\theta - \frac{1}{8}\ln|\sec\theta + \tan\theta|\right]_0^{\tan^{-1}2} = \frac{\pi}{32}[18\sqrt{5} - \ln(2+\sqrt{5})]$$

32. $V = \pi\int_0^1 y^2\sqrt{1-y^2}dy;\ y = \sin\theta,\ dy = \cos\theta\,d\theta,$

$$V = \pi\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta = \frac{\pi}{4}\int_0^{\pi/2}\sin^2 2\theta\,d\theta = \frac{\pi}{8}\int_0^{\pi/2}(1-\cos 4\theta)d\theta = \frac{\pi}{8}\left(\theta - \frac{1}{4}\sin 4\theta\right)\Big]_0^{\pi/2} = \frac{\pi^2}{16}$$

33. **(a)** $x = 3\sinh u,\ dx = 3\cosh u\,du,\ \int du = u + C = \sinh^{-1}(x/3) + C$

(b) $x = 3\tan\theta,\ dx = 3\sec^2\theta\,d\theta,$

$$\int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left(\sqrt{x^2+9}/3 + x/3\right) + C$$

but $\sinh^{-1}(x/3) = \ln\left(x/3 + \sqrt{x^2/9+1}\right) = \ln\left(x/3 + \sqrt{x^2+9}/3\right)$ so the results agree.

(c) $x = \cosh u,\ dx = \sinh u\,du,$

$$\int \sinh^2 u\,du = \frac{1}{2}\int(\cosh 2u - 1)du = \frac{1}{4}\sinh 2u - \frac{1}{2}u + C$$

$$= \frac{1}{2}\sinh u\cosh u - \frac{1}{2}u + C = \frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\cosh^{-1}x + C$$

because $\cosh u = x$, and $\sinh u = \sqrt{\cosh^2 u - 1} = \sqrt{x^2-1}$

34. $A = \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx;\ x = a\cos\theta, dx = -a\sin\theta\,d\theta,$

$$A = -\frac{4b}{a}\int_{\pi/2}^0 a^2\sin^2\theta\,d\theta = 4ab\int_0^{\pi/2}\sin^2\theta\,d\theta = 2ab\int_0^{\pi/2}(1-\cos 2\theta)\,d\theta = \pi ab$$

35. $\int \frac{1}{(x-2)^2+9}dx = \frac{1}{3}\tan^{-1}\left(\frac{x-2}{3}\right) + C$

36. $\int \frac{1}{\sqrt{1-(x-1)^2}}dx = \sin^{-1}(x-1) + C$

37. $\int \frac{1}{\sqrt{9-(x-1)^2}}dx = \sin^{-1}\left(\frac{x-1}{3}\right) + C$

38. $\int \frac{1}{16(x+1/2)^2+1}dx = \frac{1}{16}\int\frac{1}{(x+1/2)^2+1/16}dx = \frac{1}{4}\tan^{-1}(4x+2) + C$

39. $\int \frac{1}{\sqrt{(x-3)^2+1}}dx = \ln\left(x - 3 + \sqrt{(x-3)^2+1}\right) + C$

40. $\displaystyle\int \frac{x}{(x+3)^2+1}dx$, let $u = x+3$,

$$\int \frac{u-3}{u^2+1}du = \int\left(\frac{u}{u^2+1} - \frac{3}{u^2+1}\right)du = \frac{1}{2}\ln(u^2+1) - 3\tan^{-1}u + C$$
$$= \frac{1}{2}\ln(x^2+6x+10) - 3\tan^{-1}(x+3) + C$$

41. $\displaystyle\int \sqrt{4-(x+1)^2}dx$, let $x+1 = 2\sin\theta$,

$$4\int \cos^2\theta\,d\theta = 2\theta + \sin 2\theta + C = 2\theta + 2\sin\theta\cos\theta + C$$
$$= 2\sin^{-1}\left(\frac{x+1}{2}\right) + \frac{1}{2}(x+1)\sqrt{3-2x-x^2} + C$$

42. $\displaystyle\int \frac{e^x}{\sqrt{(e^x+1/2)^2+3/4}}dx$, let $u = e^x + 1/2$,

$$\int \frac{1}{\sqrt{u^2+3/4}}du = \sinh^{-1}(2u/\sqrt{3}) + C = \sinh^{-1}\left(\frac{2e^x+1}{\sqrt{3}}\right) + C$$

Alternate solution: let $e^x + 1/2 = \dfrac{\sqrt{3}}{2}\tan\theta$,

$$\int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left(\frac{2\sqrt{e^{2x}+e^x+1}}{\sqrt{3}} + \frac{2e^x+1}{\sqrt{3}}\right) + C_1$$
$$= \ln(2\sqrt{e^{2x}+e^x+1} + 2e^x + 1) + C$$

43. $\displaystyle\int \frac{1}{2(x+1)^2+5}dx = \frac{1}{2}\int \frac{1}{(x+1)^2+5/2}dx = \frac{1}{\sqrt{10}}\tan^{-1}\sqrt{2/5}(x+1) + C$

44. $\displaystyle\int \frac{2x+3}{4(x+1/2)^2+4}dx$, let $u = x+1/2$,

$$\int \frac{2u+2}{4u^2+4}du = \frac{1}{2}\int\left(\frac{u}{u^2+1} + \frac{1}{u^2+1}\right)du = \frac{1}{4}\ln(u^2+1) + \frac{1}{2}\tan^{-1}u + C$$
$$= \frac{1}{4}\ln(x^2+x+5/4) + \frac{1}{2}\tan^{-1}(x+1/2) + C$$

45. $\displaystyle\int_1^2 \frac{1}{\sqrt{4x-x^2}}\,dx = \int_1^2 \frac{1}{\sqrt{4-(x-2)^2}}\,dx = \sin^{-1}\frac{x-2}{2}\Big]_1^2 = \pi/6$

46. $\displaystyle\int_0^1 \sqrt{4x-x^2}dx = \int_0^1 \sqrt{4-(x-2)^2}dx$, let $x-2 = 2\sin\theta$,

$$4\int_{-\pi/2}^{-\pi/6} \cos^2\theta\,d\theta = \Big[2\theta + \sin 2\theta\Big]_{-\pi/2}^{-\pi/6} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

47. $u = \sin^2 x, du = 2\sin x\cos x\,dx$;

$$\frac{1}{2}\int \sqrt{1-u^2}\,du = \frac{1}{4}\left[u\sqrt{1-u^2} + \sin^{-1}u\right] + C = \frac{1}{4}\left[\sin^2 x\sqrt{1-\sin^4 x} + \sin^{-1}(\sin^2 x)\right] + C$$

48. $u = x\sin x, du = (x\cos x + \sin x)\,dx;$

$$\int \sqrt{1+u^2}\,du = \frac{1}{2}u\sqrt{1+u^2} + \frac{1}{2}\sinh^{-1}u + C = \frac{1}{2}x\sin x\sqrt{1+x^2\sin^2 x} + \frac{1}{2}\sinh^{-1}(x\sin x) + C$$

EXERCISE SET 8.5

1. $\dfrac{A}{(x-2)} + \dfrac{B}{(x+5)}$

2. $\dfrac{5}{x(x-3)(x+3)} = \dfrac{A}{x} + \dfrac{B}{x-3} + \dfrac{C}{x+3}$

3. $\dfrac{2x-3}{x^2(x-1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-1}$

4. $\dfrac{A}{x+2} + \dfrac{B}{(x+2)^2} + \dfrac{C}{(x+2)^3}$

5. $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{Dx+E}{x^2+1}$

6. $\dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+5}$

7. $\dfrac{Ax+B}{x^2+5} + \dfrac{Cx+D}{(x^2+5)^2}$

8. $\dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+1} + \dfrac{Dx+E}{(x^2+1)^2}$

9. $\dfrac{1}{(x+4)(x-1)} = \dfrac{A}{x+4} + \dfrac{B}{x-1};\ A = -\dfrac{1}{5},\ B = \dfrac{1}{5}$ so

$$-\frac{1}{5}\int \frac{1}{x+4}dx + \frac{1}{5}\int \frac{1}{x-1}dx = -\frac{1}{5}\ln|x+4| + \frac{1}{5}\ln|x-1| + C = \frac{1}{5}\ln\left|\frac{x-1}{x+4}\right| + C$$

10. $\dfrac{1}{(x+1)(x+7)} = \dfrac{A}{x+1} + \dfrac{B}{x+7};\ A = \dfrac{1}{6},\ B = -\dfrac{1}{6}$ so

$$\frac{1}{6}\int \frac{1}{x+1}dx - \frac{1}{6}\int \frac{1}{x+7}dx = \frac{1}{6}\ln|x+1| - \frac{1}{6}\ln|x+7| + C = \frac{1}{6}\ln\left|\frac{x+1}{x+7}\right| + C$$

11. $\dfrac{11x+17}{(2x-1)(x+4)} = \dfrac{A}{2x-1} + \dfrac{B}{x+4};\ A = 5,\ B = 3$ so

$$5\int \frac{1}{2x-1}dx + 3\int \frac{1}{x+4}dx = \frac{5}{2}\ln|2x-1| + 3\ln|x+4| + C$$

12. $\dfrac{5x-5}{(x-3)(3x+1)} = \dfrac{A}{x-3} + \dfrac{B}{3x+1};\ A = 1,\ B = 2$ so

$$\int \frac{1}{x-3}dx + 2\int \frac{1}{3x+1}dx = \ln|x-3| + \frac{2}{3}\ln|3x+1| + C$$

13. $\dfrac{2x^2-9x-9}{x(x+3)(x-3)} = \dfrac{A}{x} + \dfrac{B}{x+3} + \dfrac{C}{x-3};\ A = 1,\ B = 2,\ C = -1$ so

$$\int \frac{1}{x}\,dx + 2\int \frac{1}{x+3}\,dx - \int \frac{1}{x-3}\,dx = \ln|x| + 2\ln|x+3| - \ln|x-3| + C = \ln\left|\frac{x(x+3)^2}{x-3}\right| + C$$

Note that the symbol C has been recycled; to save space this recycling is usually not mentioned.

14. $\dfrac{1}{x(x+1)(x-1)} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x-1}$; $A = -1,\ B = \dfrac{1}{2},\ C = \dfrac{1}{2}$ so

$$-\int \frac{1}{x}\,dx + \frac{1}{2}\int \frac{1}{x+1}\,dx + \frac{1}{2}\int \frac{1}{x-1}\,dx = -\ln|x| + \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C$$

$$= \frac{1}{2}\ln\left|\frac{(x+1)(x-1)}{x^2}\right| + C = \frac{1}{2}\ln\frac{|x^2-1|}{x^2} + C$$

15. $\dfrac{x^2+2}{x+2} = x - 2 + \dfrac{6}{x+2}$, $\displaystyle\int \left(x - 2 + \frac{6}{x+2}\right) dx = \frac{1}{2}x^2 - 2x + 6\ \ln|x+2| + C$

16. $\dfrac{x^2-4}{x-1} = x + 1 - \dfrac{3}{x-1}$, $\displaystyle\int \left(x + 1 - \frac{3}{x-1}\right) dx = \frac{1}{2}x^2 + x - 3\ \ln|x-1| + C$

17. $\dfrac{3x^2-10}{x^2-4x+4} = 3 + \dfrac{12x-22}{x^2-4x+4}$, $\dfrac{12x-22}{(x-2)^2} = \dfrac{A}{x-2} + \dfrac{B}{(x-2)^2}$; $A = 12,\ B = 2$ so

$$\int 3dx + 12\int \frac{1}{x-2}dx + 2\int \frac{1}{(x-2)^2}dx = 3x + 12\ln|x-2| - 2/(x-2) + C$$

18. $\dfrac{x^2}{x^2-3x+2} = 1 + \dfrac{3x-2}{x^2-3x+2}$, $\dfrac{3x-2}{(x-1)(x-2)} = \dfrac{A}{x-1} + \dfrac{B}{x-2}$; $A = -1,\ B = 4$ so

$$\int dx - \int \frac{1}{x-1}dx + 4\int \frac{1}{x-2}dx = x - \ln|x-1| + 4\ln|x-2| + C$$

19. $\dfrac{x^5+2x^2+1}{x^3-x} = x^2 + 1 + \dfrac{2x^2+x+1}{x^3-x}$,

$\dfrac{2x^2+x+1}{x(x+1)(x-1)} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x-1}$; $A = -1,\ B = 1,\ C = 2$ so

$$\int (x^2+1)dx - \int \frac{1}{x}dx + \int \frac{1}{x+1}dx + 2\int \frac{1}{x-1}dx$$

$$= \frac{1}{3}x^3 + x - \ln|x| + \ln|x+1| + 2\ln|x-1| + C = \frac{1}{3}x^3 + x + \ln\left|\frac{(x+1)(x-1)^2}{x}\right| + C$$

20. $\dfrac{2x^5-x^3-1}{x^3-4x} = 2x^2 + 7 + \dfrac{28x-1}{x^3-4x}$,

$\dfrac{28x-1}{x(x+2)(x-2)} = \dfrac{A}{x} + \dfrac{B}{x+2} + \dfrac{C}{x-2}$; $A = \dfrac{1}{4},\ B = -\dfrac{57}{8},\ C = \dfrac{55}{8}$ so

$$\int (2x^2+7)\,dx + \frac{1}{4}\int \frac{1}{x}\,dx - \frac{57}{8}\int \frac{1}{x+2}\,dx + \frac{55}{8}\int \frac{1}{x-2}\,dx$$

$$= \frac{2}{3}x^3 + 7x + \frac{1}{4}\ln|x| - \frac{57}{8}\ln|x+2| + \frac{55}{8}\ln|x-2| + C$$

21. $\dfrac{2x^2+3}{x(x-1)^2} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$; $A = 3,\ B = -1,\ C = 5$ so

$$3\int \frac{1}{x}\,dx - \int \frac{1}{x-1}\,dx + 5\int \frac{1}{(x-1)^2}dx = 3\ln|x| - \ln|x-1| - 5/(x-1) + C$$

22. $\dfrac{3x^2-x+1}{x^2(x-1)}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x-1}$; $A=0,\ B=-1,\ C=3$ so

$$-\int\frac{1}{x^2}dx+3\int\frac{1}{x-1}dx=1/x+3\ln|x-1|+C$$

23. $\dfrac{x^2+x-16}{(x+1)(x-3)^2}=\dfrac{A}{x+1}+\dfrac{B}{x-3}+\dfrac{C}{(x-3)^2}$; $A=-1,\ B=2,\ C=-1$ so

$$-\int\frac{1}{x+1}\,dx+2\int\frac{1}{x-3}\,dx-\int\frac{1}{(x-3)^2}\,dx$$

$$=-\ln|x+1|+2\ln|x-3|+\frac{1}{x-3}+C=\ln\frac{(x-3)^2}{|x+1|}+\frac{1}{x-3}+C$$

24. $\dfrac{2x^2-2x-1}{x^2(x-1)}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x-1}$; $A=3,\ B=1,\ C=-1$ so

$$3\int\frac{1}{x}\,dx+\int\frac{1}{x^2}\,dx-\int\frac{1}{x-1}\,dx=3\ln|x|-\frac{1}{x}-\ln|x-1|+C$$

25. $\dfrac{x^2}{(x+2)^3}=\dfrac{A}{x+2}+\dfrac{B}{(x+2)^2}+\dfrac{C}{(x+2)^3}$; $A=1,\ B=-4,\ C=4$ so

$$\int\frac{1}{x+2}\,dx-4\int\frac{1}{(x+2)^2}\,dx+4\int\frac{1}{(x+2)^3}\,dx=\ln|x+2|+\frac{4}{x+2}-\frac{2}{(x+2)^2}+C$$

26. $\dfrac{2x^2+3x+3}{(x+1)^3}=\dfrac{A}{x+1}+\dfrac{B}{(x+1)^2}+\dfrac{C}{(x+1)^3}$; $A=2,\ B=-1,\ C=2$ so

$$2\int\frac{1}{x+1}\,dx-\int\frac{1}{(x+1)^2}\,dx+2\int\frac{1}{(x+1)^3}\,dx=2\ln|x+1|+\frac{1}{x+1}-\frac{1}{(x+1)^2}+C$$

27. $\dfrac{2x^2-1}{(4x-1)(x^2+1)}=\dfrac{A}{4x-1}+\dfrac{Bx+C}{x^2+1}$; $A=-14/17,\ B=12/17,\ C=3/17$ so

$$\int\frac{2x^2-1}{(4x-1)(x^2+1)}dx=-\frac{7}{34}\ln|4x-1|+\frac{6}{17}\ln(x^2+1)+\frac{3}{17}\tan^{-1}x+C$$

28. $\dfrac{1}{x(x^2+1)}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2+1}$; $A=1,\ B=-1,\ C=0$ so

$$\int\frac{1}{x^3+x}\,dx=\ln|x|-\frac{1}{2}\ln(x^2+1)+C=\frac{1}{2}\ln\frac{x^2}{x^2+1}+C$$

29. $\dfrac{x^3+3x^2+x+9}{(x^2+1)(x^2+3)}=\dfrac{Ax+B}{x^2+1}+\dfrac{Cx+D}{x^2+3}$; $A=0,\ B=3,\ C=1,\ D=0$ so

$$\int\frac{x^3+3x^2+x+9}{(x^2+1)(x^2+3)}\,dx=3\tan^{-1}x+\frac{1}{2}\ln(x^2+3)+C$$

30. $\dfrac{x^3+x^2+x+2}{(x^2+1)(x^2+2)}=\dfrac{Ax+B}{x^2+1}+\dfrac{Cx+D}{x^2+2}$; $A=D=0,\ B=C=1$ so

$$\int\frac{x^3+x^2+x+2}{(x^2+1)(x^2+2)}\,dx=\tan^{-1}x+\frac{1}{2}\ln(x^2+2)+C$$

31. $\dfrac{x^3-3x^2+2x-3}{x^2+1}=x-3+\dfrac{x}{x^2+1}$,

$$\int \frac{x^3-3x^2+2x-3}{x^2+1}\,dx=\frac{1}{2}x^2-3x+\frac{1}{2}\ln(x^2+1)+C$$

32. $\dfrac{x^4+6x^3+10x^2+x}{x^2+6x+10}=x^2+\dfrac{x}{x^2+6x+10}$,

$$\int \frac{x}{x^2+6x+10}\,dx=\int \frac{x}{(x+3)^2+1}\,dx=\int \frac{u-3}{u^2+1}du, \quad u=x+3$$

$$=\frac{1}{2}\ln(u^2+1)-3\tan^{-1}u+C_1$$

so $\displaystyle\int \frac{x^4+6x^3+10x^2+x}{x^2+6x+10}\,dx=\frac{1}{3}x^3+\frac{1}{2}\ln(x^2+6x+10)-3\tan^{-1}(x+3)+C$

33. Let $x=\sin\theta$ to get $\displaystyle\int \frac{1}{x^2+4x-5}\,dx$, and $\dfrac{1}{(x+5)(x-1)}=\dfrac{A}{x+5}+\dfrac{B}{x-1}$; $A=-1/6$,

$B=1/6$ so we get $\displaystyle -\frac{1}{6}\int \frac{1}{x+5}\,dx+\frac{1}{6}\int \frac{1}{x-1}\,dx=\frac{1}{6}\ln\left|\frac{x-1}{x+5}\right|+C=\frac{1}{6}\ln\left(\frac{1-\sin\theta}{5+\sin\theta}\right)+C.$

34. Let $x=e^t$; then $\displaystyle\int \frac{e^t}{e^{2t}-4}dt=\int \frac{1}{x^2-4}\,dx$,

$\dfrac{1}{(x+2)(x-2)}=\dfrac{A}{x+2}+\dfrac{B}{x-2}$; $A=-1/4$, $B=1/4$ so

$$-\frac{1}{4}\int \frac{1}{x+2}\,dx+\frac{1}{4}\int \frac{1}{x-2}\,dx=\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|+C=\frac{1}{4}\ln\left|\frac{e^t-2}{e^t+2}\right|+C.$$

35. $\displaystyle V=\pi\int_0^2 \frac{x^4}{(9-x^2)^2}\,dx$, $\dfrac{x^4}{x^4-18x^2+81}=1+\dfrac{18x^2-81}{x^4-18x^2+81}$,

$$\frac{18x^2-81}{(9-x^2)^2}=\frac{18x^2-81}{(x+3)^2(x-3)^2}=\frac{A}{x+3}+\frac{B}{(x+3)^2}+\frac{C}{x-3}+\frac{D}{(x-3)^2};$$

$A=-\dfrac{9}{4}$, $B=\dfrac{9}{4}$, $C=\dfrac{9}{4}$, $D=\dfrac{9}{4}$ so

$$V=\pi\left[x-\frac{9}{4}\ln|x+3|-\frac{9/4}{x+3}+\frac{9}{4}\ln|x-3|-\frac{9/4}{x-3}\right]_0^2=\pi\left(\frac{19}{5}-\frac{9}{4}\ln 5\right)$$

36. Let $u=e^x$ to get $\displaystyle\int_{-\ln 5}^{\ln 5}\frac{dx}{1+e^x}=\int_{-\ln 5}^{\ln 5}\frac{e^x dx}{e^x(1+e^x)}=\int_{1/5}^{5}\frac{du}{u(1+u)}$,

$$\frac{1}{u(1+u)}=\frac{A}{u}+\frac{B}{1+u};\ A=1,\ B=-1;\ \int_{1/5}^{5}\frac{du}{u(1+u)}=(\ln u-\ln(1+u))\Big]_{1/5}^{5}=\ln 5$$

37. $\dfrac{x^2+1}{(x^2+2x+3)^2}=\dfrac{Ax+B}{x^2+2x+3}+\dfrac{Cx+D}{(x^2+2x+3)^2}$; $A=0$, $B=1$, $C=D=-2$ so

$$\int \frac{x^2+1}{(x^2+2x+3)^2}dx=\int \frac{1}{(x+1)^2+2}dx-\int \frac{2x+2}{(x^2+2x+3)^2}dx$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\frac{x+1}{\sqrt{2}}+1/(x^2+2x+3)+C$$

38. $\dfrac{x^5+x^4+4x^3+4x^2+4x+4}{(x^2+2)^3}=\dfrac{Ax+B}{x^2+2}+\dfrac{Cx+D}{(x^2+2)^2}+\dfrac{Ex+F}{(x^2+2)^3}$;

$A=B=1$, $C=D=E=F=0$ so

$$\int \frac{x+1}{x^2+2}dx=\frac{1}{2}\ln(x^2+2)+\frac{1}{\sqrt{2}}\tan^{-1}(x/\sqrt{2})+C$$

39. $x^4-3x^3-7x^2+27x-18=(x-1)(x-2)(x-3)(x+3)$,

$\dfrac{1}{(x-1)(x-2)(x-3)(x+3)}=\dfrac{A}{x-1}+\dfrac{B}{x-2}+\dfrac{C}{x-3}+\dfrac{D}{x+3}$;

$A=1/8$, $B=-1/5$, $C=1/12$, $D=-1/120$ so

$$\int \frac{dx}{x^4-3x^3-7x^2+27x-18}=\frac{1}{8}\ln|x-1|-\frac{1}{5}\ln|x-2|+\frac{1}{12}\ln|x-3|-\frac{1}{120}\ln|x+3|+C$$

40. $16x^3-4x^2+4x-1=(4x-1)(4x^2+1)$,

$\dfrac{1}{(4x-1)(4x^2+1)}=\dfrac{A}{4x-1}+\dfrac{Bx+C}{4x^2+1}$; $A=4/5$, $B=-4/5$, $C=-1/5$ so

$$\int \frac{dx}{16x^3-4x^2+4x-1}=\frac{1}{5}\ln|4x-1|-\frac{1}{10}\ln(4x^2+1)-\frac{1}{10}\tan^{-1}(2x)+C$$

41. **(a)**
$$\begin{aligned}x^4+1&=(x^4+2x^2+1)-2x^2=(x^2+1)^2-2x^2\\&=[(x^2+1)+\sqrt{2}x][(x^2+1)-\sqrt{2}x]\\&=(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1); a=\sqrt{2}, b=-\sqrt{2}\end{aligned}$$

(b) $\dfrac{x}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)}=\dfrac{Ax+B}{x^2+\sqrt{2}x+1}+\dfrac{Cx+D}{x^2-\sqrt{2}x+1}$;

$A=0$, $B=-\dfrac{\sqrt{2}}{4}$, $C=0$, $D=\dfrac{\sqrt{2}}{4}$ so

$$\begin{aligned}\int_0^1 \frac{x}{x^4+1}dx&=-\frac{\sqrt{2}}{4}\int_0^1\frac{1}{x^2+\sqrt{2}x+1}dx+\frac{\sqrt{2}}{4}\int_0^1\frac{1}{x^2-\sqrt{2}x+1}dx\\
&=-\frac{\sqrt{2}}{4}\int_0^1\frac{1}{(x+\sqrt{2}/2)^2+1/2}dx+\frac{\sqrt{2}}{4}\int_0^1\frac{1}{(x-\sqrt{2}/2)^2+1/2}dx\\
&=-\frac{\sqrt{2}}{4}\int_{\sqrt{2}/2}^{1+\sqrt{2}/2}\frac{1}{u^2+1/2}du+\frac{\sqrt{2}}{4}\int_{-\sqrt{2}/2}^{1-\sqrt{2}/2}\frac{1}{u^2+1/2}du\\
&=-\frac{1}{2}\tan^{-1}\sqrt{2}u\Big]_{\sqrt{2}/2}^{1+\sqrt{2}/2}+\frac{1}{2}\tan^{-1}\sqrt{2}u\Big]_{-\sqrt{2}/2}^{1-\sqrt{2}/2}\\
&=-\frac{1}{2}\tan^{-1}(\sqrt{2}+1)+\frac{1}{2}\left(\frac{\pi}{4}\right)+\frac{1}{2}\tan^{-1}(\sqrt{2}-1)-\frac{1}{2}\left(-\frac{\pi}{4}\right)\\
&=\frac{\pi}{4}-\frac{1}{2}[\tan^{-1}(\sqrt{2}+1)-\tan^{-1}(\sqrt{2}-1)]\\
&=\frac{\pi}{4}-\frac{1}{2}[\tan^{-1}(1+\sqrt{2})+\tan^{-1}(1-\sqrt{2})]\\
&=\frac{\pi}{4}-\frac{1}{2}\tan^{-1}\left[\frac{(1+\sqrt{2})+(1-\sqrt{2})}{1-(1+\sqrt{2})(1-\sqrt{2})}\right]\quad\text{(Exercise 78, Section 7.6)}\\
&=\frac{\pi}{4}-\frac{1}{2}\tan^{-1}1=\frac{\pi}{4}-\frac{1}{2}\left(\frac{\pi}{4}\right)=\frac{\pi}{8}\end{aligned}$$

42. $\dfrac{1}{a^2-x^2}=\dfrac{A}{a-x}+\dfrac{B}{a+x}$; $A=\dfrac{1}{2a}, B=\dfrac{1}{2a}$ so

$$\frac{1}{2a}\int\left(\frac{1}{a-x}+\frac{1}{a+x}\right)dx=\frac{1}{2a}\left(-\ln|a-x|+\ln|a+x|\right)+C=\frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|+C$$

EXERCISE SET 8.6

1. Formula (60): $\dfrac{3}{16}\left[4x+\ln|-1+4x|\right]+C$

2. Formula (62): $\dfrac{1}{9}\left[\dfrac{2}{2-3x}+\ln|2-3x|\right]+C$

3. Formula (65): $\dfrac{1}{5}\ln\left|\dfrac{x}{5+2x}\right|+C$

4. Formula (66): $-\dfrac{1}{x}-5\ln\left|\dfrac{1-5x}{x}\right|+C$

5. Formula (102): $\dfrac{1}{5}(x+1)(-3+2x)^{3/2}+C$

6. Formula (105): $\dfrac{2}{3}(-x-4)\sqrt{2-x}+C$

7. Formula (108): $\dfrac{1}{2}\ln\left|\dfrac{\sqrt{4-3x}-2}{\sqrt{4-3x}+2}\right|+C$

8. Formula (108): $\tan^{-1}\dfrac{\sqrt{3x-4}}{2}+C$

9. Formula (69): $\dfrac{1}{2\sqrt{5}}\ln\left|\dfrac{x+\sqrt{5}}{x-\sqrt{5}}\right|+C$

10. Formula (70): $\dfrac{1}{6}\ln\left|\dfrac{x-3}{x+3}\right|+C$

11. Formula (73): $\dfrac{x}{2}\sqrt{x^2-3}-\dfrac{3}{2}\ln\left|x+\sqrt{x^2-3}\right|+C$

12. Formula (93): $-\dfrac{\sqrt{x^2+5}}{x}+\ln(x+\sqrt{x^2+5})+C$

13. Formula (95): $\dfrac{x}{2}\sqrt{x^2+4}-2\ln(x+\sqrt{x^2+4})+C$

14. Formula (90): $-\dfrac{\sqrt{x^2-2}}{2x}+C$

15. Formula (74): $\dfrac{x}{2}\sqrt{9-x^2}+\dfrac{9}{2}\sin^{-1}\dfrac{x}{3}+C$

16. Formula (80): $-\dfrac{\sqrt{4-x^2}}{x}-\sin^{-1}\dfrac{x}{2}+C$

17. Formula (79): $\sqrt{3-x^2}-\sqrt{3}\ln\left|\dfrac{\sqrt{3}+\sqrt{9-x^2}}{x}\right|+C$

18. Formula (117): $-\dfrac{\sqrt{6x-x^2}}{3x}+C$

19. Formula (38): $-\dfrac{1}{10}\sin(5x)+\dfrac{1}{2}\sin x+C$

20. Formula (40): $-\dfrac{1}{14}\cos(7x)+\dfrac{1}{6}\cos(3x)+C$

21. Formula (50): $\dfrac{x^4}{16}\left[4\ln x-1\right]+C$

22. Formula (50): $4\sqrt{x}\left[\dfrac{1}{2}\ln x-1\right]+C$

23. Formula (42): $\dfrac{e^{-2x}}{13}(-2\sin(3x)-3\cos(3x))+C$

24. Formula (43): $\dfrac{e^x}{5}(\cos(2x) + 2\sin(2x)) + C$

25. $u = e^{2x}, du = 2e^{2x}dx$, Formula (62): $\dfrac{1}{2}\displaystyle\int \frac{u\,du}{(4-3u)^2} = \frac{1}{18}\left[\frac{4}{4-3e^{2x}} + \ln\left|4-3e^{2x}\right|\right] + C$

26. $u = \sin 2x, du = 2\cos 2x dx$, Formula (116): $\displaystyle\int \frac{du}{2u(3-u)} = \frac{1}{6}\ln\left|\frac{\sin 2x}{3-\sin 2x}\right| + C$

27. $u = 3\sqrt{x}, du = \dfrac{3}{2\sqrt{x}}dx$, Formula (68): $\dfrac{2}{3}\displaystyle\int \frac{du}{u^2+4} = \frac{1}{3}\tan^{-1}\frac{3\sqrt{x}}{2} + C$

28. $u = \sin 4x, du = 4\cos 4x dx$, Formula (68): $\dfrac{1}{4}\displaystyle\int \frac{du}{9+u^2} = \frac{1}{12}\tan^{-1}\frac{\sin 4x}{3} + C$

29. $u = 3x, du = 3dx$, Formula (76): $\dfrac{1}{3}\displaystyle\int \frac{du}{\sqrt{u^2-4}} = \frac{1}{3}\ln\left|3x + \sqrt{9x^2-4}\right| + C$

30. $u = \sqrt{2}x^2, du = 2\sqrt{2}x dx$, Formula (72):

$$\frac{1}{2\sqrt{2}}\int \sqrt{u^2+3}\,du = \frac{x^2}{4}\sqrt{2x^4+3} + \frac{3}{4\sqrt{2}}\ln\left(\sqrt{2}x^2 + \sqrt{2x^4+3}\right) + C$$

31. $u = 3x^2, du = 6x dx, u^2 du = 54x^5 dx$, Formula (81):

$$\frac{1}{54}\int \frac{u^2\,du}{\sqrt{5-u^2}} = -\frac{x^2}{36}\sqrt{5-9x^4} + \frac{5}{108}\sin^{-1}\frac{3x^2}{\sqrt{5}} + C$$

32. $u = 2x, du = 2dx$, Formula (83): $2\displaystyle\int \frac{du}{u^2\sqrt{3-u^2}} = -\frac{1}{3x}\sqrt{3-4x^2} + C$

33. $u = \ln x, du = dx/x$, Formula (26): $\displaystyle\int \sin^2 u\,du = \frac{1}{2}\ln x + \frac{1}{4}\sin(2\ln x) + C$

34. $u = e^{-2x}, du = -2e^{-2x}$, Formula (27): $-\dfrac{1}{2}\displaystyle\int \cos^2 u\,du = -\frac{1}{4}e^{-2x} - \frac{1}{8}\sin\left(2e^{-2x}\right) + C$

35. $u = -2x, du = -2dx$, Formula (51): $\dfrac{1}{4}\displaystyle\int ue^u\,du = \frac{1}{4}(-2x-1)e^{-2x} + C$

36. $u = 5x-1, du = 5dx$, Formula (50): $\dfrac{1}{5}\displaystyle\int \ln u\,du = \frac{1}{5}(u\ln u - u) + C = \frac{1}{5}(5x-1)[\ln(5x-1)-1] + C$

37. $u = \cos 3x, du = -3\sin 3x$, Formula (67): $-\displaystyle\int \frac{du}{u(u+1)^2} = -\frac{1}{3}\left[\frac{1}{1+\cos 3x} + \ln\left|\frac{\cos 3x}{1+\cos 3x}\right|\right] + C$

38. $u = \ln x, du = \dfrac{1}{x}dx$, Formula (105): $\displaystyle\int \frac{u\,du}{\sqrt{4u-1}} = \frac{1}{12}(2\ln x + 1)\sqrt{4\ln x - 1} + C$

39. $u = 4x^2, du = 8x dx$, Formula (70): $\dfrac{1}{8}\displaystyle\int \frac{du}{u^2-1} = \frac{1}{16}\ln\left|\frac{4x^2-1}{4x^2+1}\right| + C$

40. $u = 2e^x, du = 2e^x dx$, Formula (69): $\dfrac{1}{2}\displaystyle\int \frac{du}{3-u^2} = \frac{1}{4\sqrt{3}}\ln\left|\frac{2e^x+\sqrt{3}}{2e^x-\sqrt{3}}\right| + C$

41. $u = 2e^x, du = 2e^x dx$, Formula (74):

$$\frac{1}{2}\int \sqrt{3-u^2}\,du = \frac{1}{4}u\sqrt{3-u^2} + \frac{3}{4}\sin^{-1}(u/\sqrt{3}) + C = \frac{1}{2}e^x\sqrt{3-4e^{2x}} + \frac{3}{4}\sin^{-1}(2e^x/\sqrt{3}) + C$$

42. $u = 3x, du = 3dx$, Formula (80):

$$3\int \frac{\sqrt{4-u^2}du}{u^2} = -3\frac{\sqrt{4-u^2}}{u} - 3\sin^{-1}(u/2) + C = -\frac{\sqrt{4-9x^2}}{x} - 3\sin^{-1}(3x/2) + C$$

43. $u = 3x, du = 3dx$, Formula (112):

$$\frac{1}{3}\int \sqrt{\frac{5}{3}u - u^2}\,du = \frac{1}{6}\left(u - \frac{5}{6}\right)\sqrt{\frac{5}{3}u - u^2} + \frac{25}{216}\sin^{-1}\left(\frac{u-5}{5}\right) + C$$

$$= \frac{18x-5}{36}\sqrt{5x-9x^2} + \frac{25}{216}\sin^{-1}\left(\frac{18x-5}{5}\right) + C$$

44. $u = \sqrt{5}x, du = \sqrt{5}\,dx$, Formula (117):

$$\int \frac{du}{u\sqrt{(u/\sqrt{5}) - u^2}} = -\frac{\sqrt{(u/\sqrt{5}) - u^2}}{u/(2\sqrt{5})} + C = -2\frac{\sqrt{x-5x^2}}{x} + C$$

45. $u = 3x, du = 3dx$, Formula (44):

$$\frac{1}{9}\int u\sin u\,du = \frac{1}{9}(\sin u - u\cos u) + C = \frac{1}{9}(\sin 3x - 3x\cos 3x) + C$$

46. $u = \sqrt{x}, u^2 = x, 2udu = dx$, Formula (45): $2\int u\cos u\,du = 2\cos\sqrt{x} + 2\sqrt{x}\sin\sqrt{x} + C$

47. $u = -\sqrt{x}, u^2 = x, 2udu = dx$, Formula (51): $2\int ue^u du = -2(\sqrt{x}+1)e^{-\sqrt{x}} + C$

48. $u = 2 - 3x^2, du = -6xdx$, Formula (50):

$$-\frac{1}{6}\int \ln u\,du = -\frac{1}{6}(u\ln u - u) + C = -\frac{1}{6}((2-3x^2)\ln(2-3x^2) + \frac{1}{6}(2-3x^2) + C$$

49. $x^2 + 4x - 5 = (x+2)^2 - 9; u = x + 2, du = dx$, Formula (70):

$$\int \frac{du}{u^2-9} = \frac{1}{6}\ln\left|\frac{u-3}{u+3}\right| + C = \frac{1}{6}\ln\left|\frac{x-1}{x+5}\right| + C$$

50. $x^2 + 2x - 3 = (x+1)^2 - 4, u = x + 1, du = dx$, Formula (77):

$$\int \sqrt{4-u^2}\,du = \frac{1}{2}u\sqrt{4-u^2} + 2\sin^{-1}(u/2) + C$$

$$= \frac{1}{2}(x+1)\sqrt{3-2x-x^2} + 2\sin^{-1}((x+1)/2) + C$$

51. $x^2 - 4x - 5 = (x-2)^2 - 9, u = x - 2, du = dx$, Formula (77):

$$\int \frac{u+2}{\sqrt{9-u^2}}\,du = \int \frac{u\,du}{\sqrt{9-u^2}} + 2\int \frac{du}{\sqrt{9-u^2}} = -\sqrt{9-u^2} + 2\sin^{-1}\frac{u}{3} + C$$

$$= -\sqrt{5+4x-x^2} + 2\sin^{-1}\left(\frac{x-2}{3}\right) + C$$

52. $x^2+6x+13=(x+3)^2+4, u=x+3, du=dx$, Formula (71):

$$\int \frac{(u-3)\,du}{u^2+4} = \frac{1}{2}\ln(u^2+4) - \frac{3}{2}\tan^{-1}(u/2)+C = \frac{1}{2}\ln(x^2+6x+13) - \frac{3}{2}\tan^{-1}((x+3)/2)+C$$

53. $u=\sqrt{x-2},\ x=u^2+2,\ dx=2u\,du;$

$$\int 2u^2(u^2+2)du = 2\int (u^4+2u^2)du = \frac{2}{5}u^5+\frac{4}{3}u^3+C = \frac{2}{5}(x-2)^{5/2}+\frac{4}{3}(x-2)^{3/2}+C$$

54. $u=\sqrt{x+1},\ x=u^2-1,\ dx=2u\,du;$

$$2\int (u^2-1)du = \frac{2}{3}u^3-2u+C = \frac{2}{3}(x+1)^{3/2}-2\sqrt{x+1}+C$$

55. $u=\sqrt{x^3+1},\ x^3=u^2-1,\ 3x^2dx=2u\,du;$

$$\frac{2}{3}\int u^2(u^2-1)du = \frac{2}{3}\int (u^4-u^2)du = \frac{2}{15}u^5-\frac{2}{9}u^3+C = \frac{2}{15}(x^3+1)^{5/2}-\frac{2}{9}(x^3+1)^{3/2}+C$$

56. $u=\sqrt{x^3-1},\ x^3=u^2+1,\ 3x^2dx=2u\,du;$

$$\frac{2}{3}\int \frac{1}{u^2+1}du = \frac{2}{3}\tan^{-1}u+C = \frac{2}{3}\tan^{-1}\sqrt{x^3-1}+C$$

57. $u=x^{1/6},\ x=u^6,\ dx=6u^5du;$

$$\int \frac{6u^5}{u^3+u^2}du = 6\int \frac{u^3}{u+1}du = 6\int \left[u^2-u+1-\frac{1}{u+1}\right]du$$
$$= 2x^{1/2}-3x^{1/3}+6x^{1/6}-6\ \ln(x^{1/6}+1)+C$$

58. $u=x^{1/5},\ x=u^5,\ dx=5u^4du;\ \int \frac{5u^4}{u^5-u^3}du = 5\int \frac{u}{u^2-1}du = \frac{5}{2}\ln|x^{2/5}-1|+C$

59. $u=x^{1/4},\ x=u^4,\ dx=4u^3du;\ 4\int \frac{1}{u(1-u)}du = 4\int \left[\frac{1}{u}+\frac{1}{1-u}\right]du = 4\ln\frac{x^{1/4}}{|1-x^{1/4}|}+C$

60. $u=x^{1/3},\ x=u^3,\ dx=3u^2du;\ 3\int \frac{u^4}{u^3+1}du = 3\int \left(u-\frac{u}{u^3+1}\right)du,$

$$\frac{u}{u^3+1} = \frac{u}{(u+1)(u^2-u+1)} = \frac{-1/3}{u+1}+\frac{(1/3)u+1/3}{u^2-u+1} \text{ so}$$

$$3\int\left(u-\frac{u}{u^3+1}\right)du = \int\left(3u+\frac{1}{u+1}-\frac{u+1}{u^2-u+1}\right)du$$
$$= \frac{3}{2}u^2+\ln|u+1|-\frac{1}{2}\ln(u^2-u+1)-\sqrt{3}\tan^{-1}\frac{2u-1}{\sqrt{3}}+C$$
$$= \frac{3}{2}x^{2/3}+\ln|x^{1/3}+1|-\frac{1}{2}\ln(x^{2/3}-x^{1/3}+1)-\sqrt{3}\tan^{-1}\frac{2x^{1/3}-1}{\sqrt{3}}+C$$

61. $u=x^{1/6},\ x=u^6,\ dx=6u^5du;$

$$6\int \frac{u^3}{u-1}du = 6\int \left[u^2+u+1+\frac{1}{u-1}\right]du = 2x^{1/2}+3x^{1/3}+6x^{1/6}+6\ \ln|x^{1/6}-1|+C$$

62. $u = \sqrt{x}$, $x = u^2$, $dx = 2u\,du$;

$$-2\int \frac{u^2+u}{u-1}du = -2\int \left(u+2+\frac{2}{u-1}\right)du = -x - 4\sqrt{x} - 4\ln|\sqrt{x}-1| + C$$

63. $u = \sqrt{1+x^2}$, $x^2 = u^2 - 1$, $2x\,dx = 2u\,du$, $x\,dx = u\,du$;

$$\int (u^2-1)du = \frac{1}{3}(1+x^2)^{3/2} - (1+x^2)^{1/2} + C$$

64. $u = (x+3)^{1/5}$, $x = u^5 - 3$, $dx = 5u^4 du$;

$$5\int (u^8 - 3u^3)du = \frac{5}{9}(x+3)^{9/5} - \frac{15}{4}(x+3)^{4/5} + C$$

65. $u = \sqrt{x}$, $x = u^2$, $dx = 2u\,du$, Formula (44): $2\int u\sin u\,du = 2\sin\sqrt{x} - 2\sqrt{x}\cos\sqrt{x} + C$

66. $u = \sqrt{x}$, $x = u^2$, $dx = 2u\,du$, Formula (51): $2\int ue^u du = 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$

67. $$\int \frac{1}{1+\dfrac{2u}{1+u^2}+\dfrac{1-u^2}{1+u^2}}\,\frac{2}{1+u^2}du = \int \frac{1}{u+1}du = \ln|\tan(x/2)+1| + C$$

68. $$\int \frac{1}{2+\dfrac{2u}{1+u^2}}\,\frac{2}{1+u^2}du = \int \frac{1}{u^2+u+1}du$$

$$= \int \frac{1}{(u+1/2)^2+3/4}du = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan(x/2)+1}{\sqrt{3}}\right) + C$$

69. $u = \tan(\theta/2)$, $\displaystyle\int \frac{d\theta}{1-\cos\theta} = \int \frac{1}{u^2}du = -\frac{1}{u} + C = -\cot(\theta/2) + C$

70. $u = \tan(x/2)$,

$$\int \frac{2}{3u^2+8u-3}du = \frac{2}{3}\int \frac{1}{(u+4/3)^2-25/9}du = \frac{2}{3}\int \frac{1}{z^2-25/9}dz \quad (z = u+4/3)$$

$$= \frac{1}{5}\ln\left|\frac{z-5/3}{z+5/3}\right| + C = \frac{1}{5}\ln\left|\frac{\tan(x/2)-1/3}{\tan(x/2)+3}\right| + C$$

71. $u = \tan(x/2)$, $\displaystyle 2\int \frac{1-u^2}{(3u^2+1)(u^2+1)}du$;

$$\frac{1-u^2}{(3u^2+1)(u^2+1)} = \frac{(0)u+2}{3u^2+1} + \frac{(0)u-1}{u^2+1} = \frac{2}{3u^2+1} - \frac{1}{u^2+1} \text{ so}$$

$$\int \frac{\cos x}{2-\cos x}dx = \frac{4}{\sqrt{3}}\tan^{-1}\left[\sqrt{3}\tan(x/2)\right] - x + C$$

72. $u = \tan(x/2)$, $\displaystyle\frac{1}{2}\int \frac{1-u^2}{u}du = \frac{1}{2}\int (1/u - u)du = \frac{1}{2}\ln|\tan(x/2)| - \frac{1}{4}\tan^2(x/2) + C$

73. $\displaystyle\int_2^x \frac{1}{t(4-t)}\,dt = \frac{1}{4}\ln\frac{t}{4-t}\bigg]_2^x$ (Formula (65), $a=4, b=-1$)

$$= \frac{1}{4}\left[\ln\frac{x}{4-x} - \ln 1\right] = \frac{1}{4}\ln\frac{x}{4-x}, \frac{1}{4}\ln\frac{x}{4-x} = 0.5, \ln\frac{x}{4-x} = 2,$$

$\dfrac{x}{4-x} = e^2$, $x = 4e^2 - e^2 x$, $x(1+e^2) = 4e^2$, $x = 4e^2/(1+e^2) \approx 3.523188312$

74. $\displaystyle\int_1^x \frac{1}{t\sqrt{2t-1}}\,dt = 2\tan^{-1}\sqrt{2t-1}\bigg]_1^x$ (Formula (108), $a=-1, b=2$)

$$= 2\left(\tan^{-1}\sqrt{2x-1} - \tan^{-1} 1\right) = 2\left(\tan^{-1}\sqrt{2x-1} - \pi/4\right),$$

$2(\tan^{-1}\sqrt{2x-1} - \pi/4) = 1$, $\tan^{-1}\sqrt{2x-1} = 1/2 + \pi/4$, $\sqrt{2x-1} = \tan(1/2+\pi/4)$,

$x = [1 + \tan^2(1/2+\pi/4)]/2 \approx 6.307993516$

75. $\displaystyle A = \int_0^4 \sqrt{25-x^2}\,dx = \left(\frac{1}{2}x\sqrt{25-x^2} + \frac{25}{2}\sin^{-1}\frac{x}{5}\right)\bigg]_0^4$ (Formula (74), $a=5$)

$$= 6 + \frac{25}{2}\sin^{-1}\frac{4}{5} \approx 17.59119023$$

76. $\displaystyle A = \int_{2/3}^2 \sqrt{9x^2-4}\,dx$; $u = 3x$,

$\displaystyle A = \frac{1}{3}\int_2^6 \sqrt{u^2-4}\,du = \frac{1}{3}\left(\frac{1}{2}u\sqrt{u^2-4} - 2\ln\left|u + \sqrt{u^2-4}\,\right|\right)\bigg]_2^6$ (Formula (73), $a^2 = 4$)

$$= \frac{1}{3}\left(3\sqrt{32} - 2\ln(6+\sqrt{32}) + 2\ln 2\right) = 4\sqrt{2} - \frac{2}{3}\ln(3+2\sqrt{2}) \approx 4.481689467$$

77. $\displaystyle A = \int_0^1 \frac{1}{25-16x^2}\,dx$; $u = 4x$,

$\displaystyle A = \frac{1}{4}\int_0^4 \frac{1}{25-u^2}\,du = \frac{1}{40}\ln\left|\frac{u+5}{u-5}\right|\bigg]_0^4 = \frac{1}{40}\ln 9 \approx 0.054930614$ (Formula (69), $a=5$)

78. $\displaystyle A = \int_1^4 \sqrt{x}\ln x\,dx = \frac{4}{9}x^{3/2}\left(\frac{3}{2}\ln x - 1\right)\bigg]_1^4$ (Formula (50), $n = 1/2$)

$$= \frac{4}{9}(12\ln 4 - 7) \approx 4.282458815$$

79. $\displaystyle V = 2\pi\int_0^{\pi/2} x\cos x\,dx = 2\pi(\cos x + x\sin x)\bigg]_0^{\pi/2} = \pi(\pi-2) \approx 3.586419094$ (Formula (45))

80. $\displaystyle V = 2\pi\int_4^8 x\sqrt{x-4}\,dx = \frac{4\pi}{15}(3x+8)(x-4)^{3/2}\bigg]_4^8$ (Formula (102), $a=-4, b=1$)

$$= \frac{1024}{15}\pi \approx 214.4660585$$

81. $V = 2\pi \int_0^3 xe^{-x} dx$; $u = -x$,

$$V = 2\pi \int_0^{-3} ue^u du = 2\pi e^u(u-1)\Big]_0^{-3} = 2\pi(1 - 4e^{-3}) \approx 5.031899801 \quad \text{(Formula (51))}$$

82. $V = 2\pi \int_1^5 x \ln x \, dx = \frac{\pi}{2} x^2 (2\ln x - 1)\Big]_1^5$

$$= \pi(25\ln 5 - 12) \approx 88.70584621 \quad \text{(Formula (50), } n = 1\text{)}$$

83. $L = \int_0^2 \sqrt{1 + 16x^2}\, dx$; $u = 4x$,

$$L = \frac{1}{4}\int_0^8 \sqrt{1+u^2}\, du = \frac{1}{4}\left(\frac{u}{2}\sqrt{1+u^2} + \frac{1}{2}\ln\left(u + \sqrt{1+u^2}\right)\right)\Bigg]_0^8 \quad \text{(Formula (72), } a^2 = 1\text{)}$$

$$= \sqrt{65} + \frac{1}{8}\ln(8 + \sqrt{65}) \approx 8.409316783$$

84. $L = \int_1^3 \sqrt{1 + 9/x^2}\, dx = \int_1^3 \frac{\sqrt{x^2+9}}{x} dx = \left(\sqrt{x^2+9} - 3\ln\left|\frac{3 + \sqrt{x^2+9}}{x}\right|\right)\Bigg]_1^3$

$$= 3\sqrt{2} - \sqrt{10} + 3\ln\frac{3+\sqrt{10}}{1+\sqrt{2}} \approx 3.891581644 \quad \text{(Formula (89), } a = 3\text{)}$$

85. $S = 2\pi \int_0^\pi (\sin x)\sqrt{1 + \cos^2 x}\, dx$; $u = \cos x$,

$$S = -2\pi \int_1^{-1} \sqrt{1+u^2}\, du = 4\pi \int_0^1 \sqrt{1+u^2}\, du = 4\pi\left(\frac{u}{2}\sqrt{1+u^2} + \frac{1}{2}\ln\left(u + \sqrt{1+u^2}\right)\right)\Bigg]_0^1 a^2 = 1$$

$$= 2\pi\left[\sqrt{2} + \ln(1+\sqrt{2})\right] \approx 14.42359945 \quad \text{(Formula (72))}$$

86. $S = 2\pi \int_1^4 \frac{1}{x}\sqrt{1 + 1/x^4}\, dx = 2\pi \int_1^4 \frac{\sqrt{x^4+1}}{x^3} dx$; $u = x^2$,

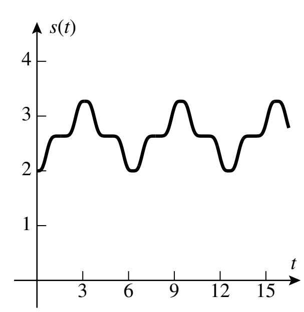
$$S = \pi \int_1^{16} \frac{\sqrt{u^2+1}}{u^2} du = \pi\left(-\frac{\sqrt{u^2+1}}{u} + \ln\left(u + \sqrt{u^2+1}\right)\right)\Bigg]_1^{16}$$

$$= \pi\left(\sqrt{2} - \frac{\sqrt{257}}{16} + \ln\frac{16+\sqrt{257}}{1+\sqrt{2}}\right) \approx 9.417237485 \text{ (Formula (93), } a^2 = 1\text{)}$$

87. (a) $s(t) = 2 + \int_0^t 20\cos^6 u \sin^3 u\, du$

$$= -\frac{20}{9}\sin^2 t\cos^7 t - \frac{40}{63}\cos^7 t + \frac{166}{63}$$

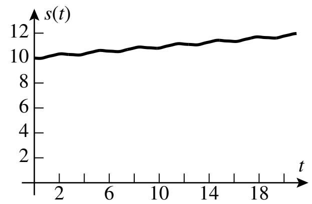
(b)



88. **(a)** $v(t) = \int_0^t a(u)\,du = -\frac{1}{10}e^{-t}\cos 2t + \frac{1}{5}e^{-t}\sin 2t + \frac{1}{74}e^{-t}\cos 6t - \frac{3}{37}e^{-t}\sin 6t + \frac{1}{10} - \frac{1}{74}$

$$s(t) = 10 + \int_0^t v(u)\,du$$
$$= -\frac{3}{50}e^{-t}\cos 2t - \frac{2}{25}e^{-t}\sin 2t + \frac{35}{2738}e^{-t}\cos 6t + \frac{6}{1369}e^{-t}\sin 6t + \frac{16}{185}t + \frac{343866}{34225}$$

(b)



89. **(a)** $\int \sec x\,dx = \int \frac{1}{\cos x}dx = \int \frac{2}{1-u^2}du = \ln\left|\frac{1+u}{1-u}\right| + C = \ln\left|\frac{1+\tan(x/2)}{1-\tan(x/2)}\right| + C$

$$= \ln\left\{\left|\frac{\cos(x/2)+\sin(x/2)}{\cos(x/2)-\sin(x/2)}\right|\left|\frac{\cos(x/2)+\sin(x/2)}{\cos(x/2)+\sin(x/2)}\right|\right\} + C = \ln\left|\frac{1+\sin x}{\cos x}\right| + C$$
$$= \ln|\sec x + \tan x| + C$$

(b) $\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) = \frac{\tan\frac{\pi}{4} + \tan\frac{x}{2}}{1 - \tan\frac{\pi}{4}\tan\frac{x}{2}} = \frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}$

90. $\int \csc x\,dx = \int \frac{1}{\sin x}dx = \int 1/u\,du = \ln|\tan(x/2)| + C$ but

$\ln|\tan(x/2)| = \frac{1}{2}\ln\frac{\sin^2(x/2)}{\cos^2(x/2)} = \frac{1}{2}\ln\frac{(1-\cos x)/2}{(1+\cos x)/2} = \frac{1}{2}\ln\frac{1-\cos x}{1+\cos x}$; also,

$\frac{1-\cos x}{1+\cos x} = \frac{1-\cos^2 x}{(1+\cos x)^2} = \frac{1}{(\csc x + \cot x)^2}$ so $\frac{1}{2}\ln\frac{1-\cos x}{1+\cos x} = -\ln|\csc x + \cot x|$

91. Let $u = \tanh(x/2)$ then $\cosh(x/2) = 1/\operatorname{sech}(x/2) = 1/\sqrt{1-\tanh^2(x/2)} = 1/\sqrt{1-u^2}$,

$\sinh(x/2) = \tanh(x/2)\cosh(x/2) = u/\sqrt{1-u^2}$, so $\sinh x = 2\sinh(x/2)\cosh(x/2) = 2u/(1-u^2)$,

$\cosh x = \cosh^2(x/2) + \sinh^2(x/2) = (1+u^2)/(1-u^2)$, $x = 2\tanh^{-1}u$, $dx = [2/(1-u^2)]du$;

$$\int \frac{dx}{2\cosh x + \sinh x} = \int \frac{1}{u^2+u+1}du = \frac{2}{\sqrt{3}}\tan^{-1}\frac{2u+1}{\sqrt{3}} + C = \frac{2}{\sqrt{3}}\tan^{-1}\frac{2\tanh(x/2)+1}{\sqrt{3}} + C.$$

EXERCISE SET 8.7

1. exact value $= 14/3 \approx 4.666666667$

(a) 4.667600663, $|E_M| \approx 0.000933996$
(b) 4.664795679, $|E_T| \approx 0.001870988$
(c) 4.666651630, $|E_S| \approx 0.000015037$

2. exact value $= 2$

(a) 1.998377048, $|E_M| \approx 0.001622952$
(b) 2.003260982, $|E_T| \approx 0.003260982$
(c) 2.000072698, $|E_S| \approx 0.000072698$

3. exact value = 2

(a) 2.008248408, $|E_M| \approx 0.008248408$
(b) 1.983523538, $|E_T| \approx 0.016476462$
(c) 2.000109517, $|E_S| \approx 0.000109517$

4. exact value $= \sin(1) \approx 0.841470985$

(a) 0.841821700, $|E_M| \approx 0.000350715$
(b) 0.840769642, $|E_T| \approx 0.000701343$
(c) 0.841471453, $|E_S| \approx 0.000000468$

5. exact value $= e^{-1} - e^{-3} \approx 0.318092373$

(a) 0.317562837, $|E_M| \approx 0.000529536$
(b) 0.319151975, $|E_T| \approx 0.001059602$
(c) 0.318095187, $|E_S| \approx 0.000002814$

6. exact value $= \frac{1}{2}\ln 5 \approx 0.804718956$

(a) 0.801605339, $|E_M| \approx 0.003113617$
(b) 0.811019505, $|E_T| \approx 0.006300549$
(c) 0.805041497, $|E_S| \approx 0.000322541$

7. $f(x) = \sqrt{x+1}$, $f''(x) = -\frac{1}{4}(x+1)^{-3/2}$, $f^{(4)}(x) = -\frac{15}{16}(x+1)^{-7/2}$; $K_2 = 1/4$, $K_4 = 15/16$

(a) $|E_M| \le \frac{27}{2400}(1/4) = 0.002812500$

(b) $|E_T| \le \frac{27}{1200}(1/4) = 0.005625000$

(c) $|E_S| \le \frac{243}{180 \times 10^4}(15/16) \approx 0.000126563$

8. $f(x) = 1/\sqrt{x}$, $f''(x) = \frac{3}{4}x^{-5/2}$, $f^{(4)}(x) = \frac{105}{16}x^{-9/2}$; $K_2 = 3/4$, $K_4 = 105/16$

(a) $|E_M| \le \frac{27}{2400}(3/4) = 0.008437500$

(b) $|E_T| \le \frac{27}{1200}(3/4) = 0.016875000$

(c) $|E_S| \le \frac{243}{180 \times 10^4}(105/16) \approx 0.000885938$

9. $f(x) = \sin x$, $f''(x) = -\sin x$, $f^{(4)}(x) = \sin x$; $K_2 = K_4 = 1$

(a) $|E_M| \le \frac{\pi^3}{2400}(1) \approx 0.012919282$

(b) $|E_T| \le \frac{\pi^3}{1200}(1) \approx 0.025838564$

(c) $|E_S| \le \frac{\pi^5}{180 \times 10^4}(1) \approx 0.000170011$

10. $f(x) = \cos x$, $f''(x) = -\cos x$, $f^{(4)}(x) = \cos x$; $K_2 = K_4 = 1$

(a) $|E_M| \le \frac{1}{2400}(1) \approx 0.000416667$

(b) $|E_T| \le \frac{1}{1200}(1) \approx 0.000833333$

(c) $|E_S| \le \frac{1}{180 \times 10^4}(1) \approx 0.000000556$

11. $f(x) = e^{-x}$, $f''(x) = f^{(4)}(x) = e^{-x}$; $K_2 = K_4 = e^{-1}$

(a) $|E_M| \le \frac{8}{2400}(e^{-1}) \approx 0.001226265$

(b) $|E_T| \le \frac{8}{1200}(e^{-1}) \approx 0.002452530$

(c) $|E_S| \le \frac{32}{180 \times 10^4}(e^{-1}) \approx 0.000006540$

12. $f(x) = 1/(2x+3)$, $f''(x) = 8(2x+3)^{-3}$, $f^{(4)}(x) = 384(2x+3)^{-5}$; $K_2 = 8$, $K_4 = 384$

(a) $|E_M| \le \frac{8}{2400}(8) \approx 0.026666667$

(b) $|E_T| \le \frac{8}{1200}(8) \approx 0.053333333$

(c) $|E_S| \le \frac{32}{180 \times 10^4}(384) \approx 0.006826667$

13. **(a)** $n > \left[\dfrac{(27)(1/4)}{(24)(5\times 10^{-4})}\right]^{1/2} \approx 23.7$; $n = 24$ **(b)** $n > \left[\dfrac{(27)(1/4)}{(12)(5\times 10^{-4})}\right]^{1/2} \approx 33.5$; $n=34$

(c) $n > \left[\dfrac{(243)(15/16)}{(180)(5\times 10^{-4})}\right]^{1/4} \approx 7.1$; $n = 8$

14. **(a)** $n > \left[\dfrac{(27)(3/4)}{(24)(5\times 10^{-4})}\right]^{1/2} \approx 41.1$; $n = 42$ **(b)** $n > \left[\dfrac{(27)(3/4)}{(12)(5\times 10^{-4})}\right]^{1/2} \approx 58.1$; $n=59$

(c) $n > \left[\dfrac{(243)(105/16)}{(180)(5\times 10^{-4})}\right]^{1/4} \approx 11.5$; $n = 12$

15. **(a)** $n > \left[\dfrac{(\pi^3)(1)}{(24)(10^{-3})}\right]^{1/2} \approx 35.9$; $n = 36$ **(b)** $n > \left[\dfrac{(\pi^3)(1)}{(12)(10^{-3})}\right]^{1/2} \approx 50.8$; $n = 51$

(c) $n > \left[\dfrac{(\pi^5)(1)}{(180)(10^{-3})}\right]^{1/4} \approx 6.4$; $n = 8$

16. **(a)** $n > \left[\dfrac{(1)(1)}{(24)(10^{-3})}\right]^{1/2} \approx 6.5$; $n = 7$ **(b)** $n > \left[\dfrac{(1)(1)}{(12)(10^{-3})}\right]^{1/2} \approx 9.1$; $n = 10$

(c) $n > \left[\dfrac{(1)(1)}{(180)(10^{-3})}\right]^{1/4} \approx 1.5$; $n = 2$

17. **(a)** $n > \left[\dfrac{(8)(e^{-1})}{(24)(10^{-6})}\right]^{1/2} \approx 350.2$; $n = 351$ **(b)** $n > \left[\dfrac{(8)(e^{-1})}{(12)(10^{-6})}\right]^{1/2} \approx 495.2$; $n = 496$

(c) $n > \left[\dfrac{(32)(e^{-1})}{(180)(10^{-6})}\right]^{1/4} \approx 15.99$; $n = 16$

18. **(a)** $n > \left[\dfrac{(8)(8)}{(24)(10^{-6})}\right]^{1/2} \approx 1632.99$; $n = 1633$ **(b)** $n > \left[\dfrac{(8)(8)}{(12)(10^{-6})}\right]^{1/2} \approx 2309.4$; $n = 2310$

(c) $n > \left[\dfrac{(32)(384)}{(180)(10^{-6})}\right]^{1/4} \approx 90.9$; $n = 92$

19. $g(X_0) = aX_0^2 + bX_0 + c = 4a + 2b + c = f(X_0) = 1/X_0 = 1/2$; similarly $9a + 3b + c = 1/3, 16a + 4b + c = 1/4$. Three equations in three unknowns, with solution $a = 1/24, b = -3/8, c = 13/12, g(x) = x^2/24 - 3x/8 + 13/12$.

$$\int_0^4 g(x)\,dx = \int \left(\frac{x^2}{24} - \frac{3x}{8} + \frac{13}{12}\right) dx = \frac{25}{36}$$

$$\frac{\Delta x}{3}[f(X_0) + 4f(X_1) + f(X_2)] = \frac{1}{3}\left[\frac{1}{2} + \frac{4}{3} + \frac{1}{4}\right] = \frac{25}{36}$$

20. $f(X_0) = 1 = g(X_0) = c, f(X_1) = 3/4 = g(X_1) = a/36 + b/6 + c$,
$f(X_2) = 1/4 = g(X_2) = a/9 + b/3 + c$,
with solution $a = -9/2, b = -3/4, c = 1, g(x) = -9x^2/2 - 3x/4 + 1$,

$$\int_0^{1/3} g(x)\,dx = 17/72$$

$$\frac{\Delta x}{3}[f(X_0) + 4f(X_1) + f(X_2) = \frac{1}{18}[1 + 3 + 1/4] = 17/72$$

21. 0.746824948, 0.746824133

22. 1.137631378, 1.137630147

23. 2.129861595, 2.129861293

24. 2.418388347, 2.418399152

25. 0.805376152, 0.804776489

26. 1.536963087, 1.544294774

27. **(a)** 3.142425985, $|E_M| \approx 0.000833331$
(b) 3.139925989, $|E_T| \approx 0.001666665$
(c) 3.141592614, $|E_S| \approx 0.000000040$

28. **(a)** 3.152411433, $|E_M| \approx 0.010818779$
(b) 3.104518326, $|E_T| \approx 0.037074328$
(c) 3.127008159, $|E_S| \approx 0.014584495$

29. $S_{14} = 0.693147984$, $|E_S| \approx 0.000000803 = 8.03 \times 10^{-7}$; the method used in Example 6 results in a value of n which ensures that the magnitude of the error will be less than 10^{-6}, this is not necessarily the *smallest* value of n.

30. **(a)** greater, because the graph of e^{-x^2} is concave up on the interval $(1, 2)$
(b) less, because the graph of e^{-x^2} is concave down on the interval $(0, 0.5)$

31. $f(x) = x\sin x$, $f''(x) = 2\cos x - x\sin x$, $|f''(x)| \le 2|\cos x| + |x|\,|\sin x| \le 2 + 2 = 4$ so $K_2 \le 4$,

$$n > \left[\frac{(8)(4)}{(24)(10^{-4})}\right]^{1/2} \approx 115.5;\ n = 116 \text{ (a smaller } n \text{ might suffice)}$$

32. $f(x) = e^{\cos x}$, $f''(x) = (\sin^2 x)e^{\cos x} - (\cos x)e^{\cos x}$, $|f''(x)| \le e^{\cos x}(\sin^2 x + |\cos x|) \le 2e$ so

$$K_2 \le 2e,\ n > \left[\frac{(1)(2e)}{(24)(10^{-4})}\right]^{1/2} \approx 47.6;\ n = 48 \text{ (a smaller } n \text{ might suffice)}$$

33. $f(x) = \sqrt{x}$, $f''(x) = -\dfrac{1}{4x^{3/2}}$, $\lim_{x\to 0^+} |f''(x)| = +\infty$

34. $f(x) = \sin\sqrt{x}$, $f''(x) = -\dfrac{\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}}{4x^{3/2}}$, $\lim_{x\to 0^+} |f''(x)| = +\infty$

35. $L = \int_0^\pi \sqrt{1 + \cos^2 x}\,dx \approx 3.820187623$

36. $L = \int_1^3 \sqrt{1 + 1/x^4}\,dx \approx 2.146822803$

37.

t (s)	0	5	10	15	20
v (mi/hr)	0	40	60	73	84
v (ft/s)	0	58.67	88	107.07	123.2

$$\int_0^{20} v\,dt \approx \frac{20}{(3)(4)}[0 + 4(58.67) + 2(88) + 4(107.07) + 123.2] \approx 1604 \text{ ft}$$

38.

t	0	1	2	3	4	5	6	7	8
a	0	0.02	0.08	0.20	0.40	0.60	0.70	0.60	0

$$\int_0^8 a\,dt \approx \frac{8}{(3)(8)}[0 + 4(0.02) + 2(0.08) + 4(0.20) + 2(0.40) + 4(0.60) + 2(0.70) + 4(0.60) + 0]$$
$$\approx 2.7 \text{ cm/s}$$

39. $\displaystyle\int_0^{180} v\,dt \approx \frac{180}{(3)(6)}[0.00 + 4(0.03) + 2(0.08) + 4(0.16) + 2(0.27) + 4(0.42) + 0.65] = 37.9 \text{ mi}$

40. $\displaystyle\int_0^{1800} (1/v)dx \approx \frac{1800}{(3)(6)}\left[\frac{1}{3100} + \frac{4}{2908} + \frac{2}{2725} + \frac{4}{2549} + \frac{2}{2379} + \frac{4}{2216} + \frac{1}{2059}\right] \approx 0.71 \text{ s}$

41. $\displaystyle V = \int_0^{16} \pi r^2 dy = \pi\int_0^{16} r^2 dy \approx \pi\frac{16}{(3)(4)}[(8.5)^2 + 4(11.5)^2 + 2(13.8)^2 + 4(15.4)^2 + (16.8)^2]$
$\approx 9270 \text{ cm}^3 \approx 9.3 \text{ L}$

42. $\displaystyle A = \int_0^{600} h\,dx \approx \frac{600}{(3)(6)}[0 + 4(7) + 2(16) + 4(24) + 2(25) + 4(16) + 0] = 9000 \text{ ft}^2,$
$V = 75A \approx 75(9000) = 675,000 \text{ ft}^3$

43. $\displaystyle\int_a^b f(x)\,dx \approx A_1 + A_2 + \cdots + A_n = \frac{b-a}{n}\left[\frac{1}{2}(y_0 + y_1) + \frac{1}{2}(y_1 + y_2) + \cdots + \frac{1}{2}(y_{n-1} + y_n)\right]$
$\displaystyle = \frac{b-a}{2n}[y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n]$

44. right endpoint, trapezoidal, midpoint, left endpoint

45. **(a)** The maximum value of $|f''(x)|$ is approximately 3.844880.
(b) $n = 18$
(c) 0.904741

46. **(a)** The maximum value of $|f''(x)|$ is approximately 1.467890.
(b) $n = 12$
(c) 1.112062

47. **(a)** The maximum value of $|f^{(4)}(x)|$ is approximately 42.551816.
(b) $n = 8$
(c) 0.904524

48. **(a)** The maximum value of $|f^{(4)}(x)|$ is approximately 7.022710.
(b) $n = 8$
(c) 1.111443

EXERCISE SET 8.8

1. **(a)** improper; infinite discontinuity at $x = 3$
(b) continuous integrand, not improper
(c) improper; infinite discontinuity at $x = 0$
(d) improper; infinite interval of integration
(e) improper; infinite interval of integration and infinite discontinuity at $x = 1$
(f) continuous integrand, not improper

2. **(a)** improper if $p > 0$ **(b)** improper if $1 \le p \le 2$
(c) integrand is continuous for all p, not improper

3. $\lim_{\ell\to+\infty}(-e^{-x})\Big]_0^\ell = \lim_{\ell\to+\infty}(-e^{-\ell}+1) = 1$

4. $\lim_{\ell\to+\infty}\frac{1}{2}\ln(1+x^2)\Big]_{-1}^\ell = \lim_{\ell\to+\infty}\frac{1}{2}[\ln(1+\ell^2)-\ln 2] = +\infty$, divergent

5. $\lim_{\ell\to+\infty}\ln\frac{x-1}{x+1}\Big]_4^\ell = \lim_{\ell\to+\infty}\left(\ln\frac{\ell-1}{\ell+1}-\ln\frac{3}{5}\right) = -\ln\frac{3}{5} = \ln\frac{5}{3}$

6. $\lim_{\ell\to+\infty}-\frac{1}{2}e^{-x^2}\Big]_0^\ell = \lim_{\ell\to+\infty}\frac{1}{2}\left(-e^{-\ell^2}+1\right) = 1/2$

7. $\lim_{\ell\to+\infty}-\frac{1}{2\ln^2 x}\Big]_e^\ell = \lim_{\ell\to+\infty}\left[-\frac{1}{2\ln^2\ell}+\frac{1}{2}\right] = \frac{1}{2}$

8. $\lim_{\ell\to+\infty}2\sqrt{\ln x}\Big]_2^\ell = \lim_{\ell\to+\infty}(2\sqrt{\ln\ell}-2\sqrt{\ln 2}) = +\infty$, divergent

9. $\lim_{\ell\to-\infty}-\frac{1}{4(2x-1)^2}\Big]_\ell^0 = \lim_{\ell\to-\infty}\frac{1}{4}[-1+1/(2\ell-1)^2] = -1/4$

10. $\lim_{\ell\to-\infty}\frac{1}{2}\tan^{-1}\frac{x}{2}\Big]_\ell^2 = \lim_{\ell\to-\infty}\frac{1}{2}\left[\frac{\pi}{4}-\tan^{-1}\frac{\ell}{2}\right] = \frac{1}{2}[\pi/4-(-\pi/2)] = 3\pi/8$

11. $\lim_{\ell\to-\infty}\frac{1}{3}e^{3x}\Big]_\ell^0 = \lim_{\ell\to-\infty}\left[\frac{1}{3}-\frac{1}{3}e^{3\ell}\right] = \frac{1}{3}$

12. $\lim_{\ell\to-\infty}-\frac{1}{2}\ln(3-2e^x)\Big]_\ell^0 = \lim_{\ell\to-\infty}\frac{1}{2}\ln(3-2e^\ell) = \frac{1}{2}\ln 3$

13. $\int_{-\infty}^{+\infty}x^3dx$ converges if $\int_{-\infty}^{0}x^3dx$ and $\int_{0}^{+\infty}x^3dx$ both converge; it diverges if either (or both) diverges. $\int_0^{+\infty}x^3dx = \lim_{\ell\to+\infty}\frac{1}{4}x^4\Big]_0^\ell = \lim_{\ell\to+\infty}\frac{1}{4}\ell^4 = +\infty$ so $\int_{-\infty}^{+\infty}x^3dx$ is divergent.

14. $\int_0^{+\infty}\frac{x}{\sqrt{x^2+2}}dx = \lim_{\ell\to+\infty}\sqrt{x^2+2}\Big]_0^\ell = \lim_{\ell\to+\infty}(\sqrt{\ell^2+2}-\sqrt{2}) = +\infty$
so $\int_{-\infty}^{\infty}\frac{x}{\sqrt{x^2+2}}dx$ is divergent.

15. $\int_0^{+\infty}\frac{x}{(x^2+3)^2}dx = \lim_{\ell\to+\infty}-\frac{1}{2(x^2+3)}\Big]_0^\ell = \lim_{\ell\to+\infty}\frac{1}{2}[-1/(\ell^2+3)+1/3] = \frac{1}{6}$,
similarly $\int_{-\infty}^{0}\frac{x}{(x^2+3)^2}dx = -1/6$ so $\int_{-\infty}^{\infty}\frac{x}{(x^2+3)^2}dx = 1/6+(-1/6) = 0$

16. $\displaystyle\int_0^{+\infty} \frac{e^{-t}}{1+e^{-2t}}dt = \lim_{\ell\to+\infty} -\tan^{-1}(e^{-t})\Big]_0^{\ell} = \lim_{\ell\to+\infty}\left[-\tan^{-1}(e^{-\ell})+\frac{\pi}{4}\right] = \frac{\pi}{4},$

$\displaystyle\int_{-\infty}^{0} \frac{e^{-t}}{1+e^{-2t}}dt = \lim_{\ell\to-\infty} -\tan^{-1}(e^{-t})\Big]_{\ell}^{0} = \lim_{\ell\to-\infty}\left[-\frac{\pi}{4}+\tan^{-1}(e^{-\ell})\right] = \frac{\pi}{4},$

$\displaystyle\int_{-\infty}^{+\infty} \frac{e^{-t}}{1+e^{-2t}}dt = \frac{\pi}{4}+\frac{\pi}{4} = \frac{\pi}{2}$

17. $\displaystyle\lim_{\ell\to3^+} -\frac{1}{x-3}\Big]_{\ell}^{4} = \lim_{\ell\to3^+}\left[-1+\frac{1}{\ell-3}\right] = +\infty$, divergent

18. $\displaystyle\lim_{\ell\to0^+} \frac{3}{2}x^{2/3}\Big]_{\ell}^{8} = \lim_{\ell\to0^+} \frac{3}{2}(4-\ell^{2/3}) = 6$

19. $\displaystyle\lim_{\ell\to\pi/2^-} -\ln(\cos x)\Big]_0^{\ell} = \lim_{\ell\to\pi/2^-} -\ln(\cos\ell) = +\infty$, divergent

20. $\displaystyle\lim_{\ell\to9^-} -2\sqrt{9-x}\Big]_0^{\ell} = \lim_{\ell\to9^-} 2(-\sqrt{9-\ell}+3) = 6$

21. $\displaystyle\lim_{\ell\to1^-} \sin^{-1}x\Big]_0^{\ell} = \lim_{\ell\to1^-} \sin^{-1}\ell = \pi/2$

22. $\displaystyle\lim_{\ell\to-3^+} -\sqrt{9-x^2}\Big]_{\ell}^{1} = \lim_{\ell\to-3^+}(-\sqrt{8}+\sqrt{9-\ell^2}) = -\sqrt{8}$

23. $\displaystyle\lim_{\ell\to\pi/6^-} -\sqrt{1-2\sin x}\Big]_0^{\ell} = \lim_{\ell\to\pi/6^-}(-\sqrt{1-2\sin\ell}+1) = 1$

24. $\displaystyle\lim_{\ell\to\pi/4^-} -\ln(1-\tan x)\Big]_0^{\ell} = \lim_{\ell\to\pi/4^-} -\ln(1-\tan\ell) = +\infty$, divergent

25. $\displaystyle\int_0^2 \frac{dx}{x-2} = \lim_{\ell\to2^-} \ln|x-2|\Big]_0^{\ell} = \lim_{\ell\to2^-}(\ln|\ell-2|-\ln 2) = -\infty$, divergent

26. $\displaystyle\int_0^2 \frac{dx}{x^2} = \lim_{\ell\to0^+} -1/x\Big]_{\ell}^{2} = \lim_{\ell\to0^+}(-1/2+1/\ell) = +\infty$ so $\displaystyle\int_{-2}^{2}\frac{dx}{x^2}$ is divergent

27. $\displaystyle\int_0^8 x^{-1/3}dx = \lim_{\ell\to0^+}\frac{3}{2}x^{2/3}\Big]_{\ell}^{8} = \lim_{\ell\to0^+}\frac{3}{2}(4-\ell^{2/3}) = 6,$

$\displaystyle\int_{-1}^{0} x^{-1/3}dx = \lim_{\ell\to0^-}\frac{3}{2}x^{2/3}\Big]_{-1}^{\ell} = \lim_{\ell\to0^-}\frac{3}{2}(\ell^{2/3}-1) = -3/2$

so $\displaystyle\int_{-1}^{8} x^{-1/3}dx = 6+(-3/2) = 9/2$

28. $\displaystyle\int_0^2 \frac{dx}{(x-2)^{2/3}} = \lim_{\ell\to2^-} 3(x-2)^{1/3}\Big]_0^{\ell} = \lim_{\ell\to2^-} 3[(\ell-2)^{1/3}-(-2)^{1/3}] = 3\sqrt[3]{2},$

similarly $\displaystyle\int_2^4 \frac{dx}{(x-2)^{2/3}} = \lim_{\ell\to2^+} 3(x-2)^{1/3}\Big]_{\ell}^{4} = 3\sqrt[3]{2}$ so $\displaystyle\int_0^4 \frac{dx}{(x-2)^{2/3}} = 6\sqrt[3]{2}$

29. Define $\int_0^{+\infty} \frac{1}{x^2}dx = \int_0^a \frac{1}{x^2}dx + \int_a^{+\infty} \frac{1}{x^2}dx$ where $a > 0$; take $a = 1$ for convenience,

$\int_0^1 \frac{1}{x^2}dx = \lim_{\ell\to 0^+} (-1/x)\Big]_\ell^1 = \lim_{\ell\to 0^+} (1/\ell - 1) = +\infty$ so $\int_0^{+\infty} \frac{1}{x^2}dx$ is divergent.

30. Define $\int_1^{+\infty} \frac{dx}{x\sqrt{x^2-1}} = \int_1^a \frac{dx}{x\sqrt{x^2-1}} + \int_a^{+\infty} \frac{dx}{x\sqrt{x^2-1}}$ where $a > 1$,

take $a = 2$ for convenience to get

$\int_1^2 \frac{dx}{x\sqrt{x^2-1}} = \lim_{\ell\to 1^+} \sec^{-1} x\Big]_\ell^2 = \lim_{\ell\to 1^+} (\pi/3 - \sec^{-1}\ell) = \pi/3,$

$\int_2^{+\infty} \frac{dx}{x\sqrt{x^2-1}} = \lim_{\ell\to +\infty} \sec^{-1} x\Big]_2^\ell = \pi/2 - \pi/3$ so $\int_1^{+\infty} \frac{dx}{x\sqrt{x^2-1}} = \pi/2.$

31. $\int_0^{+\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx = 2\int_0^{+\infty} e^{-u}du = 2\lim_{\ell\to+\infty}\left(-e^{-u}\right)\Big]_0^\ell = 2\lim_{\ell\to+\infty}\left(1 - e^{-\ell}\right) = 2$

32. $\int_0^{+\infty} \frac{dx}{\sqrt{x}(x+4)} = 2\int_0^{+\infty} \frac{du}{u^2+4} = 2\lim_{\ell\to+\infty} \frac{1}{2}\tan^{-1}\frac{u}{2}\Big]_0^\ell = \lim_{\ell\to+\infty} \tan^{-1}\frac{\ell}{2} = \frac{\pi}{2}$

33. $\int_0^{+\infty} \frac{e^{-x}}{\sqrt{1-e^{-x}}}dx = \int_0^1 \frac{du}{\sqrt{u}} = \lim_{\ell\to 0^+} 2\sqrt{u}\Big]_\ell^1 = \lim_{\ell\to 0^+} 2(1-\sqrt{\ell}) = 2$

34. $\int_0^{+\infty} \frac{e^{-x}}{\sqrt{1-e^{-2x}}}dx = -\int_1^0 \frac{du}{\sqrt{1-u^2}} = \int_0^1 \frac{du}{\sqrt{1-u^2}} = \lim_{\ell\to 1} \sin^{-1} u\Big]_0^\ell = \lim_{\ell\to 1} \sin^{-1}\ell = \frac{\pi}{2}$

35. $\lim_{\ell\to+\infty} \int_0^\ell e^{-x}\cos x\,dx = \lim_{\ell\to+\infty} \frac{1}{2}e^{-x}(\sin x - \cos x)\Big]_0^\ell = 1/2$

36. $A = \int_0^{+\infty} xe^{-3x}dx = \lim_{\ell\to+\infty} -\frac{1}{9}(3x+1)e^{-3x}\Big]_0^\ell = 1/3$

37. **(a)** 2.726585 **(b)** 2.804364 **(c)** 0.219384 **(d)** 0.504067

39. $1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{4 - x^{2/3}}{x^{2/3}} = \frac{4}{x^{2/3}}$; the arc length is $\int_0^8 \frac{2}{x^{1/3}}\,dx = 3x^{2/3}\Big|_0^8 = 12$

40. $1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{16x^2}{9-4x^2} = \frac{9+12x^2}{9-4x^2}$; the arc length is $\int_0^{3/2} \sqrt{\frac{9+12x^2}{9-4x^2}}\,dx \approx 3.633168$

41. $\int \ln x\,dx = x\ln x - x + C,$

$\int_0^1 \ln x\,dx = \lim_{\ell\to 0^+} \int_\ell^1 \ln x\,dx = \lim_{\ell\to 0^+} (x\ln x - x)\Big]_\ell^1 = \lim_{\ell\to 0^+} (-1 - \ell\ln\ell + \ell),$

but $\lim_{\ell\to 0^+} \ell\ln\ell = \lim_{\ell\to 0^+} \frac{\ln\ell}{1/\ell} = \lim_{\ell\to 0^+} (-\ell) = 0$ so $\int_0^1 \ln x\,dx = -1$

42. $\displaystyle\int \frac{\ln x}{x^2}dx = -\frac{\ln x}{x} - \frac{1}{x} + C,$

$\displaystyle\int_1^{+\infty} \frac{\ln x}{x^2}dx = \lim_{\ell\to+\infty}\int_1^{\ell} \frac{\ln x}{x^2}dx = \lim_{\ell\to+\infty}\left(-\frac{\ln x}{x} - \frac{1}{x}\right)\Big]_1^{\ell} = \lim_{\ell\to+\infty}\left(-\frac{\ln \ell}{\ell} - \frac{1}{\ell} + 1\right),$

but $\displaystyle\lim_{\ell\to+\infty}\frac{\ln\ell}{\ell} = \lim_{\ell\to+\infty}\frac{1}{\ell} = 0$ so $\displaystyle\int_1^{+\infty}\frac{\ln x}{x^2} = 1$

43. $\displaystyle\int_0^{+\infty} e^{-3x}dx = \lim_{\ell\to+\infty}\int_0^{\ell} e^{-3x}dx = \lim_{\ell\to+\infty}\left(-\frac{1}{3}e^{-3x}\right)\Big]_0^{\ell} = \lim_{\ell\to+\infty}\left(-\frac{1}{3}e^{-3\ell} + \frac{1}{9}\right) = \frac{1}{3}$

44. $\displaystyle A = \int_3^{+\infty}\frac{8}{x^2-4}dx = \lim_{\ell\to+\infty} 2\ln\frac{x-2}{x+2}\Big]_3^{\ell} = \lim_{\ell\to+\infty} 2\left[\ln\frac{\ell-2}{\ell+2} - \ln\frac{1}{5}\right] = 2\ln 5$

45. **(a)** $\displaystyle V = \pi\int_0^{+\infty} e^{-2x}dx = -\frac{\pi}{2}\lim_{\ell\to+\infty} e^{-2x}\Big]_0^{\ell} = \pi/2$

(b) $\displaystyle S = 2\pi\int_0^{+\infty} e^{-x}\sqrt{1+e^{-2x}}dx$, let $u = e^{-x}$ to get

$\displaystyle S = -2\pi\int_1^0 \sqrt{1+u^2}du = 2\pi\left[\frac{u}{2}\sqrt{1+u^2} + \frac{1}{2}\ln\left|u+\sqrt{1+u^2}\right|\right]_0^1 = \pi\left[\sqrt{2} + \ln(1+\sqrt{2})\right]$

47. **(a)** For $x \ge 1, x^2 \ge x, e^{-x^2} \le e^{-x}$

(b) $\displaystyle\int_1^{+\infty} e^{-x}\,dx = \lim_{\ell\to+\infty}\int_1^{\ell} e^{-x}\,dx = \lim_{\ell\to+\infty} -e^{-x}\Big]_1^{\ell} = \lim_{\ell\to+\infty}(e^{-1} - e^{-\ell}) = 1/e$

(c) By Parts (a) and (b) and Exercise 46(b), $\displaystyle\int_1^{+\infty} e^{-x^2}\,dx$ is convergent and is $\le 1/e$.

48. **(a)** If $x \ge 0$ then $\displaystyle e^x \ge 1, \frac{1}{2x+1} \le \frac{e^x}{2x+1}$

(b) $\displaystyle\lim_{\ell\to+\infty}\int_0^{\ell}\frac{dx}{2x+1} = \lim_{\ell\to+\infty}\frac{1}{2}\ln(2x+1)\Big]_0^{\ell} = +\infty$

(c) By Parts (a) and (b) and Exercise 46(a), $\displaystyle\int_0^{+\infty}\frac{e^x}{2x+1}\,dx$ is divergent.

49. $\displaystyle V = \lim_{\ell\to+\infty}\int_1^{\ell}(\pi/x^2)\,dx = \lim_{\ell\to+\infty} -(\pi/x)\Big]_1^{\ell} = \lim_{\ell\to+\infty}(\pi - \pi/\ell) = \pi$

$\displaystyle A = \lim_{\ell\to+\infty}\int_1^{\ell} 2\pi(1/x)\sqrt{1+1/x^4}\,dx$; use Exercise 46(a) with $f(x) = 2\pi/x$, $g(x) = (2\pi/x)\sqrt{1+1/x^4}$ and $a = 1$ to see that the area is infinite.

50. **(a)** $\displaystyle 1 \le \frac{\sqrt{x^3+1}}{x}$ for $x \ge 2$, $\displaystyle\int_2^{+\infty} 1dx = +\infty$

(b) $\displaystyle\int_2^{+\infty}\frac{x}{x^5+1}dx \le \int_2^{+\infty}\frac{dx}{x^4} = \lim_{\ell\to+\infty} -\frac{1}{3x^3}\Big]_2^{\ell} = 1/24$

(c) $\displaystyle\int_0^{\infty}\frac{xe^x}{2x+1}\,dx \ge \int_1^{+\infty}\frac{xe^x}{2x+1} \ge \int_1^{+\infty}\frac{dx}{2x+1} = +\infty$

51. $\displaystyle\int_0^{2x}\sqrt{1+t^3}dt \ge \int_0^{2x} t^{3/2}dt = \frac{2}{5}t^{5/2}\Big]_0^{2x} = \frac{2}{5}(2x)^{5/2}$,

$\displaystyle\lim_{x\to+\infty}\int_0^{2x} t^{3/2}dt = \lim_{x\to+\infty}\frac{2}{5}(2x)^{5/2} = +\infty$ so $\displaystyle\int_0^{+\infty}\sqrt{1+t^3}dt = +\infty$; by L'Hôpital's Rule

$$\lim_{x\to+\infty}\frac{\displaystyle\int_0^{2x}\sqrt{1+t^3}dt}{x^{5/2}} = \lim_{x\to+\infty}\frac{2\sqrt{1+(2x)^3}}{(5/2)x^{3/2}} = \lim_{x\to+\infty}\frac{2\sqrt{1/x^3+8}}{5/2} = 8\sqrt{2}/5$$

52. **(b)** $u=\sqrt{x}$, $\displaystyle\int_0^{+\infty}\frac{\cos\sqrt{x}}{\sqrt{x}}dx = 2\int_0^{+\infty}\cos u\,du$; $\displaystyle\int_0^{+\infty}\cos u\,du$ diverges by Part (a).

53. Let $x = r\tan\theta$ to get $\displaystyle\int\frac{dx}{(r^2+x^2)^{3/2}} = \frac{1}{r^2}\int\cos\theta\,d\theta = \frac{1}{r^2}\sin\theta + C = \frac{x}{r^2\sqrt{r^2+x^2}} + C$

so $\displaystyle u = \frac{2\pi NIr}{k}\lim_{\ell\to+\infty}\frac{x}{r^2\sqrt{r^2+x^2}}\Big]_a^{\ell} = \frac{2\pi NI}{kr}\lim_{\ell\to+\infty}(\ell/\sqrt{r^2+\ell^2} - a/\sqrt{r^2+a^2})$

$\displaystyle = \frac{2\pi NI}{kr}(1 - a/\sqrt{r^2+a^2})$.

54. Let $a^2 = \dfrac{M}{2RT}$ to get

(a) $\displaystyle\bar{v} = \frac{4}{\sqrt{\pi}}\left(\frac{M}{2RT}\right)^{3/2}\frac{1}{2}\left(\frac{M}{2RT}\right)^{-2} = \frac{2}{\sqrt{\pi}}\sqrt{\frac{2RT}{M}} = \sqrt{\frac{8RT}{\pi M}}$

(b) $\displaystyle v_{\text{rms}}^2 = \frac{4}{\sqrt{\pi}}\left(\frac{M}{2RT}\right)^{3/2}\frac{3\sqrt{\pi}}{8}\left(\frac{M}{2RT}\right)^{-5/2} = \frac{3RT}{M}$ so $\displaystyle v_{\text{rms}} = \sqrt{\frac{3RT}{M}}$

55. **(a)** Satellite's weight $= w(x) = k/x^2$ lb when x = distance from center of Earth; $w(4000) = 6000$ so $k = 9.6\times10^{10}$ and $\displaystyle W = \int_{4000}^{4000+\ell} 9.6\times10^{10}x^{-2}dx$ mi·lb.

(b) $\displaystyle\int_{4000}^{+\infty} 9.6\times10^{10}x^{-2}dx = \lim_{\ell\to+\infty} -9.6\times10^{10}/x\Big]_{4000}^{\ell} = 2.4\times10^7$ mi·lb

56. **(a)** $\displaystyle\mathcal{L}\{1\} = \int_0^{+\infty}e^{-st}dt = \lim_{\ell\to+\infty} -\frac{1}{s}e^{-st}\Big]_0^{\ell} = \frac{1}{s}$

(b) $\displaystyle\mathcal{L}\{e^{2t}\} = \int_0^{+\infty}e^{-st}e^{2t}dt = \int_0^{+\infty}e^{-(s-2)t}dt = \lim_{\ell\to+\infty} -\frac{1}{s-2}e^{-(s-2)t}\Big]_0^{\ell} = \frac{1}{s-2}$

(c) $\displaystyle\mathcal{L}\{\sin t\} = \int_0^{+\infty}e^{-st}\sin t\,dt = \lim_{\ell\to+\infty}\frac{e^{-st}}{s^2+1}(-s\sin t - \cos t)\Big]_0^{\ell} = \frac{1}{s^2+1}$

(d) $\displaystyle\mathcal{L}\{\cos t\} = \int_0^{+\infty}e^{-st}\cos t\,dt = \lim_{\ell\to+\infty}\frac{e^{-st}}{s^2+1}(-s\cos t + \sin t)\Big]_0^{\ell} = \frac{s}{s^2+1}$

57. **(a)** $\displaystyle\mathcal{L}\{f(t)\} = \int_0^{+\infty}te^{-st}\,dt = \lim_{\ell\to+\infty} -(t/s + 1/s^2)e^{-st}\Big]_0^{\ell} = \frac{1}{s^2}$

(b) $\displaystyle\mathcal{L}\{f(t)\} = \int_0^{+\infty}t^2e^{-st}\,dt = \lim_{\ell\to+\infty} -(t^2/s + 2t/s^2 + 2/s^3)e^{-st}\Big]_0^{\ell} = \frac{2}{s^3}$

(c) $\displaystyle\mathcal{L}\{f(t)\} = \int_3^{+\infty}e^{-st}dt = \lim_{\ell\to+\infty} -\frac{1}{s}e^{-st}\Big]_3^{\ell} = \frac{e^{-3s}}{s}$

58.

10	100	1000	10,000
0.8862269	0.8862269	0.8862269	0.8862269

59. **(a)** $u = \sqrt{a}x, du = \sqrt{a}\,dx, 2\int_0^{+\infty} e^{-ax^2}\,dx = \frac{2}{\sqrt{a}}\int_0^{+\infty} e^{-u^2}\,du = \sqrt{\pi/a}$

(b) $x = \sqrt{2}\sigma u, dx = \sqrt{2}\sigma\,du, \frac{2}{\sqrt{2\pi}\sigma}\int_0^{+\infty} e^{-x^2/2\sigma^2}\,dx = \frac{2}{\sqrt{\pi}}\int_0^{+\infty} e^{-u^2}\,du = 1$

60. **(a)** $\int_0^3 e^{-x^2}dx \approx 0.8862;\ \sqrt{\pi}/2 \approx 0.8862$

(b) $\int_0^{+\infty} e^{-x^2}dx = \int_0^3 e^{-x^2}dx + \int_3^{+\infty} e^{-x^2}dx$ so $E = \int_3^{+\infty} e^{-x^2}dx < \int_3^{+\infty} xe^{-x^2}dx = \frac{1}{2}e^{-9} < 7\times 10^{-5}$

61. **(a)** $\int_0^4 \frac{1}{x^6+1}dx \approx 1.047;\ \pi/3 \approx 1.047$

(b) $\int_0^{+\infty} \frac{1}{x^6+1}dx = \int_0^4 \frac{1}{x^6+1}dx + \int_4^{+\infty} \frac{1}{x^6+1}dx$ so

$E = \int_4^{+\infty} \frac{1}{x^6+1}dx < \int_4^{+\infty} \frac{1}{x^6}dx = \frac{1}{5(4)^5} < 2\times 10^{-4}$

62. If $p = 0$, then $\int_0^{+\infty}(1)dx = \lim_{\ell\to+\infty} x\Big]_0^\ell = +\infty$,

if $p \neq 0$, then $\int_0^{+\infty} e^{px}dx = \lim_{\ell\to+\infty}\frac{1}{p}e^{px}\Big]_0^\ell = \lim_{\ell\to+\infty}\frac{1}{p}(e^{p\ell}-1) = \begin{cases} -1/p, & p<0 \\ +\infty, & p>0 \end{cases}$.

63. If $p = 1$, then $\int_0^1 \frac{dx}{x} = \lim_{\ell\to 0^+}\ln x\Big]_\ell^1 = +\infty$;

if $p \neq 1$, then $\int_0^1 \frac{dx}{x^p} = \lim_{\ell\to 0^+}\frac{x^{1-p}}{1-p}\Big]_\ell^1 = \lim_{\ell\to 0^+}[(1-\ell^{1-p})/(1-p)] = \begin{cases} 1/(1-p), & p<1 \\ +\infty, & p>1 \end{cases}$.

64. $u = \sqrt{1-x}, u^2 = 1-x, 2u\,du = -dx$;

$-2\int_1^0 \sqrt{2-u^2}\,du = 2\int_0^1 \sqrt{2-u^2}\,du = \left[u\sqrt{2-u^2} + 2\sin^{-1}(u/\sqrt{2})\right]_0^1 = \sqrt{2} + \pi/2$

65. $2\int_0^1 \cos(u^2)du \approx 1.809$

66. $-2\int_1^0 \sin(1-u^2)du = 2\int_0^1 \sin(1-u^2)du \approx 1.187$

CHAPTER 8 SUPPLEMENTARY EXERCISES

1. **(a)** integration by parts, $u = x,\ dv = \sin x\,dx$
(b) u-substitution: $u = \sin x$
(c) reduction formula
(d) u-substitution: $u = \tan x$
(e) u-substitution: $u = x^3 + 1$
(f) u-substitution: $u = x + 1$
(g) integration by parts: $dv = dx, u = \tan^{-1} x$
(h) trigonometric substitution: $x = 2\sin\theta$
(i) u-substitution: $u = 4 - x^2$

2. **(a)** $x = 3\tan\theta$ **(b)** $x = 3\sin\theta$ **(c)** $x = \frac{1}{3}\sin\theta$
(d) $x = 3\sec\theta$ **(e)** $x = \sqrt{3}\tan\theta$ **(f)** $x = \frac{1}{9}\tan\theta$

5. **(a)** #40 **(b)** #57 **(c)** #113
(d) #108 **(e)** #52 **(f)** #71

6. **(a)** $u = x^2,\ dv = \dfrac{x}{\sqrt{x^2+1}}dx,\ du = 2x\,dx,\ v = \sqrt{x^2+1};$

$$\int_0^1 \frac{x^3}{\sqrt{x^2+1}}dx = x^2\sqrt{x^2+1}\Big]_0^1 - 2\int_0^1 x(x^2+1)^{1/2}dx$$

$$= \sqrt{2} - \frac{2}{3}(x^2+1)^{3/2}\Big]_0^1 = \sqrt{2} - \frac{2}{3}[2\sqrt{2}-1] = (2-\sqrt{2})/3$$

(b) $u^2 = x^2+1,\ x^2 = u^2-1,\ 2x\,dx = 2u\,du,\ x\,dx = u\,du;$

$$\int_0^1 \frac{x^3}{\sqrt{x^2+1}}dx = \int_0^1 \frac{x^2}{\sqrt{x^2+1}}x\,dx = \int_1^{\sqrt{2}} \frac{u^2-1}{u}u\,du$$

$$= \int_1^{\sqrt{2}}(u^2-1)du = \left(\frac{1}{3}u^3 - u\right)\Big]_1^{\sqrt{2}} = (2-\sqrt{2})/3$$

7. **(a)** $u = 2x,$

$$\int \sin^4 2x\,dx = \frac{1}{2}\int \sin^4 u\,du = \frac{1}{2}\left[-\frac{1}{4}\sin^3 u\cos u + \frac{3}{4}\int \sin^2 u\,du\right]$$

$$= -\frac{1}{8}\sin^3 u\cos u + \frac{3}{8}\left[-\frac{1}{2}\sin u\cos u + \frac{1}{2}\int du\right]$$

$$= -\frac{1}{8}\sin^3 u\cos u - \frac{3}{16}\sin u\cos u + \frac{3}{16}u + C$$

$$= -\frac{1}{8}\sin^3 2x\cos 2x - \frac{3}{16}\sin 2x\cos 2x + \frac{3}{8}x + C$$

(b) $u = x^2,$

$$\int x\cos^5(x^2)dx = \frac{1}{2}\int \cos^5 u\,du = \frac{1}{2}\int (\cos u)(1-\sin^2 u)^2\,du$$

$$= \frac{1}{2}\int \cos u\,du - \int \cos u\sin^2 u\,du + \frac{1}{2}\int \cos u\sin^4 u\,du$$

$$= \frac{1}{2}\sin u - \frac{1}{3}\sin^3 u + \frac{1}{10}\sin^5 u + C$$

$$= \frac{1}{2}\sin(x^2) - \frac{1}{3}\sin^3(x^2) + \frac{1}{10}\sin^5(x^2) + C$$

8. **(a)** With $x = \sec\theta$:

$$\int \frac{1}{x^3-x}dx = \int \cot\theta\,d\theta = \ln|\sin\theta| + C = \ln\frac{\sqrt{x^2-1}}{|x|} + C;\ \text{valid for } |x| > 1.$$

(b) With $x = \sin\theta$:

$$\int \frac{1}{x^3-x}dx = -\int \frac{1}{\sin\theta\cos\theta}d\theta = -\int 2\csc 2\theta\,d\theta$$

$$= -\ln|\csc 2\theta - \cot 2\theta| + C = \ln|\cot\theta| + C = \ln\frac{\sqrt{1-x^2}}{|x|} + C,\quad 0 < |x| < 1.$$

(c) By partial fractions:

$$\int\left(-\frac{1}{x}+\frac{1/2}{x+1}+\frac{1/2}{x-1}\right)dx = -\ln|x|+\frac{1}{2}\ln|x+1|+\frac{1}{2}\ln|x-1|+C$$

$$= \ln\frac{\sqrt{|x^2-1|}}{|x|}+C;\ \text{valid for all } x \text{ except } x=0,\pm 1.$$

9. **(a)** With $u=\sqrt{x}$:

$$\int\frac{1}{\sqrt{x}\,\sqrt{2-x}}\,dx = 2\int\frac{1}{\sqrt{2-u^2}}\,du = 2\sin^{-1}(u/\sqrt{2})+C = 2\sin^{-1}(\sqrt{x/2})+C;$$

with $u=\sqrt{2-x}$:

$$\int\frac{1}{\sqrt{x}\,\sqrt{2-x}}\,dx = -2\int\frac{1}{\sqrt{2-u^2}}\,du = -2\sin^{-1}(u/\sqrt{2})+C = -2\sin^{-1}(\sqrt{2-x}/\sqrt{2})+C;$$

completing the square:

$$\int\frac{1}{\sqrt{1-(x-1)^2}}\,dx = \sin^{-1}(x-1)+C.$$

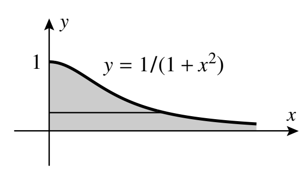
(b) In the three results in Part (a) the antiderivatives differ by a constant, in particular $2\sin^{-1}(\sqrt{x/2}) = \pi - 2\sin^{-1}(\sqrt{2-x}/\sqrt{2}) = \pi/2+\sin^{-1}(x-1)$.

10. $A=\displaystyle\int_1^2\frac{3-x}{x^3+x^2}dx,\ \frac{3-x}{x^2(x+1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1};\ A=-4,\ B=3,\ C=4$

$$A=\left[-4\ln|x|-\frac{3}{x}+4\ln|x+1|\right]_1^2$$

$$=(-4\ln 2-\frac{3}{2}+4\ln 3)-(-4\ln 1-3+4\ln 2) = \frac{3}{2}-8\ln 2+4\ln 3 = \frac{3}{2}+4\ln\frac{3}{4}$$

11. Solve $y=1/(1+x^2)$ for x to get $x=\sqrt{\dfrac{1-y}{y}}$ and integrate with respect to y to get $A=\displaystyle\int_0^1\sqrt{\frac{1-y}{y}}\,dy$ (see figure)



12. $A=\displaystyle\int_e^{+\infty}\frac{\ln x-1}{x^2}\,dx = \lim_{\ell\to+\infty} -\frac{\ln x}{x}\Bigg]_e^{\ell} = 1/e$

13. $V=2\pi\displaystyle\int_0^{+\infty}xe^{-x}dx = 2\pi\lim_{\ell\to+\infty} -e^{-x}(x+1)\Bigg]_0^{\ell} = 2\pi\lim_{\ell\to+\infty}\left[1-e^{-\ell}(\ell+1)\right]$

but $\displaystyle\lim_{\ell\to+\infty}e^{-\ell}(\ell+1) = \lim_{\ell\to+\infty}\frac{\ell+1}{e^{\ell}} = \lim_{\ell\to+\infty}\frac{1}{e^{\ell}} = 0$ so $V=2\pi$

14. $\displaystyle\int_0^{+\infty}\frac{dx}{x^2+a^2} = \lim_{\ell\to+\infty}\frac{1}{a}\tan^{-1}(x/a)\Bigg]_0^{\ell} = \lim_{\ell\to+\infty}\frac{1}{a}\tan^{-1}(\ell/a) = \frac{\pi}{2a} = 1, a=\pi/2$

15. $u=\cos\theta,\ -\displaystyle\int u^{1/2}du = -\frac{2}{3}\cos^{3/2}\theta+C$

16. Use Endpaper Formula (31) to get $\displaystyle\int\tan^7\theta d\theta = \frac{1}{6}\tan^6\theta-\frac{1}{4}\tan^4\theta+\frac{1}{2}\tan^2\theta+\ln|\cos\theta|+C.$

17. $u = \tan(x^2)$, $\dfrac{1}{2}\displaystyle\int u^2 du = \dfrac{1}{6}\tan^3(x^2) + C$

18. $x = (1/\sqrt{2})\sin\theta$, $dx = (1/\sqrt{2})\cos\theta\, d\theta$,

$$\frac{1}{\sqrt{2}}\int_{-\pi/2}^{\pi/2}\cos^4\theta\, d\theta = \frac{1}{\sqrt{2}}\left\{\frac{1}{4}\cos^3\theta\sin\theta\Big]_{-\pi/2}^{\pi/2} + \frac{3}{4}\int_{-\pi/2}^{\pi/2}\cos^2\theta\, d\theta\right\}$$

$$= \frac{3}{4\sqrt{2}}\left\{\frac{1}{2}\cos\theta\sin\theta\Big]_{-\pi/2}^{\pi/2} + \frac{1}{2}\int_{-\pi/2}^{\pi/2} d\theta\right\} = \frac{3}{4\sqrt{2}}\frac{1}{2}\pi = \frac{3\pi}{8\sqrt{2}}$$

19. $x = \sqrt{3}\tan\theta$, $dx = \sqrt{3}\sec^2\theta\, d\theta$,

$$\frac{1}{3}\int\frac{1}{\sec\theta}d\theta = \frac{1}{3}\int\cos\theta\, d\theta = \frac{1}{3}\sin\theta + C = \frac{x}{3\sqrt{3+x^2}} + C$$

20. $\displaystyle\int\frac{\cos\theta}{(\sin\theta - 3)^2 + 3}d\theta$, let $u = \sin\theta - 3$, $\displaystyle\int\frac{1}{u^2+3}du = \frac{1}{\sqrt{3}}\tan^{-1}[(\sin\theta - 3)/\sqrt{3}] + C$

21. $\displaystyle\int\frac{x+3}{\sqrt{(x+1)^2+1}}dx$, let $u = x + 1$,

$$\int\frac{u+2}{\sqrt{u^2+1}}du = \int\left[u(u^2+1)^{-1/2} + \frac{2}{\sqrt{u^2+1}}\right]du = \sqrt{u^2+1} + 2\sinh^{-1}u + C$$

$$= \sqrt{x^2+2x+2} + 2\sinh^{-1}(x+1) + C$$

<u>Alternate solution</u>: let $x + 1 = \tan\theta$,

$$\int(\tan\theta + 2)\sec\theta\, d\theta = \int\sec\theta\tan\theta\, d\theta + 2\int\sec\theta\, d\theta = \sec\theta + 2\ln|\sec\theta + \tan\theta| + C$$

$$= \sqrt{x^2+2x+2} + 2\ln(\sqrt{x^2+2x+2} + x + 1) + C.$$

22. Let $x = \tan\theta$ to get $\displaystyle\int\frac{1}{x^3 - x^2}dx$.

$\dfrac{1}{x^2(x-1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-1}$; $A = -1$, $B = -1$, $C = 1$ so

$$-\int\frac{1}{x}dx - \int\frac{1}{x^2}dx + \int\frac{1}{x-1}dx = -\ln|x| + \frac{1}{x} + \ln|x-1| + C$$

$$= \frac{1}{x} + \ln\left|\frac{x-1}{x}\right| + C = \cot\theta + \ln\left|\frac{\tan\theta - 1}{\tan\theta}\right| + C = \cot\theta + \ln|1 - \cot\theta| + C$$

23. $\dfrac{1}{(x-1)(x+2)(x-3)} = \dfrac{A}{x-1} + \dfrac{B}{x+2} + \dfrac{C}{x-3}$; $A = -\dfrac{1}{6}$, $B = \dfrac{1}{15}$, $C = \dfrac{1}{10}$ so

$$-\frac{1}{6}\int\frac{1}{x-1}dx + \frac{1}{15}\int\frac{1}{x+2}dx + \frac{1}{10}\int\frac{1}{x-3}dx$$

$$= -\frac{1}{6}\ln|x-1| + \frac{1}{15}\ln|x+2| + \frac{1}{10}\ln|x-3| + C$$

24. $\dfrac{1}{x(x^2+x+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+x+1}$; $A=1$, $B=C=-1$ so

$$\int \frac{-x-1}{x^2+x+1}dx = -\int \frac{x+1}{(x+1/2)^2+3/4}dx = -\int \frac{u+1/2}{u^2+3/4}du, \quad u = x+1/2$$

$$= -\frac{1}{2}\ln(u^2+3/4) - \frac{1}{\sqrt{3}}\tan^{-1}(2u/\sqrt{3}) + C_1$$

so $\displaystyle\int \frac{dx}{x(x^2+x+1)} = \ln|x| - \frac{1}{2}\ln(x^2+x+1) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + C$

25. $u = \sqrt{x-4}$, $x = u^2+4$, $dx = 2u\,du$,

$$\int_0^2 \frac{2u^2}{u^2+4}du = 2\int_0^2 \left[1 - \frac{4}{u^2+4}\right] du = \left[2u - 4\tan^{-1}(u/2)\right]_0^2 = 4-\pi$$

26. $u = \sqrt{x}$, $x = u^2$, $dx = 2u\,du$,

$$2\int_0^3 \frac{u^2}{u^2+9}du = 2\int_0^3 \left(1 - \frac{9}{u^2+9}\right) du = \left(2u - 6\tan^{-1}\frac{u}{3}\right)\Big]_0^3 = 6 - \frac{3}{2}\pi$$

27. $u = \sqrt{e^x+1}$, $e^x = u^2-1$, $x = \ln(u^2-1)$, $dx = \dfrac{2u}{u^2-1}du$,

$$\int \frac{2}{u^2-1}du = \int \left[\frac{1}{u-1} - \frac{1}{u+1}\right] du = \ln|u-1| - \ln|u+1| + C = \ln\frac{\sqrt{e^x+1}-1}{\sqrt{e^x+1}+1} + C$$

28. $u = \sqrt{e^x-1}$, $e^x = u^2+1$, $x = \ln(u^2+1)$, $dx = \dfrac{2u}{u^2+1}du$,

$$\int_0^1 \frac{2u^2}{u^2+1}du = 2\int_0^1 \left(1 - \frac{1}{u^2+1}\right) du = (2u - 2\tan^{-1}u)\Big]_0^1 = 2 - \frac{\pi}{2}$$

29. $\displaystyle\lim_{\ell\to+\infty} -\frac{1}{2(x^2+1)}\Big]_a^\ell = \lim_{\ell\to+\infty}\left[-\frac{1}{2(\ell^2+1)} + \frac{1}{2(a^2+1)}\right] = \frac{1}{2(a^2+1)}$

30. $\displaystyle\lim_{\ell\to+\infty} \frac{1}{ab}\tan^{-1}\frac{bx}{a}\Big]_0^\ell = \lim_{\ell\to+\infty}\frac{1}{ab}\tan^{-1}\frac{b\ell}{a} = \frac{\pi}{2ab}$

31. Let $u = x^4$ to get $\displaystyle\frac{1}{4}\int \frac{1}{\sqrt{1-u^2}}du = \frac{1}{4}\sin^{-1}u + C = \frac{1}{4}\sin^{-1}(x^4) + C.$

32. $$\int (\cos^{32}x\sin^{30}x - \cos^{30}x\sin^{32}x)dx = \int \cos^{30}x\sin^{30}x(\cos^2x - \sin^2x)dx$$

$$= \frac{1}{2^{30}}\int \sin^{30}2x\cos 2x\,dx = \frac{\sin^{31}2x}{31(2^{31})} + C$$

33. $\displaystyle\int \sqrt{x - \sqrt{x^2-4}}dx = \frac{1}{\sqrt{2}}\int(\sqrt{x+2} - \sqrt{x-2})dx = \frac{\sqrt{2}}{3}[(x+2)^{3/2} - (x-2)^{3/2}] + C$

34. $\displaystyle\int \frac{1}{x^{10}(1+x^{-9})}dx = -\frac{1}{9}\int \frac{1}{u}du = -\frac{1}{9}\ln|u| + C = -\frac{1}{9}\ln|1+x^{-9}| + C$

35. **(a)** $(x+4)(x-5)(x^2+1)^2$; $\dfrac{A}{x+4}+\dfrac{B}{x-5}+\dfrac{Cx+D}{x^2+1}+\dfrac{Ex+F}{(x^2+1)^2}$

(b) $-\dfrac{3}{x+4}+\dfrac{2}{x-5}-\dfrac{x-2}{x^2+1}-\dfrac{3}{(x^2+1)^2}$

(c) $-3\ln|x+4|+2\ln|x-5|+2\tan^{-1}x-\dfrac{1}{2}\ln(x^2+1)-\dfrac{3}{2}\left(\dfrac{x}{x^2+1}+\tan^{-1}x\right)+C$

36. **(a)** $\Gamma(1)=\displaystyle\int_0^{+\infty}e^{-t}dt=\lim_{\ell\to+\infty}-e^{-t}\Big]_0^{\ell}=\lim_{\ell\to+\infty}(-e^{-\ell}+1)=1$

(b) $\Gamma(x+1)=\displaystyle\int_0^{+\infty}t^x e^{-t}dt$; let $u=t^x$, $dv=e^{-t}dt$ to get

$$\Gamma(x+1)=-t^xe^{-t}\Big]_0^{+\infty}+x\int_0^{+\infty}t^{x-1}e^{-t}dt=-t^xe^{-t}\Big]_0^{+\infty}+x\Gamma(x)$$

$\displaystyle\lim_{t\to+\infty}t^xe^{-t}=\lim_{t\to+\infty}\frac{t^x}{e^t}=0$ (by multiple applications of L'Hôpital's rule)

so $\Gamma(x+1)=x\Gamma(x)$

(c) $\Gamma(2)=(1)\Gamma(1)=(1)(1)=1$, $\Gamma(3)=2\Gamma(2)=(2)(1)=2$, $\Gamma(4)=3\Gamma(3)=(3)(2)=6$

It appears that $\Gamma(n)=(n-1)!$ if n is a positive integer.

(d) $\Gamma\left(\dfrac{1}{2}\right)=\displaystyle\int_0^{+\infty}t^{-1/2}e^{-t}dt=2\int_0^{+\infty}e^{-u^2}du$ (with $u=\sqrt{t}$) $=2(\sqrt{\pi}/2)=\sqrt{\pi}$

(e) $\Gamma\left(\dfrac{3}{2}\right)=\dfrac{1}{2}\Gamma\left(\dfrac{1}{2}\right)=\dfrac{1}{2}\sqrt{\pi}$, $\Gamma\left(\dfrac{5}{2}\right)=\dfrac{3}{2}\Gamma\left(\dfrac{3}{2}\right)=\dfrac{3}{4}\sqrt{\pi}$

37. **(a)** $t=-\ln x, x=e^{-t}, dx=-e^{-t}dt$,

$$\int_0^1(\ln x)^n dx=-\int_{+\infty}^0(-t)^ne^{-t}dt=(-1)^n\int_0^{+\infty}t^ne^{-t}dt=(-1)^n\Gamma(n+1)$$

(b) $t=x^n, x=t^{1/n}, dx=(1/n)t^{1/n-1}dt$,

$$\int_0^{+\infty}e^{-x^n}dx=(1/n)\int_0^{+\infty}t^{1/n-1}e^{-t}dt=(1/n)\Gamma(1/n)=\Gamma(1/n+1)$$

38. **(a)** $\sqrt{\cos\theta-\cos\theta_0}=\sqrt{2\left[\sin^2(\theta_0/2)-\sin^2(\theta/2)\right]}=\sqrt{2(k^2-k^2\sin^2\phi)}=\sqrt{2k^2\cos^2\phi}$

$=\sqrt{2}\,k\,\cos\phi$; $k\sin\phi=\sin(\theta/2)$ so $k\cos\phi\,d\phi=\dfrac{1}{2}\cos(\theta/2)\,d\theta=\dfrac{1}{2}\sqrt{1-\sin^2(\theta/2)}\,d\theta$

$=\dfrac{1}{2}\sqrt{1-k^2\sin^2\phi}\,d\theta$, thus $d\theta=\dfrac{2k\cos\phi}{\sqrt{1-k^2\sin^2\phi}}\,d\phi$ and hence

$$T=\sqrt{\frac{8L}{g}}\int_0^{\pi/2}\frac{1}{\sqrt{2}k\cos\phi}\cdot\frac{2k\cos\phi}{\sqrt{1-k^2\sin^2\phi}}\,d\phi=4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{1}{\sqrt{1-k^2\sin^2\phi}}\,d\phi$$

(b) If $L=1.5$ ft and $\theta_0=(\pi/180)(20)=\pi/9$, then

$$T=\frac{\sqrt{3}}{2}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-\sin^2(\pi/18)\sin^2\phi}}\approx 1.37\text{ s.}$$

CHAPTER 8 HORIZON MODULE

1. The depth of the cut equals the terrain elevation minus the track elevation. From Figure 2, the cross sectional area of a cut of depth D meters is $10D + 2 \cdot \frac{1}{2}D^2 = D^2 + 10D$ square meters.

Distance from town A (m)	Terrain elevation (m)	Track elevation (m)	Depth of cut (m)	Cross-sectional area $f(x)$ of cut (m^2)
0	100	100	0	0
2000	105	101	4	56
4000	108	102	6	96
6000	110	103	7	119
8000	104	104	0	0
10,000	106	105	1	11
12,000	120	106	14	336
14,000	122	107	15	375
16,000	124	108	16	416
18,000	128	109	19	551
20,000	130	110	20	600

The total volume of dirt to be excavated, in cubic meters, is $\int_0^{2000} f(x)\,dx$.

By Simpson's Rule, this is approximately

$$\frac{20{,}000 - 0}{3 \cdot 10}[0 + 4 \cdot 56 + 2 \cdot 96 + 4 \cdot 119 + 2 \cdot 0 + 4 \cdot 11 + 2 \cdot 336 + 4 \cdot 375 + 2 \cdot 416 + 4 \cdot 551 + 600]$$
$$= 4{,}496{,}000 \text{ m}^3.$$

Excavation costs \$4 per m^3, so the total cost of the railroad from kA to M is about $4 \cdot 4{,}496{,}000 = 17{,}984{,}000$ dollars.

2. **(a)**

Distance from town A (m)	Terrain elevation (m)	Track elevation (m)	Depth of cut (m)	Cross-sectional area $f(x)$ of cut (m^2)
20,000	130	110	20	300
20,100	135	109.8	25.2	887.04
20,200	139	109.6	29.4	1158.36
20,300	142	109.4	32.6	1388.76
20,400	145	109.2	35.8	1639.64
20,500	147	109	38	1824
20,600	148	108.8	39.2	1928.64
20,700	146	108.6	37.4	1772.76
20,800	143	108.4	34.6	1543.16
20,900	139	108.2	30.8	1256.64
21,000	133	108	25	875

The total volume of dirt to be excavated, in cubic meters, is $\int_{20{,}000}^{21{,}000} f(x)\,dx$.

By Simpson's Rule this is approximately

$$\frac{21{,}000 - 20{,}000}{3 \cdot 10}[600 + 4 \cdot 887.04 + 2 \cdot 1158.36 + \ldots + 4 \cdot 1256.64 + 875] = 1{,}417{,}713.33 \text{ m}^3.$$

The total cost of a trench from M to N is about $4 \cdot 1{,}417{,}713.33 \approx 5{,}670{,}853$ dollars.

(b)

Distance from town A (m)	Terrain elevation (m)	Track elevation (m)	Depth of cut (m)	Cross-sectional area $f(x)$ of cut (m^2)
21,000	133	108	25	875
22,000	120	106	14	336
23,000	106	104	2	24
24,000	108	102	6	96
25,000	106	100	6	96
26,000	98	98	0	0
27,000	100	96	4	56
28,000	102	94	8	144
29,000	96	92	4	56
30,000	91	90	1	11
31,000	88	88	0	0

The total volume of dirt to be excavated, in cubic meters, is $\int_{21{,}000}^{31{,}000} f(x)\,dx$. By Simpson's Rule this is approximately

$$\frac{31{,}000 - 21{,}000}{3 \cdot 10}[875 + 4 \cdot 336 + 2 \cdot 24 + \ldots + 4 \cdot 11 + 0] = 1{,}229{,}000 \text{ m}^3.$$

The total cost of the railroad from N to B is about $4 \cdot 1{,}229{,}000 \approx 4{,}916{,}000$ dollars.

3. The total cost if trenches are used everywhere is about $17{,}984{,}000 + 5{,}670{,}853 + 4{,}916{,}000 = 28{,}570{,}853$ dollars.

4. **(a)** The cross-sectional area of a tunnel is $A_T = 80 + \frac{1}{2}\pi 5^2 \approx 119.27 \text{ m}^2$. The length of the tunnel is 1000 m, so the volume of dirt to be removed is about $1000A_T \approx 1{,}119{,}269.91 \text{ m}^3$, and the drilling and dirt-piling costs are $8 \cdot 1000A_T \approx 954{,}159$ dollars.

(b) To extend the tunnel from a length of x meters to a length of $x + dx$ meters, we must move a volume of $A_T\,dx$ cubic meters of dirt a distance of about x meters. So the cost of this extension is about $0.06 \times A_T\,dx$ dollars. The cost of moving all of the dirt in the tunnel is therefore

$$\int_0^{1000} 0.06 \times A_T\,dx = 0.06A_T \frac{x^2}{2}\Big]_0^{1000} = 30{,}000A_T \approx 3{,}578{,}097 \text{ dollars.}$$

(c) The total cost of the tunnel is about $954{,}159 + 3{,}578{,}097 \approx 4{,}532{,}257$ dollars.

5. The total cost of the railroad, using a tunnel, is $17{,}894{,}000 + 4{,}532{,}257 + 4{,}916{,}000 + 27{,}432{,}257$ dollars, which is smaller than the cost found in Exercise 3. It will be cheaper to build the railroad if a tunnel is used.