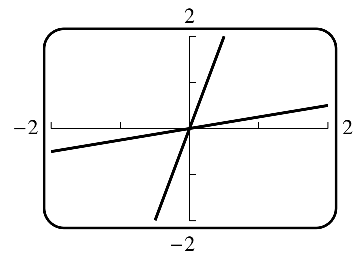
CHAPTER 7

Exponential, Logarithmic, and Inverse Trigonometric Functions

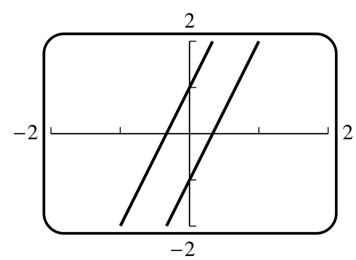
EXERCISE SET 7.1

1. **(a)** $f(g(x)) = 4(x/4) = x$, $g(f(x)) = (4x)/4 = x$, f and g are inverse functions
(b) $f(g(x)) = 3(3x - 1) + 1 = 9x - 2 \neq x$ so f and g are not inverse functions
(c) $f(g(x)) = \sqrt[3]{(x^3 + 2) - 2} = x$, $g(f(x)) = (x - 2) + 2 = x$, f and g are inverse functions
(d) $f(g(x)) = (x^{1/4})^4 = x$, $g(f(x)) = (x^4)^{1/4} = |x| \neq x$, f and g are not inverse functions

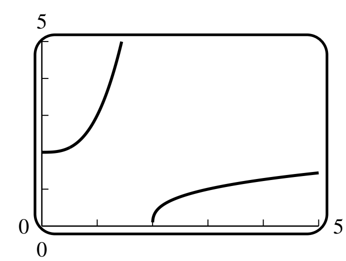
2. **(a)** They are inverse functions.



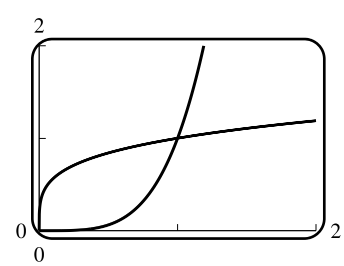
(b) The graphs are not reflections of each other about the line $y = x$.



(c) They are inverse functions provided the domain of g is restricted to $[0, +\infty)$



(d) They are inverse functions provided the domain of $f(x)$ is restricted to $[0, +\infty)$

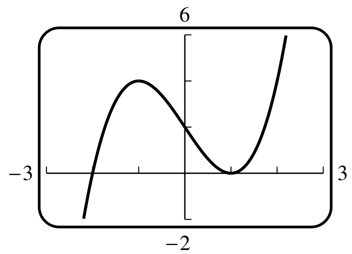


3. **(a)** yes; all outputs (the elements of row two) are distinct
(b) no; $f(1) = f(6)$

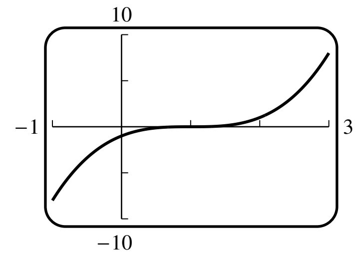
4. **(a)** no; it is easy to conceive of, say, 8 people in line at two different times
(b) no; perhaps your weight remains constant for more than a year
(c) yes, since the function is increasing, in the sense that the greater the volume, the greater the weight

5. **(a)** yes **(b)** yes **(c)** no **(d)** yes **(e)** no **(f)** no

6. **(a)** no, the horizontal line test fails



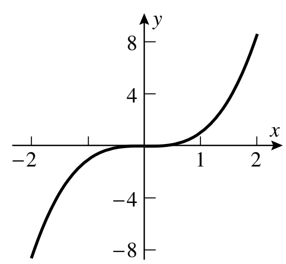
(b) yes, horizontal line test



7. **(a)** no, the horizontal line test fails
(b) no, the horizontal line test fails
(c) yes, horizontal line test

8. **(d)** no, the horizontal line test fails
(e) no, the horizontal line test fails
(f) yes, horizontal line test

9. **(a)** f has an inverse because the graph passes the horizontal line test. To compute $f^{-1}(2)$ start at 2 on the y-axis and go to the curve and then down, so $f^{-1}(2) = 8$; similarly, $f^{-1}(-1) = -1$ and $f^{-1}(0) = 0$.
(b) domain of f^{-1} is $[-2, 2]$, range is $[-8, 8]$
(c)



10. **(a)** the horizontal line test fails
(b) $-\infty < x \leq -1$; $-1 \leq x \leq 2$; and $2 \leq x < 4$.

11. **(a)** $f'(x) = 2x + 8$; $f' < 0$ on $(-\infty, -4)$ and $f' > 0$ on $(-4, +\infty)$; not one-to-one
(b) $f'(x) = 10x^4 + 3x^2 + 3 \geq 3 > 0$; $f'(x)$ is positive for all x, so f is one-to-one
(c) $f'(x) = 2 + \cos x \geq 1 > 0$ for all x, so f is one-to-one

12. **(a)** $f'(x) = 3x^2 + 6x = x(3x + 6)$ changes sign at $x = -2, 0$, so f is not one-to-one
(b) $f'(x) = 5x^4 + 24x^2 + 2 \geq 2 > 0$; f' is positive for all x, so f is one-to-one
(c) $f'(x) = \dfrac{1}{(x+1)^2}$; f is one-to-one because:
if $x_1 < x_2 < -1$ then $f' > 0$ on $[x_1, x_2]$, so $f(x_1) \neq f(x_2)$
if $-1 < x_1 < x_2$ then $f' > 0$ on $[x_1, x_2]$, so $f(x_1) \neq f(x_2)$
if $x_1 < -1 < x_2$ then $f(x_1) > 1 > f(x_2)$ since $f(x) > 1$ on $(-\infty, -1)$ and $f(x) < 1$ on $(-1, +\infty)$

13. $y = f^{-1}(x)$, $x = f(y) = y^5$, $y = x^{1/5} = f^{-1}(x)$

14. $y=f^{-1}(x)$, $x=f(y)=6y$, $y=\dfrac{1}{6}x=f^{-1}(x)$

15. $y=f^{-1}(x)$, $x=f(y)=7y-6$, $y=\dfrac{1}{7}(x+6)=f^{-1}(x)$

16. $y=f^{-1}(x)$, $x=f(y)=\dfrac{y+1}{y-1}$, $xy-x=y+1$, $(x-1)y=x+1$, $y=\dfrac{x+1}{x-1}=f^{-1}(x)$

17. $y=f^{-1}(x)$, $x=f(y)=3y^3-5$, $y=\sqrt[3]{(x+5)/3}=f^{-1}(x)$

18. $y=f^{-1}(x)$, $x=f(y)=\sqrt[5]{4y+2}$, $y=\dfrac{1}{4}(x^5-2)=f^{-1}(x)$

19. $y=f^{-1}(x)$, $x=f(y)=\sqrt[3]{2y-1}$, $y=(x^3+1)/2=f^{-1}(x)$

20. $y=f^{-1}(x)$, $x=f(y)=\dfrac{5}{y^2+1}$, $y=\sqrt{\dfrac{5-x}{x}}=f^{-1}(x)$

21. $y=f^{-1}(x)$, $x=f(y)=3/y^2$, $y=-\sqrt{3/x}=f^{-1}(x)$

22. $y=f^{-1}(x)$, $x=f(y)=\begin{cases} 2y, & y\le 0 \\ y^2, & y>0 \end{cases}$, $\quad y=f^{-1}(x)=\begin{cases} x/2, & x\le 0 \\ \sqrt{x}, & x>0 \end{cases}$

23. $y=f^{-1}(x), x=f(y)=\begin{cases} 5/2-y, & y<2 \\ 1/y, & y\ge 2 \end{cases}$, $\quad y=f^{-1}(x)=\begin{cases} 5/2-x, & x>1/2 \\ 1/x, & 0<x\le 1/2 \end{cases}$

24. $y=p^{-1}(x)$, $x=p(y)=y^3-3y^2+3y-1=(y-1)^3$, $y=x^{1/3}+1=p^{-1}(x)$

25. $y=f^{-1}(x)$, $x=f(y)=(y+2)^4$ for $y\ge 0$, $y=f^{-1}(x)=x^{1/4}-2$ for $x\ge 16$

26. $y=f^{-1}(x)$, $x=f(y)=\sqrt{y+3}$ for $y\ge -3$, $y=f^{-1}(x)=x^2-3$ for $x\ge 0$

27. $y=f^{-1}(x)$, $x=f(y)=-\sqrt{3-2y}$ for $y\le 3/2$, $y=f^{-1}(x)=(3-x^2)/2$ for $x\le 0$

28. $y=f^{-1}(x)$, $x=f(y)=3y^2+5y-2$ for $y\ge 0$, $3y^2+5y-2-x=0$ for $y\ge 0$,
$y=f^{-1}(x)=(-5+\sqrt{12x+49})/6$ for $x\ge -2$

29. $y=f^{-1}(x)$, $x=f(y)=y-5y^2$ for $y\ge 1$, $5y^2-y+x=0$ for $y\ge 1$,
$y=f^{-1}(x)=(1+\sqrt{1-20x})/10$ for $x\le -4$

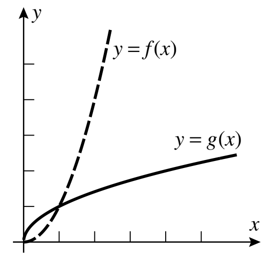
30. **(a)** $C=\dfrac{5}{9}(F-32)$
(b) how many degrees Celsius given the Fahrenheit temperature
(c) $C=-273.15^\circ$ C is equivalent to $F=-459.67^\circ$ F, so the domain is $F\ge -459.67$, the range is $C\ge -273.15$

31. **(a)** $y=f(x)=(6.214\times 10^{-4})x$ **(b)** $x=f^{-1}(y)=\dfrac{10^4}{6.214}y$
(c) how many meters in y miles

32. f and f^{-1} are continuous so $f(3)=\lim\limits_{x\to 3}f(x)=7$; then $f^{-1}(7)=3$, and
$\lim\limits_{x\to 7}f^{-1}(x)=f^{-1}\left(\lim\limits_{x\to 7}x\right)=f^{-1}(7)=3$

33. **(a)** $f(g(x)) = f(\sqrt{x})$
$= (\sqrt{x})^2 = x, x > 1;$
$g(f(x)) = g(x^2)$
$= \sqrt{x^2} = x, x > 1$

(b)



(c) no, because $f(g(x)) = x$ for every x in the domain of g is not satisfied (the domain of g is $x \geq 0$)

34. $y = f^{-1}(x)$, $x = f(y) = ay^2 + by + c$, $ay^2 + by + c - x = 0$, use the quadratic formula to get $y = \dfrac{-b \pm \sqrt{b^2 - 4a(c-x)}}{2a}$;

(a) $f^{-1}(x) = \dfrac{-b + \sqrt{b^2 - 4a(c-x)}}{2a}$ **(b)** $f^{-1}(x) = \dfrac{-b - \sqrt{b^2 - 4a(c-x)}}{2a}$

35. **(a)** $f(f(x)) = \dfrac{3 - \dfrac{3-x}{1-x}}{1 - \dfrac{3-x}{1-x}} = \dfrac{3 - 3x - 3 + x}{1 - x - 3 + x} = x$ so $f = f^{-1}$

(b) symmetric about the line $y = x$

36. $y = m(x - x_0)$ is an equation of the line. The graph of the inverse of $f(x) = m(x - x_0)$ will be the reflection of this line about $y = x$. Solve $y = m(x - x_0)$ for x to get $x = y/m + x_0 = f^{-1}(y)$ so $y = f^{-1}(x) = x/m + x_0$.

37. **(a)** $f(x) = x^3 - 3x^2 + 2x = x(x-1)(x-2)$ so $f(0) = f(1) = f(2) = 0$ thus f is not one-to-one.

(b) $f'(x) = 3x^2 - 6x + 2$, $f'(x) = 0$ when $x = \dfrac{6 \pm \sqrt{36 - 24}}{6} = 1 \pm \sqrt{3}/3$. $f'(x) > 0$ (f is increasing) if $x < 1 - \sqrt{3}/3$, $f'(x) < 0$ (f is decreasing) if $1 - \sqrt{3}/3 < x < 1 + \sqrt{3}/3$, so $f(x)$ takes on values less than $f(1 - \sqrt{3}/3)$ on both sides of $1 - \sqrt{3}/3$ thus $1 - \sqrt{3}/3$ is the largest value of k.

38. **(a)** $f(x) = x^3(x - 2)$ so $f(0) = f(2) = 0$ thus f is not one to one.

(b) $f'(x) = 4x^3 - 6x^2 = 4x^2(x - 3/2)$, $f'(x) = 0$ when $x = 0$ or $3/2$; f is decreasing on $(-\infty, 3/2]$ and increasing on $[3/2, +\infty)$ so $3/2$ is the smallest value of k.

39. if $f^{-1}(x) = 1$, then $x = f(1) = 2(1)^3 + 5(1) + 3 = 10$

40. if $f^{-1}(x) = 2$, then $x = f(2) = (2)^3/[(2)^2 + 1] = 8/5$

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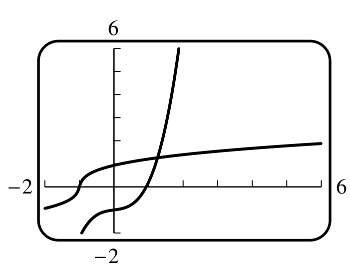

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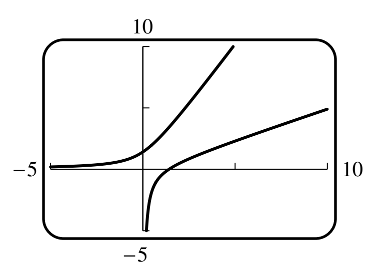

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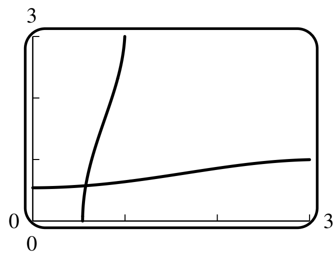

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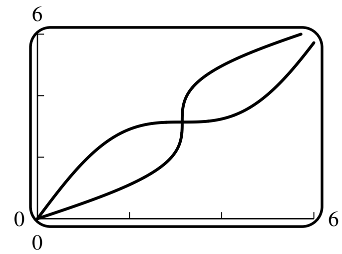


45. $y = f^{-1}(x)$, $x = f(y) = 5y^3 + y - 7$, $\dfrac{dx}{dy} = 15y^2 + 1$, $\dfrac{dy}{dx} = \dfrac{1}{15y^2+1}$;

check: $1 = 15y^2\dfrac{dy}{dx} + \dfrac{dy}{dx}$, $\dfrac{dy}{dx} = \dfrac{1}{15y^2+1}$

46. $y = f^{-1}(x)$, $x = f(y) = 1/y^2$, $\dfrac{dx}{dy} = -2y^{-3}$, $\dfrac{dy}{dx} = -y^3/2$;

check: $1 = -2y^{-3}\dfrac{dy}{dx}$, $\dfrac{dy}{dx} = -y^3/2$

47. $y = f^{-1}(x)$, $x = f(y) = 2y^5 + y^3 + 1$, $\dfrac{dx}{dy} = 10y^4 + 3y^2$, $\dfrac{dy}{dx} = \dfrac{1}{10y^4+3y^2}$;

check: $1 = 10y^4\dfrac{dy}{dx} + 3y^2\dfrac{dy}{dx}$, $\dfrac{dy}{dx} = \dfrac{1}{10y^4+3y^2}$

48. $y = f^{-1}(x)$, $x = f(y) = 5y - \sin 2y$, $\dfrac{dx}{dy} = 5 - 2\cos 2y$, $\dfrac{dy}{dx} = \dfrac{1}{5 - 2\cos 2y}$;

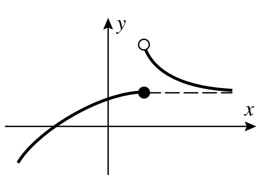
check: $1 = (5 - 2\cos 2y)\dfrac{dy}{dx}$, $\dfrac{dy}{dx} = \dfrac{1}{5 - 2\cos 2y}$

49. $f(f(x)) = x$ thus $f = f^{-1}$ so the graph is symmetric about $y = x$.

50. **(a)** Suppose $x_1 \neq x_2$ where x_1 and x_2 are in the domain of g and $g(x_1)$, $g(x_2)$ are in the domain of f then $g(x_1) \neq g(x_2)$ because g is one-to-one so $f(g(x_1)) \neq f(g(x_2))$ because f is one-to-one thus $f \circ g$ is one-to-one because $(f \circ g)(x_1) \neq (f \circ g)(x_2)$ if $x_1 \neq x_2$.

(b) f, g, and $f \circ g$ all have inverses because they are all one-to-one. Let $h = (f \circ g)^{-1}$ then $(f \circ g)(h(x)) = f[g(h(x))] = x$, apply f^{-1} to both sides to get $g(h(x)) = f^{-1}(x)$, then apply g^{-1} to get $h(x) = g^{-1}(f^{-1}(x)) = (g^{-1} \circ f^{-1})(x)$, so $h = g^{-1} \circ f^{-1}$

51.



52. Suppose that g and h are both inverses of f then $f(g(x)) = x$, $h[f(g(x))] = h(x)$, but $h[f(g(x))] = g(x)$ because h is an inverse of f so $g(x) = h(x)$.

53. $F'(x) = 2f'(2g(x))g'(x)$ so $F'(3) = 2f'(2g(3))g'(3)$. By inspection $f(1) = 3$, so $g(3) = f^{-1}(3) = 1$ and $g'(3) = (f^{-1})'(3) = 1/f'(f^{-1}(3)) = 1/f'(1) = 1/7$ because $f'(x) = 4x^3 + 3x^2$. Thus $F'(3) = 2f'(2)(1/7) = 2(44)(1/7) = 88/7$.
$F(3) = f(2g(3)), g(3) = f^{-1}(3)$; by inspection $f(1) = 3$, so $g(3) = f^{-1}(3) = 1, F(3) = f(2) = 25$.

EXERCISE SET 7.2

1. **(a)** -4 **(b)** 4 **(c)** $1/4$

2. **(a)** $1/16$ **(b)** 8 **(c)** $1/3$

3. **(a)** 2.9690 **(b)** 0.0341

4. **(a)** 1.8882 **(b)** 0.9381

5. **(a)** $\log_2 16 = \log_2(2^4) = 4$ **(b)** $\log_2\left(\dfrac{1}{32}\right) = \log_2(2^{-5}) = -5$
(c) $\log_4 4 = 1$ **(d)** $\log_9 3 = \log_9(9^{1/2}) = 1/2$

6. **(a)** $\log_{10}(0.001) = \log_{10}(10^{-3}) = -3$ **(b)** $\log_{10}(10^4) = 4$
(c) $\ln(e^3) = 3$ **(d)** $\ln(\sqrt{e}) = \ln(e^{1/2}) = 1/2$

7. **(a)** 1.3655 **(b)** -0.3011

8. **(a)** -0.5229 **(b)** 1.1447

9. **(a)** $2\ln a + \dfrac{1}{2}\ln b + \dfrac{1}{2}\ln c = 2r + s/2 + t/2$ **(b)** $\ln b - 3\ln a - \ln c = s - 3r - t$

10. **(a)** $\dfrac{1}{3}\ln c - \ln a - \ln b = t/3 - r - s$ **(b)** $\dfrac{1}{2}(\ln a + 3\ln b - 2\ln c) = r/2 + 3s/2 - t$

11. **(a)** $1 + \log x + \dfrac{1}{2}\log(x-3)$ **(b)** $2\ln|x| + 3\ln\sin x - \dfrac{1}{2}\ln(x^2+1)$

12. **(a)** $\dfrac{1}{3}\log(x+2) - \log\cos 5x$ **(b)** $\dfrac{1}{2}\ln(x^2+1) - \dfrac{1}{2}\ln(x^3+5)$

13. $\log\dfrac{2^4(16)}{3} = \log(256/3)$

14. $\log\sqrt{x} - \log(\sin^3 2x) + \log 100 = \log\dfrac{100\sqrt{x}}{\sin^3 2x}$

15. $\ln\dfrac{\sqrt[3]{x}(x+1)^2}{\cos x}$

16. $1 + x = 10^3 = 1000,\ x = 999$

17. $\sqrt{x} = 10^{-1} = 0.1,\ x = 0.01$

18. $x^2 = e^4,\ x = \pm e^2$

19. $1/x = e^{-2},\ x = e^2$

20. $x = 7$

21. $2x = 8,\ x = 4$

22. $\log_{10} x^3 = 30,\ x^3 = 10^{30},\ x = 10^{10}$

23. $\log_{10} x = 5,\ x = 10^5$

24. $\ln 4x - \ln x^6 = \ln 2,\ \ln\dfrac{4}{x^5} = \ln 2,\ \dfrac{4}{x^5} = 2,\ x^5 = 2,\ x = \sqrt[5]{2}$

25. $\ln 2x^2 = \ln 3,\ 2x^2 = 3,\ x^2 = 3/2,\ x = \sqrt{3/2}$ (we discard $-\sqrt{3/2}$ because it does not satisfy the original equation)

26. $\ln 3^x = \ln 2,\ x\ln 3 = \ln 2,\ x = \dfrac{\ln 2}{\ln 3}$

27. $\ln 5^{-2x} = \ln 3,\ -2x\ln 5 = \ln 3,\ x = -\dfrac{\ln 3}{2\ln 5}$

28. $e^{-2x} = 5/3,\ -2x = \ln(5/3),\ x = -\dfrac{1}{2}\ln(5/3)$

29. $e^{3x} = 7/2,\ 3x = \ln(7/2),\ x = \dfrac{1}{3}\ln(7/2)$

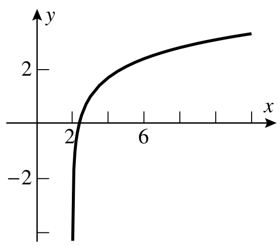
30. $e^x(1-2x) = 0$ so $e^x = 0$ (impossible) or $1 - 2x = 0,\ x = 1/2$

31. $e^{-x}(x+2) = 0$ so $e^{-x} = 0$ (impossible) or $x + 2 = 0,\ x = -2$

32. $e^{2x} - e^x - 6 = (e^x - 3)(e^x + 2) = 0$ so $e^x = -2$ (impossible) or $e^x = 3,\ x = \ln 3$

33. $e^{-2x} - 3e^{-x} + 2 = (e^{-x} - 2)(e^{-x} - 1) = 0$ so $e^{-x} = 2,\ x = -\ln 2$ or $e^{-x} = 1,\ x = 0$

34. **(a)**



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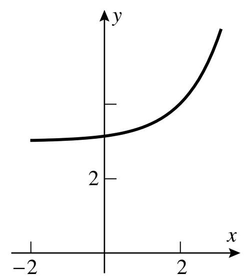
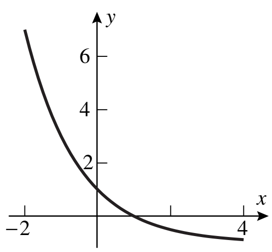


35. **(a)**



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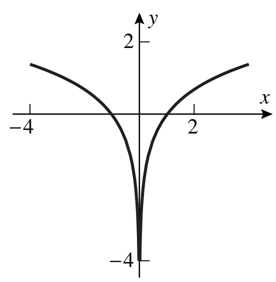
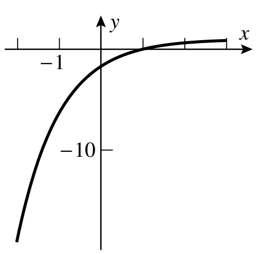


36. **(a)**



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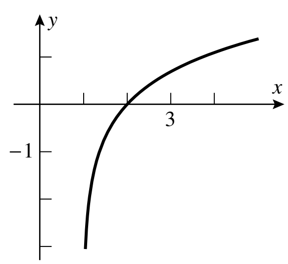
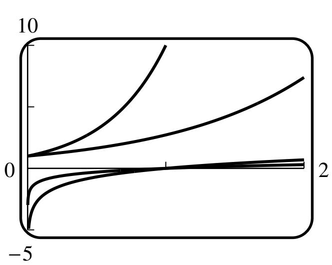


37. $\log_2 7.35 = (\log 7.35)/(\log 2) = (\ln 7.35)/(\ln 2) \approx 2.8777$;
$\log_5 0.6 = (\log 0.6)/(\log 5) = (\ln 0.6)/(\ln 5) \approx -0.3174$

38.



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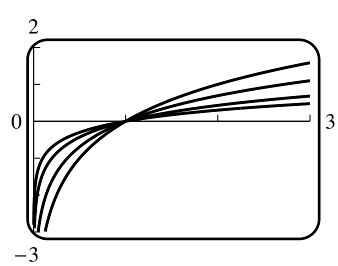
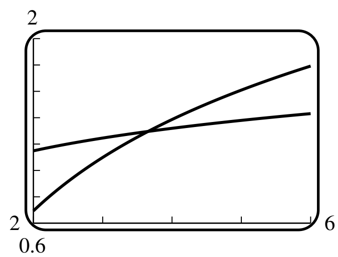


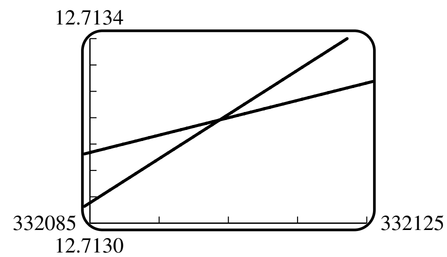
40. **(a)** Let $X = \log_b x$ and $Y = \log_a x$. Then $b^X = x$ and $a^Y = x$ so $a^Y = b^X$, or $a^{Y/X} = b$, which means $\log_a b = Y/X$. Substituting for Y and X yields $\dfrac{\log_a x}{\log_b x} = \log_a b, \log_b x = \dfrac{\log_a x}{\log_a b}$.

(b) Let $x = a$ to get $\log_b a = (\log_a a)/(\log_a b) = 1/(\log_a b)$ so $(\log_a b)(\log_b a) = 1$.
$(\log_2 81)(\log_3 32) = (\log_2[3^4])(\log_3[2^5]) = (4\log_2 3)(5\log_3 2) = 20(\log_2 3)(\log_3 2) = 20$

41. **(a)** $x = 3.6541, y = 1.2958$



(b) $x \approx 332105.11, y \approx 12.7132$



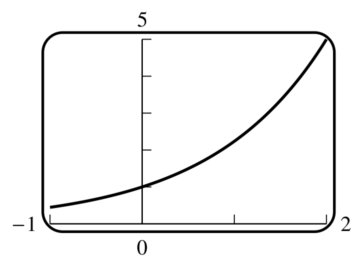
42. Since the units are billions, one trillion is 1,000 units. Solve $1000 = 0.051517(1.1306727)^x$ for x by taking common logarithms, resulting in $3 = \log 0.051517 + x \log 1.1306727$, which yields $x \approx 77.4$, so the debt first reached one trillion dollars around 1977.

43. **(a)** no, the curve passes through the origin

(b) $y = 2^{x/4}$

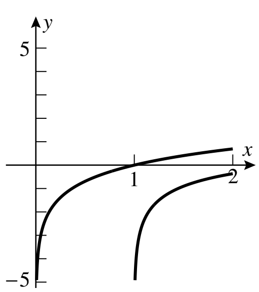
(c) $y = 2^{-x}$

(d) $y = (\sqrt{5})^x$



44. **(a)** As $x \to +\infty$ the function grows very slowly, but it is always increasing and tends to $+\infty$. As $x \to 1^+$ the function tends to $-\infty$.

(b)



45. $\log(1/2) < 0$ so $3\log(1/2) < 2\log(1/2)$

46. Let $x = \log_b a$ and $y = \log_b c$, so $a = b^x$ and $c = b^y$.
First, $ac = b^x b^y = b^{x+y}$ or equivalently, $\log_b(ac) = x + y = \log_b a + \log_b c$.
Secondly, $a/c = b^x/b^y = b^{x-y}$ or equivalently, $\log_b(a/c) = x - y = \log_b a - \log_b c$.
Next, $a^r = (b^x)^r = b^{rx}$ or equivalently, $\log_b a^r = rx = r\log_b a$.
Finally, $1/c = 1/b^y = b^{-y}$ or equivalently, $\log_b(1/c) = -y = -\log_b c$.

47. $75e^{-t/125} = 15, t = -125\ln(1/5) = 125\ln 5 \approx 201$ days.

48. **(a)** a If $t = 0$, then $Q = 12$ grams

(b) b $Q = 12e^{-0.055(4)} = 12e^{-0.22} \approx 9.63$ grams

(c) $12e^{-0.055t} = 6, e^{-0.055t} = 0.5, t = -(\ln 0.5)/(0.055) \approx 12.6$ hours

49. **(a)** 7.4; basic **(b)** 4.2; acidic **(c)** 6.4; acidic **(d)** 5.9; acidic

50. **(a)** $\log[H^+] = -2.44, [H^+] = 10^{-2.44} \approx 3.6 \times 10^{-3}$ mol/L
(b) $\log[H^+] = -8.06, [H^+] = 10^{-8.06} \approx 8.7 \times 10^{-9}$ mol/L

51. **(a)** 140 dB; damage **(b)** 120 dB; damage
(c) 80 dB; no damage **(d)** 75 dB; no damage

52. Suppose that $I_1 = 3I_2$ and $\beta_1 = 10\log_{10} I_1/I_0$, $\beta_2 = 10\log_{10} I_2/I_0$. Then $I_1/I_0 = 3I_2/I_0$, $\log_{10} I_1/I_0 = \log_{10} 3I_2/I_0 = \log_{10} 3 + \log_{10} I_2/I_0$, $\beta_1 = 10\log_{10} 3 + \beta_2$, $\beta_1 - \beta_2 = 10\log_{10} 3 \approx 4.8$ decibels.

53. Let I_A and I_B be the intensities of the automobile and blender, respectively. Then $\log_{10} I_A/I_0 = 7$ and $\log_{10} I_B/I_0 = 9.3$, $I_A = 10^7 I_0$ and $I_B = 10^{9.3} I_0$, so $I_B/I_A = 10^{2.3} \approx 200$.

54. The decibel level of the nth echo is $120(2/3)^n$;

$120(2/3)^n < 10$ if $(2/3)^n < 1/12, n < \dfrac{\log(1/12)}{\log(2/3)} = \dfrac{\log 12}{\log 1.5} \approx 6.13$ so 6 echoes can be heard.

55. **(a)** $\log E = 4.4 + 1.5(8.2) = 16.7, E = 10^{16.7} \approx 5 \times 10^{16}$ J

(b) Let M_1 and M_2 be the magnitudes of earthquakes with energies of E and $10E$, respectively. Then $1.5(M_2 - M_1) = \log(10E) - \log E = \log 10 = 1$, $M_2 - M_1 = 1/1.5 = 2/3 \approx 0.67$.

56. Let E_1 and E_2 be the energies of earthquakes with magnitudes M and $M+1$, respectively. Then $\log E_2 - \log E_1 = \log(E_2/E_1) = 1.5, E_2/E_1 = 10^{1.5} \approx 31.6$.

57. If $t = -2x$, then $x = -t/2$ and $\lim_{x\to 0}(1-2x)^{1/x} = \lim_{t\to 0}(1+t)^{-2/t} = \lim_{t\to 0}[(1+t)^{1/t}]^{-2} = e^{-2}$.

58. If $t = 3/x$, then $x = 3/t$ and $\lim_{x\to +\infty}(1+3/x)^x = \lim_{t\to 0^+}(1+t)^{3/t} = \lim_{t\to 0^+}[(1+t)^{1/t}]^3 = e^3$.

EXERCISE SET 7.3

1. $\dfrac{1}{2x}(2) = 1/x$

2. $\dfrac{1}{x^3}(3x^2) = 3/x$

3. $2(\ln x)\left(\dfrac{1}{x}\right) = \dfrac{2\ln x}{x}$

4. $\dfrac{1}{\sin x}(\cos x) = \cot x$

5. $\dfrac{1}{\tan x}(\sec^2 x) = \dfrac{\sec^2 x}{\tan x}$

6. $\dfrac{1}{2+\sqrt{x}}\left(\dfrac{1}{2\sqrt{x}}\right) = \dfrac{1}{2\sqrt{x}(2+\sqrt{x})}$

7. $\dfrac{1}{x/(1+x^2)}\left[\dfrac{(1+x^2)(1) - x(2x)}{(1+x^2)^2}\right] = \dfrac{1-x^2}{x(1+x^2)}$

8. $\dfrac{1}{\ln x}\left(\dfrac{1}{x}\right) = \dfrac{1}{x\ln x}$

9. $\dfrac{3x^2 - 14x}{x^3 - 7x^2 - 3}$

10. $x^3\left(\frac{1}{x}\right)+(3x^2)\ln x=x^2(1+3\ln x)$

11. $\frac{1}{2}(\ln x)^{-1/2}\left(\frac{1}{x}\right)=\frac{1}{2x\sqrt{\ln x}}$

12. $\frac{\frac{1}{2}2(\ln x)(1/x)}{\sqrt{1+\ln^2 x}}=\frac{\ln x}{x\sqrt{1+\ln^2 x}}$

13. $-\frac{1}{x}\sin(\ln x)$

14. $2\sin(\ln x)\cos(\ln x)\frac{1}{x}=\frac{\sin(2\ln x)}{x}=\frac{\sin(\ln x^2)}{x}$

15. $3x^2\log_2(3-2x)+\frac{-2x^3}{(\ln 2)(3-2x)}$

16. $\left[\log_2(x^2-2x)\right]^3+3x\left[\log_2(x^2-2x)\right]^2\frac{2x-2}{(x^2-2x)\ln 2}$

17. $\frac{2x(1+\log x)-x/(\ln 10)}{(1+\log x)^2}$

18. $1/[x(\ln 10)(1+\log x)^2]$

19. $7e^{7x}$

20. $-10xe^{-5x^2}$

21. $x^3e^x+3x^2e^x=x^2e^x(x+3)$

22. $-\frac{1}{x^2}e^{1/x}$

23. $\frac{dy}{dx}=\frac{(e^x+e^{-x})(e^x+e^{-x})-(e^x-e^{-x})(e^x-e^{-x})}{(e^x+e^{-x})^2}$

$=\frac{(e^{2x}+2+e^{-2x})-(e^{2x}-2+e^{-2x})}{(e^x+e^{-x})^2}=4/(e^x+e^{-x})^2$

24. $e^x\cos(e^x)$

25. $(x\sec^2 x+\tan x)e^{x\tan x}$

26. $\frac{dy}{dx}=\frac{(\ln x)e^x-e^x(1/x)}{(\ln x)^2}=\frac{e^x(x\ln x-1)}{x(\ln x)^2}$

27. $(1-3e^{3x})e^{(x-e^{3x})}$

28. $\frac{15}{2}x^2(1+5x^3)^{-1/2}\exp(\sqrt{1+5x^3})$

29. $\frac{(x-1)e^{-x}}{1-xe^{-x}}=\frac{x-1}{e^x-x}$

30. $\frac{1}{\cos(e^x)}[-\sin(e^x)]e^x=-e^x\tan(e^x)$

31. $\frac{dy}{dx}+\frac{1}{xy}\left(x\frac{dy}{dx}+y\right)=0,\ \frac{dy}{dx}=-\frac{y}{x(y+1)}$

32. $\frac{dy}{dx}=\frac{1}{x\tan y}\left(x\sec^2 y\frac{dy}{dx}+\tan y\right),\ \frac{dy}{dx}=\frac{\tan y}{x(\tan y-\sec^2 y)}$

33. $\frac{d}{dx}\left[\ln\cos x-\frac{1}{2}\ln(4-3x^2)\right]=-\tan x+\frac{3x}{4-3x^2}$

34. $\frac{d}{dx}\left(\frac{1}{2}[\ln(x-1)-\ln(x+1)]\right)=\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)$

35. $\ln|y|=\ln|x|+\frac{1}{3}\ln|1+x^2|,\ \frac{dy}{dx}=x\sqrt[3]{1+x^2}\left[\frac{1}{x}+\frac{2x}{3(1+x^2)}\right]$

36. $\ln|y| = \frac{1}{5}[\ln|x-1| - \ln|x+1|]$, $\frac{dy}{dx} = \frac{1}{5}\sqrt[5]{\frac{x-1}{x+1}}\left[\frac{1}{x-1} - \frac{1}{x+1}\right]$

37. $\ln|y| = \frac{1}{3}\ln|x^2-8| + \frac{1}{2}\ln|x^3+1| - \ln|x^6-7x+5|$

$$\frac{dy}{dx} = \frac{(x^2-8)^{1/3}\sqrt{x^3+1}}{x^6-7x+5}\left[\frac{2x}{3(x^2-8)} + \frac{3x^2}{2(x^3+1)} - \frac{6x^5-7}{x^6-7x+5}\right]$$

38. $\ln|y| = \ln|\sin x| + \ln|\cos x| + 3\ln|\tan x| - \frac{1}{2}\ln|x|$

$$\frac{dy}{dx} = \frac{\sin x\cos x\tan^3 x}{\sqrt{x}}\left[\cot x - \tan x + \frac{3\sec^2 x}{\tan x} - \frac{1}{2x}\right]$$

39. $f'(x) = 2^x\ln 2$; $y = 2^x$, $\ln y = x\ln 2$, $\frac{1}{y}y' = \ln 2$, $y' = y\ln 2 = 2^x\ln 2$

40. $f'(x) = -3^{-x}\ln 3$; $y = 3^{-x}$, $\ln y = -x\ln 3$, $\frac{1}{y}y' = -\ln 3$, $y' = -y\ln 3 = -3^{-x}\ln 3$

41. $f'(x) = \pi^{\sin x}(\ln\pi)\cos x$;

$y = \pi^{\sin x}$, $\ln y = (\sin x)\ln\pi$, $\frac{1}{y}y' = (\ln\pi)\cos x$, $y' = \pi^{\sin x}(\ln\pi)\cos x$

42. $f'(x) = \pi^{x\tan x}(\ln\pi)(x\sec^2 x + \tan x)$;

$y = \pi^{x\tan x}$, $\ln y = (x\tan x)\ln\pi$, $\frac{1}{y}y' = (\ln\pi)(x\sec^2 x + \tan x)$

$y' = \pi^{x\tan x}(\ln\pi)(x\sec^2 x + \tan x)$

43. $\ln y = (\ln x)\ln(x^3-2x)$, $\frac{1}{y}\frac{dy}{dx} = \frac{3x^2-2}{x^3-2x}\ln x + \frac{1}{x}\ln(x^3-2x)$,

$$\frac{dy}{dx} = (x^3-2x)^{\ln x}\left[\frac{3x^2-2}{x^3-2x}\ln x + \frac{1}{x}\ln(x^3-2x)\right]$$

44. $\ln y = (\sin x)\ln x$, $\frac{1}{y}\frac{dy}{dx} = \frac{\sin x}{x} + (\cos x)\ln x$, $\frac{dy}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + (\cos x)\ln x\right]$

45. $\ln y = (\tan x)\ln(\ln x)$, $\frac{1}{y}\frac{dy}{dx} = \frac{1}{x\ln x}\tan x + (\sec^2 x)\ln(\ln x)$,

$$\frac{dy}{dx} = (\ln x)^{\tan x}\left[\frac{\tan x}{x\ln x} + (\sec^2 x)\ln(\ln x)\right]$$

46. $\ln y = (\ln x)\ln(x^2+3)$, $\frac{1}{y}\frac{dy}{dx} = \frac{2x}{x^2+3}\ln x + \frac{1}{x}\ln(x^2+3)$,

$$\frac{dy}{dx} = (x^2+3)^{\ln x}\left[\frac{2x}{x^2+3}\ln x + \frac{1}{x}\ln(x^2+3)\right]$$

47. $f'(x) = ex^{e-1}$

48. **(a)** because x^x is not of the form a^x where a is constant

(b) $y = x^x$, $\ln y = x\ln x$, $\dfrac{1}{y}y' = 1 + \ln x$, $y' = x^x(1 + \ln x)$

49. **(a)** $\log_x e = \dfrac{\ln e}{\ln x} = \dfrac{1}{\ln x}$, $\dfrac{d}{dx}[\log_x e] = -\dfrac{1}{x(\ln x)^2}$

(b) $\log_x 2 = \dfrac{\ln 2}{\ln x}$, $\dfrac{d}{dx}[\log_x 2] = -\dfrac{\ln 2}{x(\ln x)^2}$

50. **(a)** e^{x^2} **(b)** $\ln x$

51. **(a)** $f'(x) = ke^{kx}$, $f''(x) = k^2e^{kx}$, $f'''(x) = k^3e^{kx}, \dots, f^{(n)}(x) = k^ne^{kx}$

(b) $f'(x) = -ke^{-kx}$, $f''(x) = k^2e^{-kx}$, $f'''(x) = -k^3e^{-kx}, \dots, f^{(n)}(x) = (-1)^nk^ne^{-kx}$

52.
$$\frac{dy}{dt} = e^{-\lambda t}(\omega A\cos\omega t - \omega B\sin\omega t) + (-\lambda)e^{-\lambda t}(A\sin\omega t + B\cos\omega t)$$
$$= e^{-\lambda t}[(\omega A - \lambda B)\cos\omega t - (\omega B + \lambda A)\sin\omega t]$$

53.
$$f'(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]\frac{d}{dx}\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$
$$= \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]\left[-\left(\frac{x-\mu}{\sigma}\right)\left(\frac{1}{\sigma}\right)\right]$$
$$= -\frac{1}{\sqrt{2\pi}\sigma^3}(x-\mu)\exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

54. $y = Ae^{kt}$, $dy/dt = kAe^{kt} = k(Ae^{kt}) = ky$

55. $y = Ae^{2x} + Be^{-4x}$, $y' = 2Ae^{2x} - 4Be^{-4x}$, $y'' = 4Ae^{2x} + 16Be^{-4x}$ so
$y'' + 2y' - 8y = (4Ae^{2x} + 16Be^{-4x}) + 2(2Ae^{2x} - 4Be^{-4x}) - 8(Ae^{2x} + Be^{-4x}) = 0$

56. **(a)** $y' = -xe^{-x} + e^{-x} = e^{-x}(1 - x)$, $xy' = xe^{-x}(1 - x) = y(1 - x)$

(b) $y' = -x^2e^{-x^2/2} + e^{-x^2/2} = e^{-x^2/2}(1 - x^2)$, $xy' = xe^{-x^2/2}(1 - x^2) = y(1 - x^2)$

57. **(a)** $f(w) = \ln w$; $f'(1) = \lim\limits_{h\to 0}\dfrac{\ln(1+h) - \ln 1}{h} = \lim\limits_{h\to 0}\dfrac{\ln(1+h)}{h} = \dfrac{1}{w}\Big|_{w=1} = 1$

(b) $f(w) = 10^w$; $f'(0) = \lim\limits_{h\to 0}\dfrac{10^h - 1}{h} = \dfrac{d}{dw}(10^w)\Big|_{w=0} = 10^w\ln 10\Big|_{w=0} = \ln 10$

58. **(a)** $f(x) = \ln x$; $f'(e^2) = \lim\limits_{\Delta x\to 0}\dfrac{\ln(e^2 + \Delta x) - 2}{\Delta x} = \dfrac{d}{dx}(\ln x)\Big|_{x=e^2} = \dfrac{1}{x}\Big|_{x=e^2} = e^{-2}$

(b) $f(w) = 2^w$; $f'(1) = \lim\limits_{w\to 1}\dfrac{2^w - 2}{w - 1} = \dfrac{d}{dw}(2^w)\Big|_{w=1} = 2^w\ln 2\Big|_{w=1} = 2\ln 2$

59. $2\ln x + 3e^x + C$

60. $\displaystyle\int\left[\frac{1}{2}t^{-1} - \sqrt{2}e^t\right]dt = \frac{1}{2}\ln t - \sqrt{2}e^t + C$

61. **(a)** $\int \frac{1}{u}du = \ln|u| + C = \ln|\ln x| + C$ **(b)** $-\frac{1}{5}\int e^u\,du = -\frac{1}{5}e^u + C = -\frac{1}{5}e^{-5x} + C$

62. **(a)** $-\frac{1}{3}\int \frac{1}{u}du = -\frac{1}{3}\ln|u| + C = -\frac{1}{3}\ln|(1+\cos 3\theta)| + C$

(b) $\int \frac{du}{u} = \ln u + C = \ln(1+e^x) + C$

63. $u = 2x,\ du = 2dx;\ \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C = \frac{1}{2}e^{2x} + C$

64. $u = 2x,\ du = 2dx;\ \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|2x| + C$

65. $u = \sin x,\ du = \cos x\,dx;\ \int e^u\,du = e^u + C = e^{\sin x} + C$

66. $u = x^4,\ du = 4x^3dx;\ \frac{1}{4}\int e^u\,du = \frac{1}{4}e^u + C = \frac{1}{4}e^{x^4} + C$

67. $u = -2x^3,\ du = -6x^2,\ -\frac{1}{6}\int e^u du = -\frac{1}{6}e^u + C = -\frac{1}{6}e^{-2x^3} + C$

68. $u = e^x - e^{-x},\ du = (e^x + e^{-x})dx,\ \int \frac{1}{u}du = \ln|u| + C = \ln\left|e^x - e^{-x}\right| + C$

69. $\int e^{-x}dx;\ u = -x,\ du = -dx;\ -\int e^u du = -e^u + C = -e^{-x} + C$

70. $\int e^{x/2}dx;\ u = x/2,\ du = dx/2;\ 2\int e^u du = 2e^u + C = 2e^{x/2} + C = 2\sqrt{e^x} + C$

71. $u = \sqrt{y+1},\ du = \frac{1}{2\sqrt{y+1}}dy,\ 2\int e^u du = 2e^u + C = 2e^{\sqrt{y+1}} + C$

72. $u = \sqrt{y},\ du = \frac{1}{2\sqrt{y}}dy,\ 2\int \frac{1}{e^u}du = 2\int e^{-u}du = -2e^{-u} + C = -2e^{-\sqrt{y}} + C$

73. $\int \left(1 + \frac{1}{t}\right)dt = t + \ln|t| + C$

74. $e^{2\ln x} = e^{\ln x^2} = x^2,\ x > 0$, so $\int e^{2\ln x}dx = \int x^2dx = \frac{1}{3}x^3 + C$

75. $\ln(e^x) + \ln(e^{-x}) = \ln(e^xe^{-x}) = \ln 1 = 0$ so $\int [\ln(e^x) + \ln(e^{-x})]dx = C$

76. $\int \frac{\cos x}{\sin x}dx;\ u = \sin x,\ du = \cos x dx;\ \int \frac{1}{u}du = \ln|u| + C = \ln|\sin x| + C$

77. $5e^x\Big]_{\ln 2}^{3} = 5e^3 - 5(2) = 5e^3 - 10$

78. $(\ln x)/2\Big]_{1/2}^{1} = (\ln 2)/2$

79. **(a)** $\frac{1}{2}\int_{-1}^{1} e^u du = \frac{1}{2}e^u\Big|_{-1}^{1} = \frac{1}{2}\left(e - e^{-1}\right)$ **(b)** $\int_1^2 u\,du = \frac{3}{2}$

80. $\int_{-6}^{6} \sqrt{36 - u^2}du = \pi(6)^2/2 = 18\pi$

81. $u = e^x + 4$, $du = e^x dx$, $u = e^{-\ln 3} + 4 = \frac{1}{3} + 4 = \frac{13}{3}$ when $x = -\ln 3$,

$u = e^{\ln 3} + 4 = 3 + 4 = 7$ when $x = \ln 3$, $\int_{13/3}^{7} \frac{1}{u}du = \ln u\Big]_{13/3}^{7} = \ln(7) - \ln(13/3) = \ln(21/13)$

82. $u = 3 - 4e^x$, $du = -4e^x dx$, $u = -1$ when $x = 0$, $u = -17$ when $x = \ln 5$

$-\frac{1}{4}\int_{-1}^{-17} u\,du = -\frac{1}{8}u^2\Big]_{-1}^{-17} = -36$

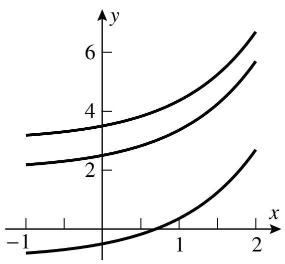
83. $\ln(x + e)\Big]_0^e = \ln(2e) - \ln e = \ln 2$

84. $-\frac{1}{2}e^{-x^2}\Big]_1^{\sqrt{2}} = (e^{-1} - e^{-2})/2$

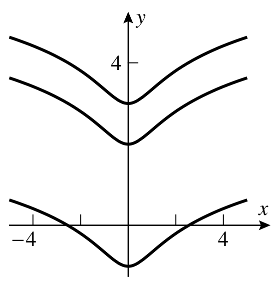
85. $-\frac{1}{3}e^{-3x}\Big]_0^{\ln 2} = -\frac{1}{3}(e^{-3\ln 2} - e^0) = -\frac{1}{3}\left(\frac{1}{8} - 1\right) = 7/24$

86. $\int_{-1}^{0}(1-e^x)dx + \int_0^1 (e^x - 1)dx = (x - e^x)\Big]_{-1}^{0} + (e^x - x)\Big]_0^1 = -1-(-1-e^{-1})+e-1-1 = e+1/e-2$

87. **(a)**



88.



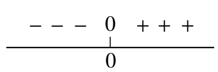
(b) $f(x) = e^x/2 + 1/2$

89. **(a)** $y(t) = \int 2e^{-t}\,dt = -2e^{-t} + C$, $y(1) = -\frac{2}{e} + C = 3 - \frac{2}{e}, C = 3$; $y(t) = -2e^{-t} + 3$

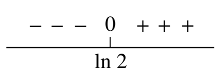
(b) $y(t) = \int t^{-1}dt = \ln|t| + C$, $y(-1) = C = 5$, $C = 5$; $y(t) = \ln|t| + 5$

EXERCISE SET 7.4

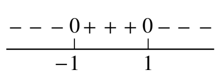
1. **(a)** critical point $x = 0$; f':
$x = 0$: relative minimum



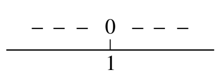
(b) critical point $x = \ln 2$: f':
$x = \ln 2$: relative minimum



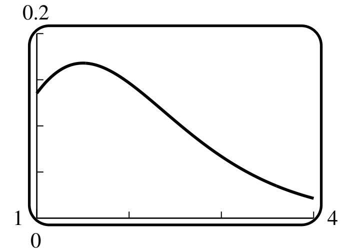
2. **(a)** critical points $x = -1, 1$: f':
$x = -1$: relative minimum;
$x = 1$: relative maximum



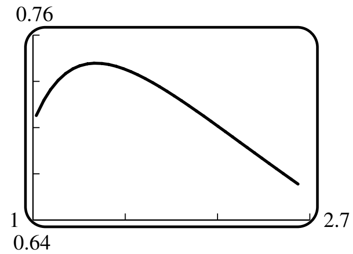
(b) $x = 1$: neither



3. $f'(x) = x^2(2x-3)e^{-2x}$, $f'(x) = 0$ for x in $[1,4]$ when $x = 3/2$;
if $x = 1, 3/2, 4$, then $f(x) = e^{-2}, \dfrac{27}{8}e^{-3}, 64e^{-8}$;
critical point at $x = 3/2$; absolute maximum of $\dfrac{27}{8}e^{-3}$ at $x = 3/2$,
absolute minimum of $64e^{-8}$ at $x = 4$

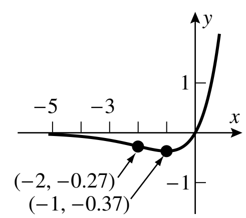


4. $f'(x) = (1-\ln 2x)/x^2$, $f'(x) = 0$ on $[1,e]$ for $x = e/2$;
if $x = 1, e/2, e$ then $f(x) = \ln 2, 2/e, (\ln 2+1)/e$;
absolute minimum of $\dfrac{1+\ln 2}{e}$ at $x = e$,
absolute maximum of $2/e$ at $x = e/2$



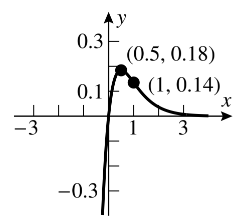
5. **(a)** $\lim\limits_{x\to+\infty} xe^x = +\infty$, $\lim\limits_{x\to-\infty} xe^x = 0$

(b) $y = xe^x$;
$y' = (x+1)e^x$;
$y'' = (x+2)e^x$



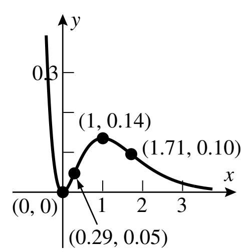
6. **(a)** $\lim\limits_{x\to+\infty} xe^{-2x} = 0$, $\lim\limits_{x\to-\infty} xe^{-2x} = -\infty$

(b) $y = xe^{-2x}$; $y' = -2\left(x - \dfrac{1}{2}\right)e^{-2x}$; $y'' = 4(x-1)e^{-2x}$



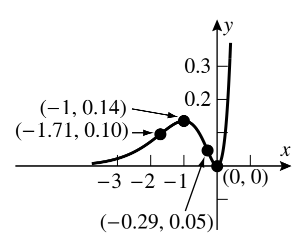
7. **(a)** $\lim\limits_{x\to+\infty} \dfrac{x^2}{e^{2x}} = 0$, $\lim\limits_{x\to-\infty} \dfrac{x^2}{e^{2x}} = +\infty$

(b) $y = x^2/e^{2x} = x^2e^{-2x}$;
$y' = 2x(1-x)e^{-2x}$;
$y'' = 2(2x^2-4x+1)e^{-2x}$;
$y'' = 0$ if $2x^2-4x+1 = 0$, when
$x = \dfrac{4 \pm \sqrt{16-8}}{4} = 1 \pm \sqrt{2}/2 \approx 0.29, 1.71$

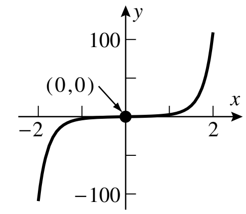


8. **(a)** $\lim\limits_{x\to+\infty} x^2e^{2x} = +\infty$, $\lim\limits_{x\to-\infty} x^2e^{2x} = 0$.

(b) $y = x^2e^{2x}$;
$y' = 2x(x+1)e^{2x}$;
$y'' = 2(2x^2+4x+1)e^{2x}$;
$y'' = 0$ if $2x^2+4x+1 = 0$, when
$x = \dfrac{-4 \pm \sqrt{16-8}}{4} = -1 \pm \sqrt{2}/2 \approx -0.29, -1.71$

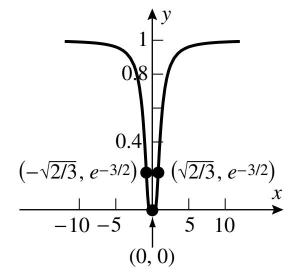


9. **(a)** $\lim_{x\to+\infty} f(x) = +\infty$, $\lim_{x\to-\infty} f(x) = -\infty$



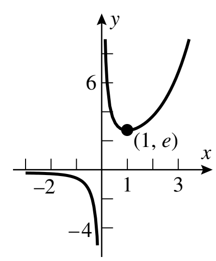
(b) $y = xe^{x^2}$;
$y' = (1+2x^2)e^{x^2}$;
$y'' = 2x(3+2x^2)e^{x^2}$
no relative extrema, inflection point at $(0,0)$

10. **(a)** $\lim_{x\to\pm\infty} f(x) = 1$



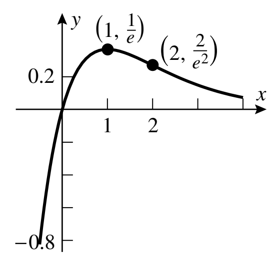
(b) $f'(x) = 2x^{-3}e^{-1/x^2}$ so $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > 0$. Set $u = x^2$ and use the given result to find $\lim_{x\to 0} f'(x) = 0$, so (by the First Derivative Test) $f(x)$ has a minimum at $x = 0$. $f''(x) = (-6x^{-4} + 4x^{-6})e^{-1/x^2}$, so $f(x)$ has points of inflection at $x = \pm\sqrt{2/3}$.

11. $\lim_{x\to+\infty} f(x) = +\infty$, $\lim_{x\to-\infty} f(x) = 0$



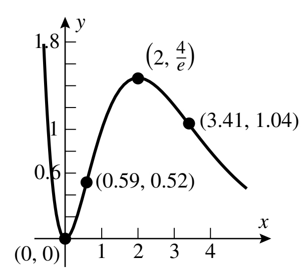
$f'(x) = e^x\dfrac{x-1}{x^2}$, $f''(x) = e^x\dfrac{x^2-2x+2}{x^3}$
critical point at $x = 1$;
relative minimum at $x = 1$
no points of inflection
vertical asymptote $x = 0$,
horizontal asymptote $y = 0$ for $x \to -\infty$

12. $\lim_{x\to+\infty} f(x) = 0$, $\lim_{x\to-\infty} f(x) = -\infty$



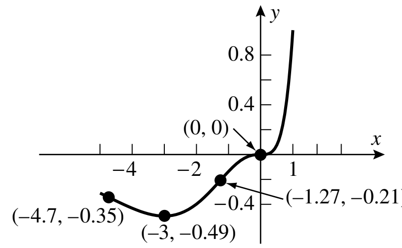
$f'(x) = (1-x)e^{-x}$, $f''(x) = (x-2)e^{-x}$
critical point at $x = 1$; relative maximum at $x = 1$
point of inflection at $x = 2$
horizontal asymptote $y = 0$ as $x \to +\infty$

13. $\lim_{x\to+\infty} f(x) = 0$, $\lim_{x\to-\infty} f(x) = +\infty$



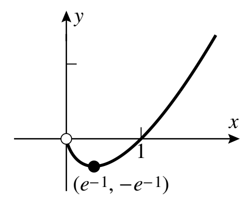
$f'(x) = x(2-x)e^{1-x}$, $f''(x) = (x^2-4x+2)e^{1-x}$
critical points at $x = 0, 2$;
relative minimum at $x = 0$,
relative maximum at $x = 2$
points of inflection at $x = 2 \pm \sqrt{2} \approx 0.59, 3.41$
horizontal asymptote $y = 0$ as $x \to +\infty$

14. $\lim\limits_{x\to+\infty} f(x) = +\infty,\ \lim\limits_{x\to-\infty} f(x) = 0$
$f'(x) = x^2(3+x)e^{x-1},\ f''(x) = x(x^2+6x+6)e^{x-1}$
critical points at $x = -3, 0$;
relative minimum at $x = -3$
points of inflection at $x = 0, -3 \pm \sqrt{3} \approx 0, -4.7, -1.27$
horizontal asymptote $y = 0$ as $x \to -\infty$



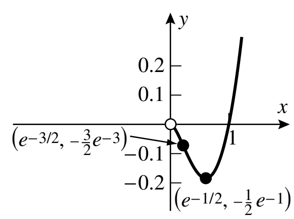
15. **(a)** $\lim\limits_{x\to0^+} y = \lim\limits_{x\to0^+} x\ln x = \lim\limits_{x\to0^+} \dfrac{\ln x}{1/x} = \lim\limits_{x\to0^+} \dfrac{1/x}{-1/x^2} = 0$;
$\lim\limits_{x\to+\infty} y = +\infty$

(b) $y = x\ln x$,
$y' = 1 + \ln x$,
$y'' = 1/x$,
$y' = 0$ when $x = e^{-1}$



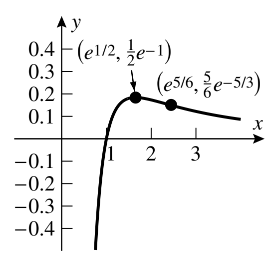
16. **(a)** $\lim\limits_{x\to0^+} y = \lim\limits_{x\to0^+} \dfrac{\ln x}{1/x^2} = \lim\limits_{x\to0^+} \dfrac{1/x}{-2/x^3} = 0$,
$\lim\limits_{x\to+\infty} y = +\infty$

(b) $y = x^2\ln x, y' = x(1+2\ln x)$,
$y'' = 3 + 2\ln x$,
$y' = 0$ if $x = e^{-1/2}$,
$y'' = 0$ if $x = e^{-3/2}$,
$\lim\limits_{x\to0^+} y' = 0$



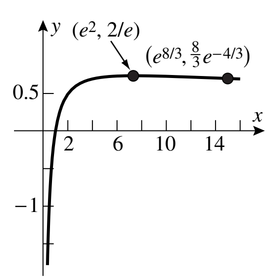
17. **(a)** $\lim\limits_{x\to0^+} y = \lim\limits_{x\to0^+} \dfrac{\ln x}{x^2} = -\infty$;
$\lim\limits_{x\to+\infty} y = \lim\limits_{x\to+\infty} \dfrac{\ln x}{x^2} = \lim\limits_{x\to+\infty} \dfrac{1/x}{2x} = 0$

(b) $y = \dfrac{\ln x}{x^2}, y' = \dfrac{1-2\ln x}{x^3}$,
$y'' = \dfrac{6\ln x - 5}{x^4}$,
$y' = 0$ if $x = e^{1/2}$,
$y'' = 0$ if $x = e^{5/6}$



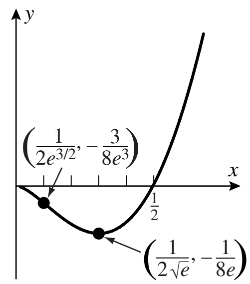
18. **(a)** Let $u = 1/x$, $\lim\limits_{x\to0^+} (\ln x)/\sqrt{x} = \lim\limits_{u\to+\infty} -\sqrt{u}\ln u = -\infty$ by inspection,
$\lim\limits_{x\to+\infty} (\ln x)/\sqrt{x} = 0$, by the rule given.

(b) $y = \dfrac{\ln x}{\sqrt{x}},\ y' = \dfrac{2-\ln x}{2x^{3/2}}$
$y'' = \dfrac{-8+3\ln x}{4x^{5/2}}$
$y' = 0$ if $x = e^2$,
$y'' = 0$ if $x = e^{8/3}$



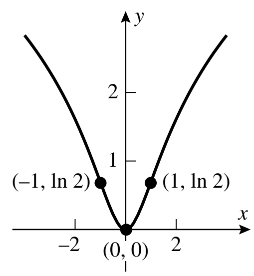
19. **(a)** $\lim_{x\to 0^+} x^2 \ln x = 0$ by the rule given, $\lim_{x\to+\infty} x^2 \ln x = +\infty$ by inspection, and $f(x)$ not defined for $x < 0$

(b) $y = x^2 \ln 2x$, $y' = 2x \ln 2x + x$
$y'' = 2\ln 2x + 3$
$y' = 0$ if $x = 1/(2\sqrt{e})$,
$y'' = 0$ if $x = 1/(2e^{3/2})$

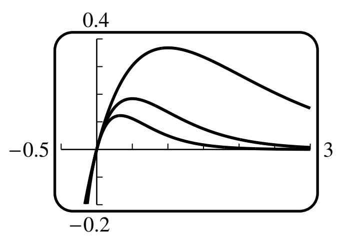


20. **(a)** $\lim_{x\to+\infty} f(x) = +\infty$; $\lim_{x\to 0} f(x) = 0$

(b) $y = \ln(x^2+1)$, $y' = 2x/(x^2+1)$
$y'' = -2\dfrac{x^2-1}{(x^2+1)^2}$
$y' = 0$ if $x = 0$
$y'' = 0$ if $x = \pm 1$

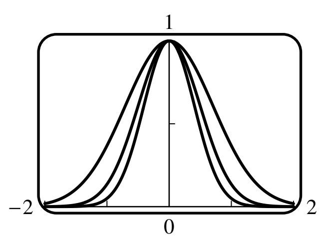


21. **(a)**



(b) $y' = (1 - bx)e^{-bx}$, $y'' = b^2(x - 2/b)e^{-bx}$; relative maximum at $x = 1/b$, $y = 1/be$; point of inflection at $x = 2/b$, $y = 2/be^2$. Increasing b moves the relative maximum and the point of inflection to the left and down, i.e. towards the origin.

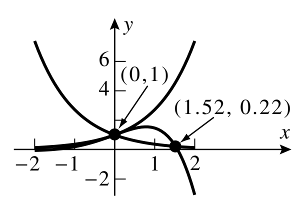
22. **(a)**



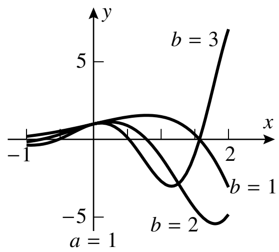
(b) $y' = -2bxe^{-bx^2}$,
$y'' = 2b(-1 + 2bx^2)e^{-bx^2}$;
relative maximum at $x = 0$, $y = 1$; points of inflection at $x = \pm\sqrt{1/2b}$, $y = 1/\sqrt{e}$. Increasing b moves the points of inflection towards the y-axis; the relative maximum doesn't move.

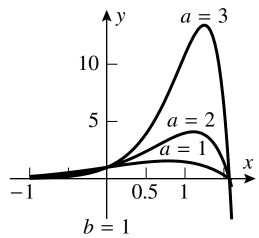
23. **(a)** The oscillations of $e^x \cos x$ about zero increase as $x \to +\infty$ so the limit does not exist, and $\lim_{x\to-\infty} e^x \cos x = 0$.

(b)



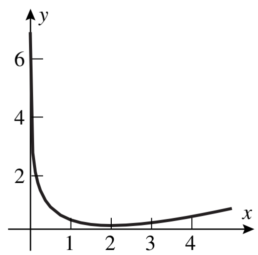
(c) The curve $y = e^{ax}\cos bx$ oscillates between $y = e^{ax}$ and $y = -e^{ax}$. The frequency of oscillation increases when b increases.





24. Let $P(x_0, y_0)$ be a point on $y = e^{3x}$ then $y_0 = e^{3x_0}$. $dy/dx = 3e^{3x}$ so $m_{\tan} = 3e^{3x_0}$ at P and an equation of the tangent line at P is $y - y_0 = 3e^{3x_0}(x - x_0)$, $y - e^{3x_0} = 3e^{3x_0}(x - x_0)$. If the line passes through the origin then $(0, 0)$ must satisfy the equation so $-e^{3x_0} = -3x_0e^{3x_0}$ which gives $x_0 = 1/3$ and thus $y_0 = e$. The point is $(1/3, e)$.

25. **(b)**



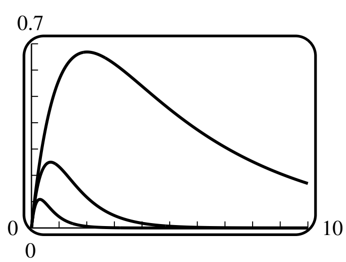
(c) $\dfrac{dy}{dx} = \dfrac{1}{2} - \dfrac{1}{x}$ so $\dfrac{dy}{dx} < 0$ at $x = 1$ and $\dfrac{dy}{dx} > 0$ at $x = e$

(d) The slope is a continuous function which goes from a negative value to a positive value; therefore it must take the value zero in between, by the Intermediate Value Theorem.

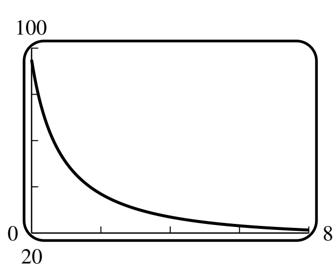
(e) $\dfrac{dy}{dx} = 0$ when $x = 2$

26. **(a)** $\dfrac{dC}{dt} = \dfrac{K}{a-b}\left(ae^{-at} - be^{-bt}\right)$ so $\dfrac{dC}{dt} = 0$ at $t = \dfrac{\ln(a/b)}{a-b}$. This is the only stationary point and $C(0) = 0$, $\lim\limits_{t\to+\infty} C(t) = 0$, $C(t) > 0$ for $0 < t < +\infty$, so it is an absolute maximum.

(b)



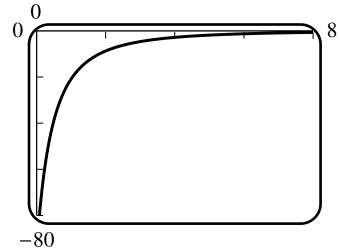
27. **(a)**



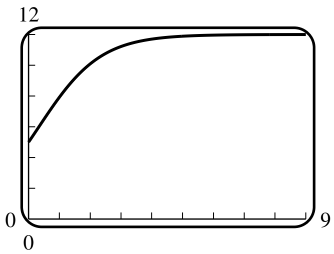
(b) as t tends to $+\infty$, the population tends to 19

$$\lim_{t\to+\infty} P(t) = \lim_{t\to+\infty} \frac{95}{5-4e^{-t/4}} = \frac{95}{5-4\lim\limits_{t\to+\infty} e^{-t/4}} = \frac{95}{5} = 19$$

(c) the rate of population growth tends to zero

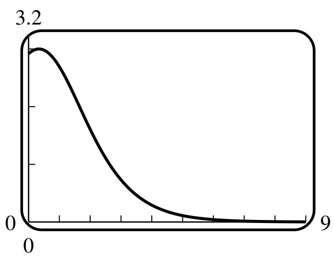


28. **(a)**

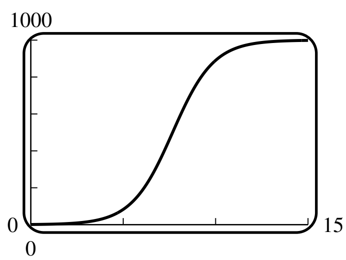


(b) P tends to 12 as t gets large; $\lim\limits_{t\to+\infty} P(t) = \lim\limits_{t\to+\infty} \dfrac{60}{5+7e^{-t}} = \dfrac{60}{5+7\lim\limits_{t\to+\infty} e^{-t}} = \dfrac{60}{5} = 12$

(c) the rate of population growth tends to zero



29. $t = 7.67$



30. **(a)** $\dfrac{dN}{dt} = 250(20-t)e^{-t/20} = 0$ at $t = 20$, $N(0) = 125{,}000$, $N(20) \approx 161{,}788$, and $N(100) \approx 128{,}369$; the absolute maximum is $N = 161{,}788$ at $t = 20$, the absolute minimum is $N = 125{,}000$ at $t = 0$.

(b) The absolute minimum of $\dfrac{dN}{dt}$ occurs when $\dfrac{d^2N}{dt^2} = 12.5(t-40)e^{-t/20} = 0$, $t = 40$.

31. **(a)** $y'(t) = \dfrac{LAke^{-kt}}{(1+Ae^{-kt})^2}S$, so $y'(0) = \dfrac{LAk}{(1+A)^2}$

(b) The rate of growth increases to its maximum, which occurs when y is halfway between 0 and L, or when $t = \dfrac{1}{k}\ln A$; it then decreases back towards zero.

(c) From (2) one sees that $\dfrac{dy}{dt}$ is maximized when y lies half way between 0 and L, i.e. $y = L/2$. This follows since the right side of (2) is a parabola (with y as independent variable) with y-intercepts $y = 0, L$. The value $y = L/2$ corresponds to $t = \dfrac{1}{k}\ln A$, from (4).

32. Since $0 < y < L$ the right-hand side of (3) can change sign only if the factor $L - 2y$ changes sign, which it does when $y = L/2$. From (1) we have $\dfrac{L}{2} = \dfrac{L}{1+Ae^{-kt}}$, $1 = Ae^{-kt}$, $t = \dfrac{1}{k}\ln A$.

33. $\dfrac{dk}{dT} = k_0 \exp\left[-\dfrac{q(T-T_0)}{2T_0T}\right]\left(-\dfrac{q}{2T^2}\right) = -\dfrac{qk_0}{2T^2}\exp\left[-\dfrac{q(T-T_0)}{2T_0T}\right]$

34. $\beta = 10\log I - 10\log I_0$, $\dfrac{d\beta}{dI} = \dfrac{10}{I\ln 10}$

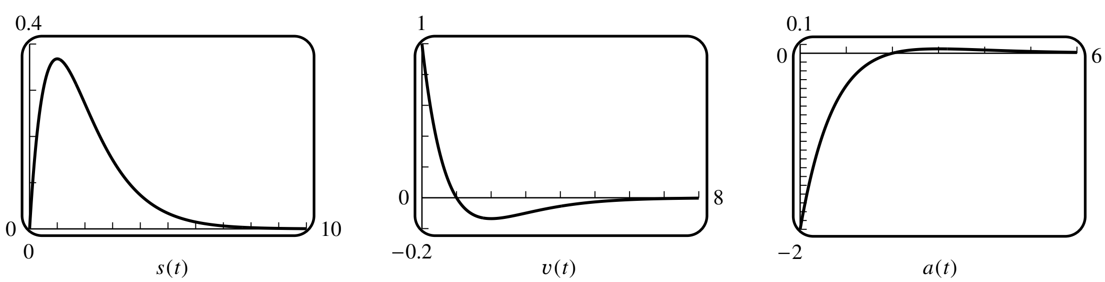
(a) $\left.\dfrac{d\beta}{dI}\right]_{I=10I_0} = \dfrac{1}{I_0\ln 10}$ db/W/m^2

(b) $\left.\dfrac{d\beta}{dI}\right]_{I=100I_0} = \dfrac{1}{10I_0\ln 10}$ db/W/m^2

(c) $\left.\dfrac{d\beta}{dI}\right]_{I=100I_0} = \dfrac{1}{100I_0\ln 10}$ db/W/m^2

35. Solve $\dfrac{dy}{dt} = 3\dfrac{dx}{dt}$ given $y = x\ln x$. Then $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} = (1+\ln x)\dfrac{dx}{dt}$, so $1+\ln x = 3$, $\ln x = 2$, $x = e^2$.

36. $v(t) = (1-t)e^{-t}$, $a(t) = (t-2)e^{-t}$



(a) $v = 0$ at $t = 1$

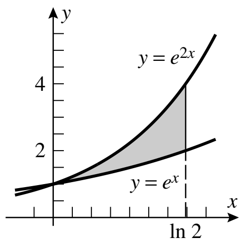
(b) $s = 1/e$ at $t = 1$

(c) a changes sign at $t = 2$, so the particle is speeding up for $1 < t < 2$ and slowing down for $0 < t < 1$ and $2 < t$

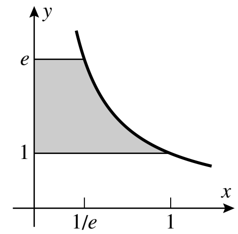
37. $\displaystyle\int_1^3 e^x dx = e^x\Big]_1^3 = e^3 - e$

38. $\displaystyle\int_1^5 \frac{1}{x}dx = \ln x\Big]_1^5 = \ln 5 - \ln 1 = \ln 5$

39. $A = \int_0^{\ln 2} \left(e^{2x} - e^x\right)\, dx$
$= \left(\frac{1}{2}e^{2x} - e^x\right)\Big]_0^{\ln 2} = 1/2$



40. $A = \int_1^e \frac{dy}{y} = \ln y\Big]_1^e = 1$



41. $A = A_1 + A_2 = \int_{-1}^0 (1 - e^x)dx + \int_0^1 (e^x - 1)dx = 1/e + e - 2$

42. $A = A_1 + A_2 = \int_{1/2}^1 \frac{1-x}{x}dx + \int_1^2 \frac{x-1}{x}dx = -\left(\frac{1}{2} - \ln 2\right) + (1 - \ln 2) = 1/2$

43. $f_{\text{ave}} = \frac{1}{e-1}\int_1^e \frac{1}{x}dx = \frac{1}{e-1}(\ln e - \ln 1) = \frac{1}{e-1}$

44. $f_{\text{ave}} = \frac{1}{\ln 5 - (-1)}\int_{-1}^{\ln 5} e^x dx = \frac{1}{\ln 5 + 1}(5 - e^{-1}) = \frac{5 - e^{-1}}{1 + \ln 5}$

45. $f_{\text{ave}} = \frac{1}{4-0}\int_0^4 e^{-2x}dx = -\frac{1}{8}e^{-2x}\Big]_0^4 = \frac{1 - e^{-8}}{8}$

46. $\int_0^k e^{2x}dx = 3,\ \frac{1}{2}e^{2x}\Big]_0^k = 3,\ \frac{1}{2}(e^{2k} - 1) = 3, e^{2k} = 7, k = \frac{1}{2}\ln 7$

47. $y(t) = (802.137)\int e^{1.528t}dt = 524.959e^{1.528t} + C;\ y(0) = 750 = 524.959 + C,\ C = 225.041,$
$y(t) = 524.959e^{1.528t} + 225.041,\ y(12) = 48,233,525,650$

48. $V_{\text{ave}} = \frac{275000}{10-0}\int_0^{10} e^{-0.17t}dt = -161764.7059e^{-0.17t}\Big]_0^{10} = \$132,212.96$

49. $s(t) = \int (25 + 10e^{-0.05t})dt = 25t - 200e^{-0.05t} + C$

(a) $s(10) - s(0) = 250 - 200(e^{-0.5} - 1) = 450 - 200/\sqrt{e} \approx 328.69$ ft

(b) yes; without it the distance would have been 250 ft

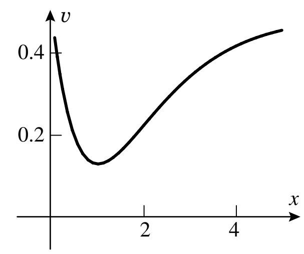
50. **(a)** displacement $= \int_0^3 (e^t - 2)dt = e^3 - 7$

distance $= \int_0^3 |v(t)|dt = -\int_0^{\ln 2} v(t)dt + \int_{\ln 2}^3 v(t)dt = e^3 - 9 + 4\ln 2$

(b) displacement $= \int_1^3 \left(\frac{1}{2} - \frac{1}{t}\right) dt = 1 - \ln 3$

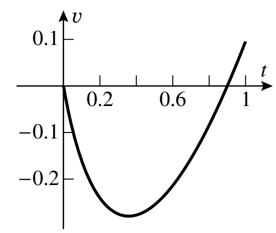
distance $= \int_1^3 |v(t)|dt = -\int_1^2 v(t)dt + \int_2^3 v(t)dt = 2\ln 2 - \ln 3$

51. **(a)** From the graph the velocity is positive, so the displacement is always increasing and is therefore positive.



(b) $s(t) = t/2 + (t+1)e^{-t}$

52. **(a)** If $t_0 < 1$ then the area between the velocity curve and the t-axis, between $t = 0$ and $t = t_0$, will always be negative, so the displacement will be negative.



(b) $s(t) = \left(\frac{t^2}{2} - \frac{1}{200}\right)\ln(t+0.1) - \frac{t^2}{4} + \frac{t}{20} - \frac{1}{200}\ln 10$

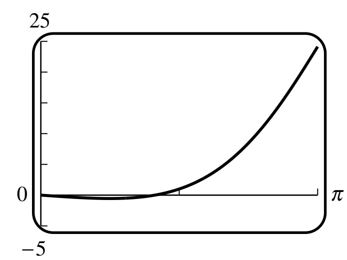
53. $x = 0$; also set

$f(x) = 1 - e^x \cos x$, $f'(x) = e^x(\sin x - \cos x)$,

$$x_{n+1} = x_n - \frac{1 - e^x \cos x}{e^x(\sin x - \cos x)}$$

$x_1 = 1$, $x_2 = 1.572512605$,

$x_3 = 1.363631415$, $x_7 = x_8 = 1.292695719$

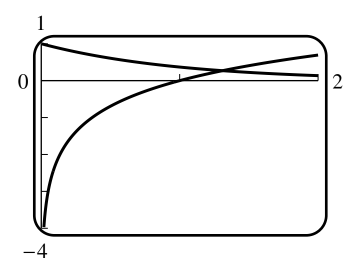


54. The graphs of $y = e^{-x}$ and $y = \ln x$ intersect near $x = 1.3$;

let $f(x) = e^{-x} - \ln x$, $f'(x) = -e^{-x} - 1/x$, $x_1 = 1.3$,

$x_{n+1} = x_n + \frac{e^{-x_n} - \ln x_n}{e^{-x_n} + 1/x_n}$, $x_2 = 1.309759929$,

$x_4 = x_5 = 1.309799586$



55. A graphing utility shows that there are two inflection points at $x \approx -0.25, 1.25$. These points are the zeros of $f''(x) = (x^4 - 4x^3 + 8x^2 - 4x - 1)\frac{e^x}{(x^2+1)^3}$. It is equivalent to find the zeros of $g(x) = x^4 - 4x^3 + 8x^2 - 4x - 1$. One root is $x = 1$ by inspection. Since $g'(x) = 4x^3 - 12x^2 + 16x - 4$, Newton's Method becomes

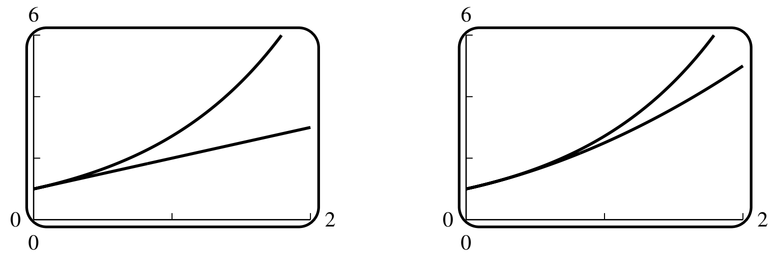
$$x_n = x_{n-1} - \frac{x_{n-1}^4 - 4x_{n-1}^3 + 8x_{n-1}^2 - 4x_{n-1} - 1}{4x_{n-1}^3 - 12x_{n-1}^2 + 16x_{n-1} - 4}$$

With $x_0 = -0.25$, $x_1 = -0.18572695, x_2 = -0.179563312, x_3 = -0.179509029$, $x_4 = x_5 = -0.179509025$. So the points of inflection are at $x \approx -0.18, x = 1$.

56. **(a)** Let $h(x) = e^x - 1 - x$ for $x \geq 0$. Then $h(0) = 0$ and $h'(x) = e^x - 1 \geq 0$ for $x \geq 0$, so $h(x)$ is increasing.

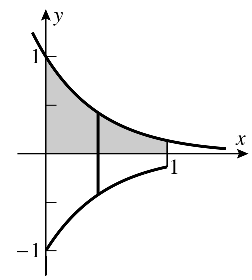
(b) Let $h(x) = e^x - 1 - x - \frac{1}{2}x^2$. Then $h(0) = 0$ and $h'(x) = e^x - 1 - x$. By Part (a), $e^x - 1 - x \geq 0$ for $x \geq 0$, so $h(x)$ is increasing.

(c)



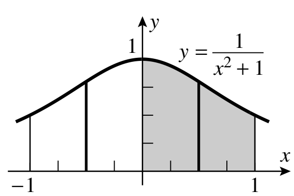
57. $V = \pi \int_0^{\ln 3} e^{2x} dx = \frac{\pi}{2} e^{2x} \Big]_0^{\ln 3} = 4\pi$

58. $V = \pi \int_0^1 e^{-4x}\, dx = \frac{\pi}{4}(1 - e^{-4})$

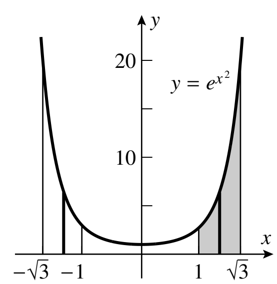


59. $V = 2\pi \int_0^1 \frac{x}{x^2+1}\, dx$

$= \pi \ln(x^2+1) \Big]_0^1 = \pi \ln 2$



60. $V = \int_1^{\sqrt{3}} 2\pi x e^{x^2}\, dx = \pi e^{x^2} \Big]_1^{\sqrt{3}} = \pi(e^3 - e)$



61. $(dx/dt)^2 + (dy/dt)^2 = [e^t(\cos t - \sin t)]^2 + [e^t(\cos t + \sin t)]^2 = 2e^{2t}$,

$L = \int_0^{\pi/2} \sqrt{2} e^t dt = \sqrt{2}(e^{\pi/2} - 1)$

62. $(dx/dt)^2 + (dy/dt)^2 = (2e^t \cos t)^2 + (-2e^t \sin t)^2 = 4e^{2t}$, $L = \int_1^4 2e^t dt = 2(e^4 - e)$

63. $dy/dx = \dfrac{\sec x \tan x}{\sec x} = \tan x$, $\sqrt{1+(y')^2} = \sqrt{1+\tan^2 x} = \sec x$ when $0 < x < \pi/4$, so

$$L = \int_0^{\pi/4} \sec x \, dx = \ln(1+\sqrt{2})$$

64. $dy/dx = \dfrac{\cos x}{\sin x} = \cot x$, $\sqrt{1+(y')^2} = \sqrt{1+\cot^2 x} = \csc x$ when $\pi/4 < x < \pi/2$, so

$$L = \int_{\pi/4}^{\pi/2} \csc x \, dx = -\ln(\sqrt{2}-1) = -\ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}(\sqrt{2}+1)\right) = \ln(1+\sqrt{2})$$

65. $f'(x) = e^x$, $\quad 1 + [f'(x)]^2 = 1 + e^{2x}$, $S = \int_0^1 2\pi e^x \sqrt{1+e^{2x}} \, dx \approx 22.94$

66. $x = g(y) = \ln y$, $g'(y) = 1/y$, $1 + [g'(y)]^2 = 1 + 1/y^2$; $S = \int_1^e 2\pi \sqrt{1+1/y^2} \ln y \, dy \approx 7.05$

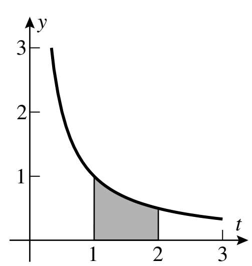
67. $x' = e^t(\cos t - \sin t), y' = e^t(\cos t + \sin t), (x')^2 + (y')^2 = 2e^{2t}$

$$S = 2\pi \int_0^{\pi/2} (e^t \sin t)\sqrt{2e^{2t}} dt = 2\sqrt{2}\pi \int_0^{\pi/2} e^{2t} \sin t \, dt$$

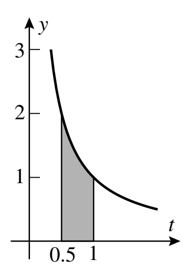
$$= 2\sqrt{2}\pi \left[\frac{1}{5}e^{2t}(2\sin t - \cos t)\right]_0^{\pi/2} = \frac{2\sqrt{2}}{5}\pi(2e^\pi + 1)$$

EXERCISE SET 7.5

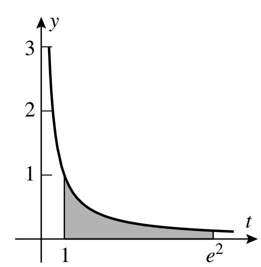
1. **(a)**



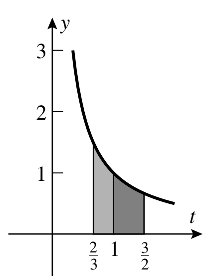
(b)



(c)



2.



3. **(a)** $\ln t\Big]_1^{ac} = \ln(ac) = \ln a + \ln c = 7$ **(b)** $\ln t\Big]_1^{1/c} = \ln(1/c) = -5$

(c) $\ln t\Big]_1^{a/c} = \ln(a/c) = 2 - 5 = -3$ **(d)** $\ln t\Big]_1^{a^3} = \ln a^3 = 3\ln a = 6$

4. **(a)** $\ln t\Big]_1^{\sqrt{a}} = \ln a^{1/2} = \frac{1}{2}\ln a = 9/2$ **(b)** $\ln t\Big]_1^{2a} = \ln 2 + 9$

(c) $\ln t\Big]_1^{2/a} = \ln 2 - 9$ **(d)** $\ln t\Big]_2^{a} = 9 - \ln 2$

5. $\ln 5 \approx 1.603210678$; $\ln 5 = 1.609437912$; magnitude of error is < 0.0063

6. $\ln 3 \approx 1.098242635$; $\ln 3 = 1.098612289$; magnitude of error is < 0.0004

7. **(a)** x^{-1}, $x > 0$ **(b)** x^2, $x \neq 0$

(c) $-x^2$, $-\infty < x < +\infty$ **(d)** $-x$, $-\infty < x < +\infty$

(e) x^3, $x > 0$ **(f)** $\ln x + x$, $x > 0$

(g) $x - \sqrt[3]{x}$, $-\infty < x < +\infty$ **(h)** $\frac{e^x}{x}$, $x > 0$

8. **(a)** $f(\ln 3) = e^{-2\ln 3} = e^{\ln(1/9)} = 1/9$

(b) $f(\ln 2) = e^{\ln 2} + 3e^{-\ln 2} = 2 + 3e^{\ln(1/2)} = 2 + 3/2 = 7/2$

9. **(a)** $3^\pi = e^{\pi \ln 3}$ **(b)** $2^{\sqrt{2}} = e^{\sqrt{2}\ln 2}$

10. **(a)** $\pi^{-x} = e^{-x\ln\pi}$ **(b)** $x^{2x} = e^{2x\ln x}$

11. **(a)** $\lim_{x\to+\infty}\left[\left(1+\frac{1}{x}\right)^x\right]^2 = \left[\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x\right]^2 = e^2$

(b) $y = 2x$, $\lim_{y\to 0}(1+y)^{2/y} = \lim_{y\to 0}\left[(1+y)^{1/y}\right]^2 = e^2$

12. **(a)** $y = 3x$, $\lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^{y/3} = \lim_{y\to+\infty}\left[\left(1+\frac{1}{y}\right)^y\right]^{1/3} = \left[\lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^y\right]^{1/3} = e^{1/3}$

(b) $\lim_{x\to 0}(1+x)^{1/3x} = \lim_{x\to 0}\left[(1+x)^{1/x}\right]^{1/3} = e^{1/3}$

13. $g'(x) = x^2 - x$ **14.** $g'(x) = 1 - \cos x$

15. **(a)** $\frac{1}{x^3}(3x^2) = \frac{3}{x}$ **(b)** $e^{\ln x}\frac{1}{x} = 1$

16. **(a)** $2x\sqrt{x^2+1}$ **(b)** $-\left(\frac{1}{x^2}\right)\sin\left(\frac{1}{x}\right)$

17. $F'(x) = \frac{\cos x}{x^2+3}$, $F''(x) = \frac{-(x^2+3)\sin\ x - 2x\cos x}{(x^2+3)^2}$

(a) 0 **(b)** 1/3 **(c)** 0

18. $F'(x) = \sqrt{3x^2+1}$, $F''(x) = \frac{3x}{\sqrt{3x^2+1}}$

(a) 0 **(b)** $\sqrt{13}$ **(c)** $6/\sqrt{13}$

19. **(a)** $\frac{d}{dx}\int_1^{x^2} t\sqrt{1+t}dt = x^2\sqrt{1+x^2}(2x) = 2x^3\sqrt{1+x^2}$

(b) $\int_1^{x^2} t\sqrt{1+t}dt = -\frac{2}{3}(x^2+1)^{3/2} + \frac{2}{5}(x^2+1)^{5/2} - \frac{4\sqrt{2}}{15}$

20. **(a)** $\frac{d}{dx}\int_x^a f(t)dt = -\frac{d}{dx}\int_a^x f(t)dt = -f(x)$

(b) $\frac{d}{dx}\int_{g(x)}^a f(t)dt = -\frac{d}{dx}\int_a^{g(x)} f(t)dt = -f(g(x))g'(x)$

21. **(a)** $-\sin x^2$ **(b)** $-\frac{\tan^2 x}{1+\tan^2 x}\sec^2 x = -\tan^2 x$

22. **(a)** $-(x^2+1)^{40}$ **(b)** $-\cos^3\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) = \frac{\cos^3(1/x)}{x^2}$

23. $-3\frac{3x-1}{9x^2+1} + 2x\frac{x^2-1}{x^4+1}$

24. If f is continuous on an open interval I and $g(x)$, $h(x)$, and a are in I then

$$\int_{h(x)}^{g(x)} f(t)dt = \int_{h(x)}^{a} f(t)dt + \int_a^{g(x)} f(t)dt = -\int_a^{h(x)} f(t)dt + \int_a^{g(x)} f(t)dt$$

so $\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)dt = -f(h(x))h'(x) + f(g(x))g'(x)$

25. **(a)** $\sin^2(x^3)(3x^2) - \sin^2(x^2)(2x) = 3x^2\sin^2(x^3) - 2x\sin^2(x^2)$

(b) $\frac{1}{1+x}(1) - \frac{1}{1-x}(-1) = \frac{2}{1-x^2}$

26. $F'(x) = \frac{1}{3x}(3) - \frac{1}{x}(1) = 0$ so $F(x)$ is constant on $(0,+\infty)$. $F(1) = \ln 3$ so $F(x) = \ln 3$ for all $x > 0$.

27. from geometry, $\int_0^3 f(t)dt = 0$, $\int_3^5 f(t)dt = 6$, $\int_5^7 f(t)dt = 0$; and $\int_7^{10} f(t)dt$

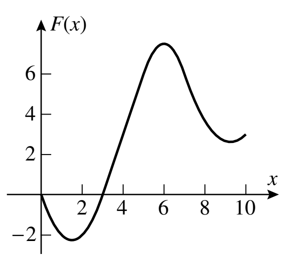
$$= \int_7^{10} (4t-37)/3dt = -3$$

(a) $F(0) = 0$, $F(3) = 0$, $F(5) = 6$, $F(7) = 6$, $F(10) = 3$

(b) F is increasing where $F' = f$ is positive, so on $[3/2, 6]$ and $[37/4, 10]$, decreasing on $[0, 3/2]$ and $[6, 37/4]$

(c) critical points when $F'(x) = f(x) = 0$, so $x = 3/2, 6, 37/4$; maximum $15/2$ at $x = 6$, minimum $-9/4$ at $x = 3/2$

(d)



28. $f_{\text{ave}} = \dfrac{1}{10-0}\displaystyle\int_0^{10} f(t)dt = \dfrac{1}{10}F(10) = 0.3$

29. $x < 0 : F(x) = \displaystyle\int_{-1}^{x} (-t)dt = -\dfrac{1}{2}t^2\Big]_{-1}^{x} = \dfrac{1}{2}(1-x^2),$

$x \geq 0 : F(x) = \displaystyle\int_{-1}^{0} (-t)dt + \int_0^x t\,dt = \dfrac{1}{2} + \dfrac{1}{2}x^2;\ F(x) = \begin{cases} (1-x^2)/2, & x < 0 \\ (1+x^2)/2, & x \geq 0 \end{cases}$

30. $0 \leq x \leq 2 : F(x) = \displaystyle\int_0^x t\,dt = \dfrac{1}{2}x^2,$

$x > 2 : F(x) = \displaystyle\int_0^2 t\,dt + \int_2^x 2\,dt = 2 + 2(x-2) = 2x-2;\ F(x) = \begin{cases} x^2/2, & 0 \leq x \leq 2 \\ 2x-2, & x > 2 \end{cases}$

31. $y(x) = 2 + \displaystyle\int_1^x t^{1/3}dt = 2 + \dfrac{3}{4}t^{4/3}\Big]_1^x = \dfrac{5}{4} + \dfrac{3}{4}x^{4/3}$

32. $y(x) = \displaystyle\int_1^x (t^{1/2} + t^{-1/2})dt = \dfrac{2}{3}x^{3/2} - \dfrac{2}{3} + 2x^{1/2} - 2 = \dfrac{2}{3}x^{3/2} + 2x^{1/2} - \dfrac{8}{3}$

33. $y(x) = 1 + \displaystyle\int_{\pi/4}^{x} (\sec^2 t - \sin t)dt = \tan x + \cos x - \sqrt{2}/2$

34. $y(x) = \displaystyle\int_0^x te^{t^2}dt = \dfrac{1}{2}e^{-x^2} - \dfrac{1}{2}$

35. $P(x) = P_0 + \displaystyle\int_0^x r(t)dt$ individuals

36. $s(T) = s_1 + \displaystyle\int_1^T v(t)dt$

37. II has a minimum at $x = 12$, and I has a zero there, so I could be the derivative of II; on the other hand I has a minimum near $x = 1/3$, but II is not zero there, so II could not be the derivative of I, so I is the graph of $f(x)$ and II is the graph of $\int_0^x f(t)\,dt$.

38. **(b)** $\lim\limits_{k\to 0} \dfrac{1}{k}(x^k - 1) = \dfrac{d}{dt}x^t\Big]_{t=0} = \ln x$

39. **(a)** where $f(t) = 0$; by the First Derivative Test, at $t = 3$

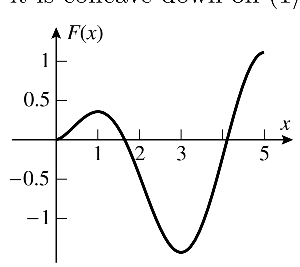
(b) where $f(t) = 0$; by the First Derivative Test, at $t = 1, 5$

(c) at $t = 0, 1$ or 5; from the graph it is evident that it is at $t = 5$

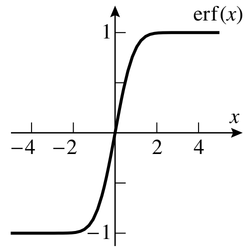
(d) at $t = 0, 3$ or 5; from the graph it is evident that it is at $t = 3$

(e) F is concave up when $F'' = f'$ is positive, i.e. where f is increasing, so on $(0, 1/2)$ and $(2, 4)$; it is concave down on $(1/2, 2)$ and $(4, 5)$

(f)



40. **(a)**



(c) $\text{erf}'(x) > 0$ for all x, so there are no relative extrema

(e) $\text{erf}''(x) = -4xe^{-x^2}/\sqrt{\pi}$ changes sign only at $x = 0$ so that is the only point of inflection

(g) $\lim_{x\to+\infty} \text{erf}(x) = +1$, $\lim_{x\to-\infty} \text{erf}(x) = -1$

41. $C'(x) = \cos(\pi x^2/2)$, $C''(x) = -\pi x \sin(\pi x^2/2)$

(a) $\cos t$ goes from negative to positive at $2k\pi - \pi/2$, and from positive to negative at $t = 2k\pi + \pi/2$, so $C(x)$ has relative minima when $\pi x^2/2 = 2k\pi - \pi/2$, $x = \pm\sqrt{4k-1}$, $k = 1, 2, \ldots$, and $C(x)$ has relative maxima when $\pi x^2/2 = (4k+1)\pi/2$, $x = \pm\sqrt{4k+1}$, $k = 0, 1, \ldots$.

(b) $\sin t$ changes sign at $t = k\pi$, so $C(x)$ has inflection points at $\pi x^2/2 = k\pi$, $x = \pm\sqrt{2k}$, $k = 1, 2, \ldots$; the case $k = 0$ is distinct due to the factor of x in $C''(x)$, but x changes sign at $x = 0$ and $\sin(\pi x^2/2)$ does not, so there is also a point of inflection at $x = 0$

42. Let $F(x) = \int_1^x \ln t\,dt$, $F'(x) = \lim_{h\to 0} \dfrac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \dfrac{1}{h}\int_x^{x+h} \ln t\,dt$; but $F'(x) = \ln x$ so $\lim_{h\to 0} \dfrac{1}{h}\int_x^{x+h} \ln t\,dt = \ln x$

43. Differentiate: $f(x) = 3e^{3x}$, so $2 + \int_a^x f(t)dt = 2 + \int_a^x 3e^{3t}dt = 2 + e^{3t}\Big]_a^x = 2 + e^{3x} - e^{3a} = e^{3x}$ provided $e^{3a} = 2$, $a = (\ln 2)/3$.

44. **(a)** The area under $1/t$ for $x \le t \le x+1$ is less than the area of the rectangle with altitude $1/x$ and base 1, but greater than the area of the rectangle with altitude $1/(x+1)$ and base 1.

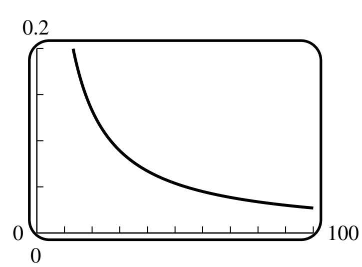
(b) $\int_x^{x+1} \dfrac{1}{t}dt = \ln t\Big]_x^{x+1} = \ln(x+1) - \ln x = \ln(1 + 1/x)$, so $1/(x+1) < \ln(1 + 1/x) < 1/x$ for $x > 0$.

(c) from Part (b), $e^{1/(x+1)} < e^{\ln(1+1/x)} < e^{1/x}$, $e^{1/(x+1)} < 1 + 1/x < e^{1/x}$, $e^{x/(x+1)} < (1 + 1/x)^x < e$; by the Squeezing Theorem, $\lim_{x\to+\infty}(1 + 1/x)^x = e$.

(d) Use the inequality $e^{x/(x+1)} < (1 + 1/x)^x$ to get $e < (1 + 1/x)^{x+1}$ so $(1 + 1/x)^x < e < (1 + 1/x)^{x+1}$.

45. From Exercise 44(d) $\left|e - \left(1 + \dfrac{1}{50}\right)^{50}\right| < y(50)$,

and from the graph $y(50) < 0.06$



46. $F'(x) = f(x)$, thus $F'(x)$ has a value at each x in I because f is continuous on I so F is continuous on I because a function that is differentiable at a point is also continuous at that point

EXERCISE SET 7.6

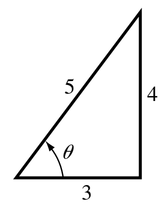
1. **(a)** $-\pi/2$ **(b)** π **(c)** $-\pi/4$ **(d)** 0

2. **(a)** $\pi/3$ **(b)** $\pi/3$ **(c)** $\pi/4$ **(d)** $2\pi/3$

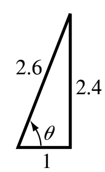
3. $\theta = -\pi/3$; $\cos\theta = 1/2$, $\tan\theta = -\sqrt{3}$, $\cot\theta = -1/\sqrt{3}$, $\sec\theta = 2$, $\csc\theta = -2/\sqrt{3}$

4. $\theta = \pi/3$; $\sin\theta = \sqrt{3}/2$, $\tan\theta = \sqrt{3}$, $\cot\theta = 1/\sqrt{3}$, $\sec\theta = 2$, $\csc\theta = 2/\sqrt{3}$

5. $\tan\theta = 4/3$, $0 < \theta < \pi/2$; use the triangle shown to get $\sin\theta = 4/5$, $\cos\theta = 3/5$, $\cot\theta = 3/4$, $\sec\theta = 5/3$, $\csc\theta = 5/4$



6. $\sec\theta = 2.6, 0 < \theta < \pi/2$; use the triangle shown to get $\sin\theta = 2.4/2.6 = 12/13, \cos\theta = 1/2.6 = 5/13,$ $\tan\theta = 2.4 = 12/5, \cot\theta = 5/12, \csc\theta = 13/12$



7. **(a)** $\pi/7$

(b) $\sin^{-1}(\sin\pi) = \sin^{-1}(\sin 0) = 0$

(c) $\sin^{-1}(\sin(5\pi/7)) = \sin^{-1}(\sin(2\pi/7)) = 2\pi/7$

(d) Note that $\pi/2 < 630 - 200\pi < \pi$ so $\sin(630) = \sin(630 - 200\pi) = \sin(\pi - (630 - 200\pi)) = \sin(201\pi - 630)$ where $0 < 201\pi - 630 < \pi/2$; $\sin^{-1}(\sin 630) = \sin^{-1}(\sin(201\pi - 630)) = 201\pi - 630.$

8. **(a)** $\pi/7$

(b) π

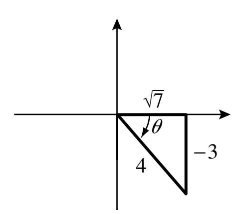
(c) $\cos^{-1}(\cos(12\pi/7)) = \cos^{-1}(\cos(2\pi/7)) = 2\pi/7$

(d) Note that $-\pi/2 < 200 - 64\pi < 0$ so $\cos(200) = \cos(200 - 64\pi) = \cos(64\pi - 200)$ where $0 < 64\pi - 200 < \pi/2$; $\cos^{-1}(\cos 200) = \cos^{-1}(\cos(64\pi - 200)) = 64\pi - 200.$

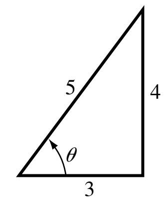
9. **(a)** $0 \le x \le \pi$ **(b)** $-1 \le x \le 1$

(c) $-\pi/2 < x < \pi/2$ **(d)** $-\infty < x < +\infty$

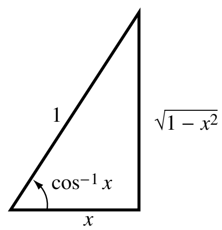
10. Let $\theta = \sin^{-1}(-3/4)$ then $\sin\theta = -3/4$, $-\pi/2 < \theta < 0$ and (see figure) $\sec\theta = 4/\sqrt{7}$



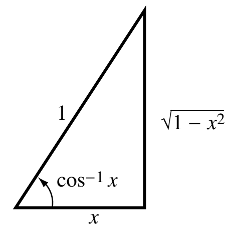
11. Let $\theta = \cos^{-1}(3/5)$, $\sin 2\theta = 2\sin\theta\cos\theta = 2(4/5)(3/5) = 24/25$



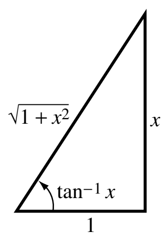
12. **(a)** $\sin(\cos^{-1} x) = \sqrt{1-x^2}$



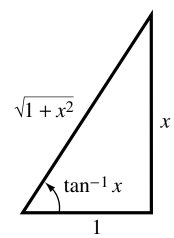
(b) $\tan(\cos^{-1} x) = \dfrac{\sqrt{1-x^2}}{x}$



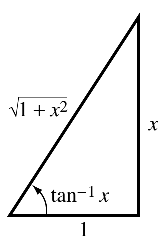
(a) $\csc(\tan^{-1} x) = \dfrac{\sqrt{1+x^2}}{x}$



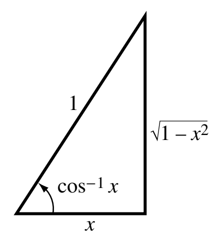
(d) $\sin(\tan^{-1} x) = \dfrac{x}{\sqrt{1+x^2}}$



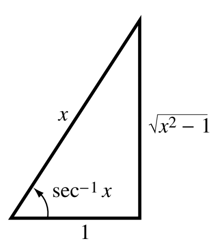
13. **(a)** $\cos(\tan^{-1} x) = \dfrac{1}{\sqrt{1+x^2}}$



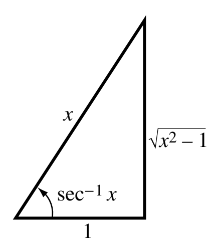
(b) $\tan(\cos^{-1} x) = \dfrac{\sqrt{1-x^2}}{x}$



(c) $\sin(\sec^{-1} x) = \dfrac{\sqrt{x^2-1}}{x}$



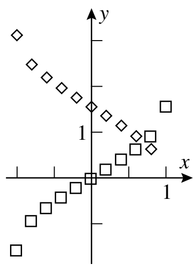
(d) $\cot(\sec^{-1} x) = \dfrac{1}{\sqrt{x^2-1}}$



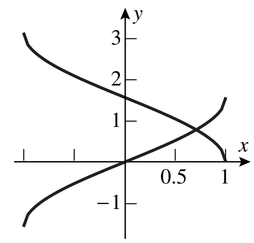
14. **(a)**

x	-1.00	-0.80	-0.6	-0.40	-0.20	0.00	0.20	0.40	0.60	0.80	1.00
$\sin^{-1} x$	-1.57	-0.93	-0.64	-0.41	-0.20	0.00	0.20	0.41	0.64	0.93	1.57
$\cos^{-1} x$	3.14	2.50	2.21	1.98	1.77	1.57	1.37	1.16	0.93	0.64	0.00

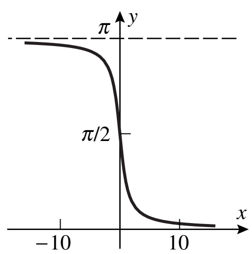
(b)

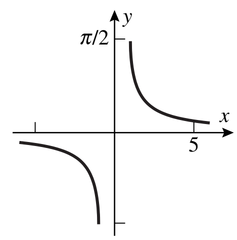


(c)



15. **(a)**





(b) The domain of $\cot^{-1} x$ is $(-\infty, +\infty)$, the range is $(0, \pi)$; the domain of $\csc^{-1} x$ is $(-\infty, -1] \cup [1, +\infty)$, the range is $[-\pi/2, 0) \cup (0, \pi/2]$.

16. **(a)** $y = \cot^{-1} x$; if $x > 0$ then $0 < y < \pi/2$ and $x = \cot y$, $\tan y = 1/x$, $y = \tan^{-1}(1/x)$; if $x < 0$ then $\pi/2 < y < \pi$ and $x = \cot y = \cot(y - \pi), \tan(y - \pi) = 1/x, y = \pi + \tan^{-1} \dfrac{1}{x}$

(b) $y = \sec^{-1} x$, $x = \sec y$, $\cos y = 1/x$, $y = \cos^{-1}(1/x)$

(c) $y = \csc^{-1} x$, $x = \csc y$, $\sin y = 1/x$, $y = \sin^{-1}(1/x)$

17. **(a)** $55.0°$ **(b)** $33.6°$ **(c)** $25.8°$

18. **(a)** Let $x = f(y) = \cot y, 0 < y < \pi, -\infty < x < +\infty$. Then f is differentiable and one-to-one and $f'(f^{-1}(x)) = \cot(\cot^{-1} x)\cos(\cot^{-1} x) = -x\dfrac{\sqrt{x^2+1}}{x} = -\sqrt{x^2+1} \neq 0$, and

$$\frac{d}{dx}[\cot^{-1} x]\bigg|_{x=0} = \lim_{x\to 0} \frac{1}{f'(f^{-1}(x))} = -\lim_{x\to 0} \sqrt{x^2+1} = -1.$$

(b) If $x \neq 0$ then, from Exercise 16(a),

$\dfrac{d}{dx}\cot^{-1} x = \dfrac{d}{dx}\tan^{-1}\dfrac{1}{x} = -\dfrac{1}{x^2}\dfrac{1}{\sqrt{1+(1/x)^2}} = -\dfrac{1}{\sqrt{x^2+1}}$. For $x = 0$, Part (a) shows the same; thus for $-\infty < x < +\infty$, $\dfrac{d}{dx}[\cot^{-1} x] = -\dfrac{1}{\sqrt{x^2+1}}$.

(c) For $-\infty < u < +\infty$, by the chain rule it follows that $\dfrac{d}{dx}[\cot^{-1} u] = -\dfrac{1}{\sqrt{u^2+1}}\dfrac{du}{dx}$.

19. **(a)** By the chain rule, $\dfrac{d}{dx}[\csc^{-1} x] = -\dfrac{1}{x^2}\dfrac{1}{\sqrt{1-(1/x)^2}} = \dfrac{-1}{|x|\sqrt{x^2-1}}$

(b) By the chain rule, $\dfrac{d}{dx}[\csc^{-1} u] = \dfrac{du}{dx}\dfrac{d}{du}[\csc^{-1} u] = \dfrac{-1}{|u|\sqrt{u^2-1}}\dfrac{du}{dx}$

20. **(a)** $x = \pi - \sin^{-1}(0.37) \approx 2.7626$ rad **(b)** $\theta = 180° + \sin^{-1}(0.61) \approx 217.6°$

21. **(a)** $x = \pi + \cos^{-1}(0.85) \approx 3.6964$ rad **(b)** $\theta = -\cos^{-1}(0.23) \approx -76.7°$

22. **(a)** $x = \tan^{-1}(3.16) - \pi \approx -1.8773$ **(b)** $\theta = 180° - \tan^{-1}(0.45) \approx 155.8°$

23. **(a)** $\dfrac{1}{\sqrt{1-x^2/9}}(1/3) = 1/\sqrt{9-x^2}$ **(b)** $-2/\sqrt{1-(2x+1)^2}$

24. **(a)** $2x/(1+x^4)$ **(b)** $-\dfrac{1}{1+x}\left(\dfrac{1}{2}x^{-1/2}\right) = -\dfrac{1}{2(1+x)\sqrt{x}}$

25. **(a)** $\dfrac{1}{|x|^7\sqrt{x^{14}-1}}(7x^6) = \dfrac{7}{|x|\sqrt{x^{14}-1}}$ **(b)** $-1/\sqrt{e^{2x}-1}$

26. **(a)** $y = 1/\tan x = \cot x$, $dy/dx = -\csc^2 x$

(b) $y = (\tan^{-1} x)^{-1}$, $dy/dx = -(\tan^{-1} x)^{-2}\left(\dfrac{1}{1+x^2}\right)$

27. **(a)** $\dfrac{1}{\sqrt{1-1/x^2}}(-1/x^2) = -\dfrac{1}{|x|\sqrt{x^2-1}}$ **(b)** $\dfrac{\sin x}{\sqrt{1-\cos^2 x}} = \dfrac{\sin x}{|\sin x|} = \begin{cases} 1, & \sin x > 0 \\ -1, & \sin x < 0 \end{cases}$

28. **(a)** $-\dfrac{1}{(\cos^{-1} x)\sqrt{1-x^2}}$ **(b)** $-\dfrac{1}{2\sqrt{\cot^{-1} x}(1+x^2)}$

29. **(a)** $\dfrac{e^x}{|x|\sqrt{x^2-1}} + e^x \sec^{-1} x$ **(b)** $\dfrac{3x^2(\sin^{-1} x)^2}{\sqrt{1-x^2}} + 2x(\sin^{-1} x)^3$

30. **(a)** 0 **(b)** 0

31. $x^3 + x\tan^{-1} y = e^y$, $3x^2 + \dfrac{x}{1+y^2}y' + \tan^{-1} y = e^y y'$, $y' = \dfrac{(3x^2+\tan^{-1} y)(1+y^2)}{(1+y^2)e^y - x}$

32. $\sin^{-1}(xy) = \cos^{-1}(x-y)$, $\dfrac{1}{\sqrt{1-x^2y^2}}(xy'+y) = -\dfrac{1}{\sqrt{1-(x-y)^2}}(1-y')$,

$y' = \dfrac{y\sqrt{1-(x-y)^2} + \sqrt{1-x^2y^2}}{\sqrt{1-x^2y^2} - x\sqrt{1-(x-y)^2}}$

33. $\sin^{-1} x\Big]_0^{1/\sqrt{2}} = \sin^{-1}(1/\sqrt{2}) - \sin^{-1} 0 = \pi/4$

34. $u = 2x$, $\dfrac{1}{2}\displaystyle\int \dfrac{1}{\sqrt{1-u^2}}du = \dfrac{1}{2}\sin^{-1}(2x) + C$

35. $\tan^{-1} x\Big]_{-1}^{1} = \tan^{-1} 1 - \tan^{-1}(-1) = \pi/4 - (-\pi/4) = \pi/2$

36. $u = 4x$, $\frac{1}{4}\int \frac{1}{1+u^2}du = \frac{1}{4}\tan^{-1}(4x) + C$

37. $\sec^{-1} x\Big]_{\sqrt{2}}^{2} = \sec^{-1} 2 - \sec^{-1}\sqrt{2} = \pi/3 - \pi/4 = \pi/12$

38. $-\sec^{-1} x\Big]_{-\sqrt{2}}^{-2/\sqrt{3}} = -\sec^{-1}(-2/\sqrt{3}) + \sec^{-1}(-\sqrt{2}) = -5\pi/6 + 3\pi/4 = -\pi/12$

39. $u = \tan x$, $\int \frac{1}{\sqrt{1-u^2}}du = \sin^{-1}(\tan x) + C$

40. $u = e^{-x}$, $-\int_{1/2}^{\sqrt{3}/2} \frac{1}{\sqrt{1-u^2}}du = -\sin^{-1} u\Big]_{1/2}^{\sqrt{3}/2} = -\sin^{-1}\frac{\sqrt{3}}{2} + \sin^{-1}\frac{1}{2} = -\frac{\pi}{3} + \frac{\pi}{6} = -\frac{\pi}{6}$

41. $u = e^x$, $\int \frac{1}{1+u^2}du = \tan^{-1}(e^x) + C$

42. $u = t^2$, $\frac{1}{2}\int \frac{1}{u^2+1}du = \frac{1}{2}\tan^{-1}(t^2) + C$

43. $u = \sqrt{x}$, $2\int_1^{\sqrt{3}} \frac{1}{u^2+1}du = 2\tan^{-1} u\Big]_1^{\sqrt{3}} = 2(\tan^{-1}\sqrt{3} - \tan^{-1} 1) = 2(\pi/3 - \pi/4) = \pi/6$

44. $u = \cos\theta$, $-\int \frac{1}{u^2+1}du = -\tan^{-1}(\cos\theta) + C$

45. $u = \ln x$, $\int \frac{1}{\sqrt{1-u^2}}du = \sin^{-1}(\ln x) + C$

46. $u = 3x$, $\int \frac{1}{u\sqrt{u^2-1}}du = \sec^{-1}(3x) + C$ if $x > 0$; $-\sec^{-1}(3x) + C$ if $x < 0$

47. $u = a\sin\theta, du = a\cos\theta\, d\theta$; $\int \frac{du}{\sqrt{a^2-u^2}} = a\theta + C = \sin^{-1}\frac{u}{a} + C$

48. If $u > 0$ then $u = a\sec\theta, du = a\sec\theta\tan\theta\, d\theta$, $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\theta = \frac{1}{a}\sec^{-1}\frac{u}{a} + C$

49. **(a)** $\sin^{-1}(x/3) + C$ **(b)** $(1/\sqrt{5})\tan^{-1}(x/\sqrt{5}) + C$
(c) $(1/\sqrt{\pi})\sec^{-1}(x/\sqrt{\pi}) + C$

50. **(a)** $u = e^x$, $\int \frac{1}{4+u^2}du = \frac{1}{2}\tan^{-1}(e^x/2) + C$

(b) $u = 2x$, $\frac{1}{2}\int \frac{1}{\sqrt{9-u^2}}du = \frac{1}{2}\sin^{-1}(2x/3) + C$,

(c) $u = \sqrt{5}y$, $\int \frac{1}{u\sqrt{u^2-3}}du = \frac{1}{\sqrt{3}}\sec^{-1}(\sqrt{5}y/\sqrt{3}) + C$

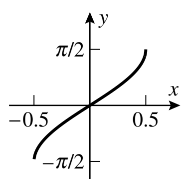
51. $u = \sqrt{3}x^2,\ \dfrac{1}{2\sqrt{3}}\displaystyle\int_0^{\sqrt{3}} \frac{1}{\sqrt{4-u^2}}du = \frac{1}{2\sqrt{3}}\sin^{-1}\frac{u}{2}\Big]_0^{\sqrt{3}} = \frac{1}{2\sqrt{3}}\left(\frac{\pi}{3}\right) = \frac{\pi}{6\sqrt{3}}$

52. $u = \sqrt{x},\ 2\displaystyle\int_1^{\sqrt{2}} \frac{1}{\sqrt{4-u^2}}du = 2\sin^{-1}\frac{u}{2}\Big]_1^{\sqrt{2}} = 2(\pi/4 - \pi/6) = \pi/6$

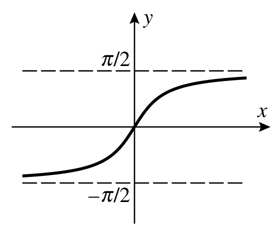
53. $u = 3x,\ \dfrac{1}{3}\displaystyle\int_0^{2\sqrt{3}} \frac{1}{4+u^2}du = \frac{1}{6}\tan^{-1}\frac{u}{2}\Big]_0^{2\sqrt{3}} = \frac{1}{6}\frac{\pi}{3} = \frac{1}{18\pi}$

54. $u = x^2,\ \dfrac{1}{2}\displaystyle\int_1^{3} \frac{1}{3+u^2}du = \frac{1}{2\sqrt{3}}\tan^{-1}\frac{u}{\sqrt{3}}\Big]_1^{3} = \frac{1}{2\sqrt{3}}(\pi/3 - \pi/6) = \frac{\pi}{12\sqrt{3}}$

55. **(a)**



(b)



56. **(a)** $\sin^{-1} 0.9 > 1$, so it is not in the domain of $\sin^{-1} x$
(b) $-1 \le \sin^{-1} x \le 1$ is necessary, or $-0.841471 \le x \le 0.841471$

57. **(b)** $\theta = \sin^{-1}\dfrac{R}{R+h} = \sin^{-1}\dfrac{6378}{16,378} \approx 23^\circ$

58. **(a)** If $\gamma = 90^\circ$, then $\sin\gamma = 1$, $\sqrt{1 - \sin^2\phi\sin^2\gamma} = \sqrt{1-\sin^2\phi} = \cos\phi$,
$D = \tan\phi\tan\lambda = (\tan 23.45^\circ)(\tan 65^\circ) \approx 0.93023374$ so $h \approx 21.1$ hours.
(b) If $\gamma = 270^\circ$, then $\sin\gamma = -1$, $D = -\tan\phi\tan\lambda \approx -0.93023374$ so $h \approx 2.9$ hours.

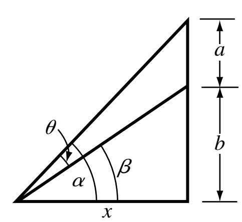
59. $\sin 2\theta = gR/v^2 = (9.8)(18)/(14)^2 = 0.9$, $2\theta = \sin^{-1}(0.9)$ or $2\theta = 180^\circ - \sin^{-1}(0.9)$ so $\theta = \frac{1}{2}\sin^{-1}(0.9) \approx 32^\circ$ or $\theta = 90^\circ - \frac{1}{2}\sin^{-1}(0.9) \approx 58^\circ$. The ball will have a lower parabolic trajectory for $\theta = 32^\circ$ and hence will result in the shorter time of flight.

60. $4^2 = 2^2 + 3^2 - 2(2)(3)\cos\theta$, $\cos\theta = -1/4$, $\theta = \cos^{-1}(-1/4) \approx 104^\circ$

61. $y = 0$ when $x^2 = 6000v^2/g$, $x = 10v\sqrt{60/g} = 1000\sqrt{30}$ for $v = 400$ and $g = 32$;
$\tan\theta = 3000/x = 3/\sqrt{30}$, $\theta = \tan^{-1}(3/\sqrt{30}) \approx 29^\circ$.

62. **(a)** $\theta = \alpha - \beta$, $\cot\alpha = \dfrac{x}{a+b}$ and $\cot\beta = \dfrac{x}{b}$ so

$$\theta = \cot^{-1}\frac{x}{a+b} - \cot^{-1}\left(\frac{x}{b}\right)$$



(b) $$\frac{d\theta}{dx} = -\frac{1}{a+b}\left(\frac{1}{1+x^2/(a+b)^2}\right) - \frac{1}{b}\frac{1}{1+(x/b)^2}$$
$$= -\frac{a+b}{(a+b)^2+x^2} - \frac{b}{b^2+x^2}$$

which is negative for all x. Thus θ is a decreasing function of x, and it has no maximum since $\lim\limits_{x\to 0^+}\theta = +\infty$.

63. **(a)** $A = \int_0^{0.8} \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x\Big]_0^{0.8} = \sin^{-1}(0.8)$

(b) The calculator was in degree mode instead of radian mode; the correct answer is 0.93.

64. $A = \int_0^{1/6} \frac{1}{\sqrt{1-9x^2}}dx = \frac{1}{3}\int_0^{1/2} \frac{1}{\sqrt{1-u^2}}du = \frac{1}{3}\sin^{-1} u\Big]_0^{1/2} = \pi/18$

65. The area is given by $\int_0^k (1/\sqrt{1-x^2} - x)dx = \sin^{-1} k - k^2/2 = 1$; solve for k to get $k = 0.997301$.

66. $x = \sin y,\ A = \int_0^{\pi/2} \sin y\, dy = -\cos y\Big]_0^{\pi/2} = 1$

67. The curves intersect at $x = a = 0$ and $x = b = 0.838422$ so the area is $\int_a^b (\sin 2x - \sin^{-1} x)dx = 0.174192$.

68. The displacement of the particle during the time interval $[0, T]$ is given by $\int_0^T v(t)dt = 3\tan^{-1} T - 0.25T^2$. The particle is 2 cm from its starting position when $3\tan^{-1} T - 0.25T^2 = 2$ or when $3\tan^{-1} T - 0.25T^2 = -2$; solve for T to get $T = 0.90$, 2.51, and 4.95 sec.

69. $V = \int_{-2}^{2} \pi \frac{1}{4+x^2}dx = \frac{\pi}{2}\tan^{-1}(x/2)\Big]_{-2}^{2} = \pi^2/4$

70. **(a)** $V = 2\pi \int_1^b \frac{x}{1+x^4}dx = \pi \tan^{-1}(x^2)\Big]_1^b = \pi\left[\tan^{-1}(b^2) - \frac{\pi}{4}\right]$

(b) $\lim_{b\to+\infty} V = \pi\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{1}{4}\pi^2$

71. The area is given by $\int_0^2 1/(1+kx^2)dx = (1/\sqrt{k})\tan^{-1}(2\sqrt{k}) = 0.6$; solve for k to get $k = 5.081435$.

72. **(a)** $\pi\int_0^1 (\sin^{-1} x)^2 dx = 1.468384$.

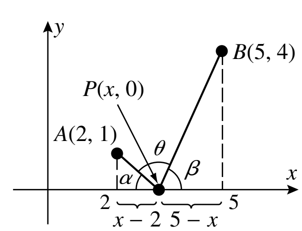
(b) $2\pi\int_0^{\pi/2} y(1-\sin y)dy = 1.468384$.

73. $\theta = \pi - (\alpha + \beta)$

$$= \pi - \cot^{-1}(x-2) - \cot^{-1}\frac{5-x}{4},$$

$$\frac{d\theta}{dx} = \frac{1}{1+(x-2)^2} + \frac{-1/4}{1+(5-x)^2/16}$$

$$= -\frac{3(x^2-2x-7)}{[1+(x-2)^2][16+(5-x)^2]}$$

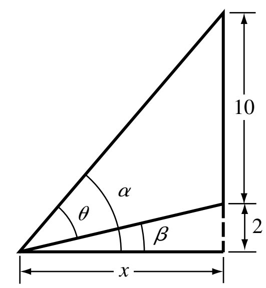


$d\theta/dx = 0$ when $x = \frac{2 \pm \sqrt{4+28}}{2} = 1 \pm 2\sqrt{2}$, only $1 + 2\sqrt{2}$ is in $[2, 5]$; $d\theta/dx > 0$ for x in $[2, 1+2\sqrt{2})$, $d\theta/dx < 0$ for x in $(1+2\sqrt{2}, 5]$, θ is maximum when $x = 1+2\sqrt{2}$.

74. $\theta = \alpha - \beta$
$= \cot^{-1}(x/12) - \cot^{-1}(x/2)$

$$\frac{d\theta}{dx} = -\frac{12}{144+x^2} + \frac{2}{4+x^2}$$
$$= \frac{10(24-x^2)}{(144+x^2)(4+x^2)}$$

$d\theta/dx = 0$ when $x = \sqrt{24} = 2\sqrt{6}$, by the first derivative test θ is maximum there.



75. By the Mean-Value Theorem on the interval $[0, x]$,

$$\frac{\tan^{-1} x - \tan^{-1} 0}{x - 0} = \frac{\tan^{-1} x}{x} = \frac{1}{1+c^2} \text{ for } c \text{ in } (0, x), \text{ but}$$

$$\frac{1}{1+x^2} < \frac{1}{1+c^2} < 1 \text{ for } c \text{ in } (0, x) \text{ so } \frac{1}{1+x^2} < \frac{\tan^{-1} x}{x} < 1,\ \frac{x}{1+x^2} < \tan^{-1} x < x.$$

76. $\dfrac{n}{n^2+k^2} = \dfrac{1}{1+k^2/n^2}\dfrac{1}{n}$ so $\displaystyle\sum_{k=1}^{n} \frac{n}{n^2+k^2} = \sum_{k=1}^{n} f(x_k^*)\Delta x$ where $f(x) = \dfrac{1}{1+x^2}$, $x_k^* = \dfrac{k}{n}$, and $\Delta x = \dfrac{1}{n}$ for $0 \le x \le 1$. Thus $\displaystyle\lim_{n\to+\infty} \sum_{k=1}^{n} \frac{n}{n^2+k^2} = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \int_0^1 \frac{1}{1+x^2}dx = \frac{\pi}{4}$.

77. **(a)** Let $\theta = \sin^{-1}(-x)$ then $\sin\theta = -x$, $-\pi/2 \le \theta \le \pi/2$. But $\sin(-\theta) = -\sin\theta$ and $-\pi/2 \le -\theta \le \pi/2$ so $\sin(-\theta) = -(-x) = x$, $-\theta = \sin^{-1} x$, $\theta = -\sin^{-1} x$.

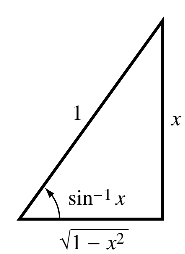
(b) proof is similar to that in Part (a)

78. **(a)** Let $\theta = \cos^{-1}(-x)$ then $\cos\theta = -x$, $0 \le \theta \le \pi$. But $\cos(\pi - \theta) = -\cos\theta$ and $0 \le \pi - \theta \le \pi$ so $\cos(\pi - \theta) = x$, $\pi - \theta = \cos^{-1} x$, $\theta = \pi - \cos^{-1} x$

(b) Let $\theta = \sec^{-1}(-x)$ for $x \ge 1$; then $\sec\theta = -x$ and $\pi/2 < \theta \le \pi$. So $0 \le \pi - \theta < \pi/2$ and $\pi - \theta = \sec^{-1}\sec(\pi - \theta) = \sec^{-1}(-\sec\theta) = \sec^{-1} x$, or $\sec^{-1}(-x) = \pi - \sec^{-1} x$.

79. **(a)** $\sin^{-1} x = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$ (see figure)

(b) $\sin^{-1} x + \cos^{-1} x = \pi/2$; $\cos^{-1} x = \pi/2 - \sin^{-1} x = \pi/2 - \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$



80. $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$,

$$\tan(\tan^{-1} x + \tan^{-1} y) = \frac{\tan(\tan^{-1} x) + \tan(\tan^{-1} y)}{1 - \tan(\tan^{-1} x)\tan(\tan^{-1} y)} = \frac{x+y}{1-xy}$$

so $\quad \tan^{-1} x + \tan^{-1} y = \tan^{-1}\dfrac{x+y}{1-xy}$

81. **(a)** $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{1/2+1/3}{1-(1/2)(1/3)}=\tan^{-1}1=\pi/4$

(b) $2\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{1/3+1/3}{1-(1/3)(1/3)}=\tan^{-1}\frac{3}{4}$,

$2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{3/4+1/7}{1-(3/4)(1/7)}=\tan^{-1}1=\pi/4$

82. $\sin(\sec^{-1}x)=\sin(\cos^{-1}(1/x))=\sqrt{1-\left(\frac{1}{x}\right)^2}=\frac{\sqrt{x^2-1}}{|x|}$

EXERCISE SET 7.7

1. **(a)** $\lim_{x\to 2}\frac{x^2-4}{x^2+2x-8}=\lim_{x\to 2}\frac{(x-2)(x+2)}{(x+4)(x-2)}=\lim_{x\to 2}\frac{x+2}{x+4}=\frac{2}{3}$

(b) $\lim_{x\to+\infty}\frac{2x-5}{3x+7}=\frac{2-\lim_{x\to+\infty}\frac{5}{x}}{3+\lim_{x\to+\infty}\frac{7}{x}}=\frac{2}{3}$

2. **(a)** $\frac{\sin x}{\tan x}=\sin x\frac{\cos x}{\sin x}=\cos x$ so $\lim_{x\to 0}\frac{\sin x}{\tan x}=\lim_{x\to 0}\cos x=1$

(b) $\frac{x^2-1}{x^3-1}=\frac{(x-1)(x+1)}{(x-1)(x^2+x+1)}=\frac{x+1}{x^2+x+1}$ so $\lim_{x\to 1}\frac{x^2-1}{x^3-1}=\frac{2}{3}$

3. $\lim_{x\to 1}\frac{1/x}{1}=1$

4. $\lim_{x\to 0}\frac{2\cos 2x}{5\cos 5x}=2/5$

5. $\lim_{x\to 0}\frac{e^x}{\cos x}=1$

6. $\lim_{x\to 3}\frac{1}{6x-13}=1/5$

7. $\lim_{\theta\to 0}\frac{\sec^2\theta}{1}=1$

8. $\lim_{t\to 0}\frac{te^t+e^t}{-e^t}=-1$

9. $\lim_{x\to\pi^+}\frac{\cos x}{1}=-1$

10. $\lim_{x\to 0^+}\frac{\cos x}{2x}=+\infty$

11. $\lim_{x\to+\infty}\frac{1/x}{1}=0$

12. $\lim_{x\to+\infty}\frac{3e^{3x}}{2x}=\lim_{x\to+\infty}\frac{9e^{3x}}{2}=+\infty$

13. $\lim_{x\to 0^+}\frac{-\csc^2 x}{1/x}=\lim_{x\to 0^+}\frac{-x}{\sin^2 x}=\lim_{x\to 0^+}\frac{-1}{2\sin x\cos x}=-\infty$

14. $\lim_{x\to 0^+}\frac{-1/x}{(-1/x^2)e^{1/x}}=\lim_{x\to 0^+}\frac{x}{e^{1/x}}=0$

15. $\lim_{x\to+\infty}\frac{100x^{99}}{e^x}=\lim_{x\to+\infty}\frac{(100)(99)x^{98}}{e^x}=\cdots=\lim_{x\to+\infty}\frac{(100)(99)(98)\cdots(1)}{e^x}=0$

16. $\lim_{x\to 0^+} \dfrac{\cos x/\sin x}{\sec^2 x/\tan x} = \lim_{x\to 0^+} \cos^2 x = 1$

17. $\lim_{x\to 0} \dfrac{2/\sqrt{1-4x^2}}{1} = 2$

18. $\lim_{x\to 0} \dfrac{1-\dfrac{1}{1+x^2}}{3x^2} = \lim_{x\to 0} \dfrac{1}{3(1+x^2)} = \dfrac{1}{3}$

19. $\lim_{x\to+\infty} xe^{-x} = \lim_{x\to+\infty} \dfrac{x}{e^x} = \lim_{x\to+\infty} \dfrac{1}{e^x} = 0$

20. $\lim_{x\to\pi} (x-\pi)\tan(x/2) = \lim_{x\to\pi} \dfrac{x-\pi}{\cot(x/2)} = \lim_{x\to\pi} \dfrac{1}{-(1/2)\csc^2(x/2)} = -2$

21. $\lim_{x\to+\infty} x\sin(\pi/x) = \lim_{x\to+\infty} \dfrac{\sin(\pi/x)}{1/x} = \lim_{x\to+\infty} \dfrac{(-\pi/x^2)\cos(\pi/x)}{-1/x^2} = \lim_{x\to+\infty} \pi\cos(\pi/x) = \pi$

22. $\lim_{x\to 0^+} \tan x \ln x = \lim_{x\to 0^+} \dfrac{\ln x}{\cot x} = \lim_{x\to 0^+} \dfrac{1/x}{-\csc^2 x} = \lim_{x\to 0^+} \dfrac{-\sin^2 x}{x} = \lim_{x\to 0^+} \dfrac{-2\sin x\cos x}{1} = 0$

23. $\lim_{x\to(\pi/2)^-} \sec 3x\cos 5x = \lim_{x\to(\pi/2)^-} \dfrac{\cos 5x}{\cos 3x} = \lim_{x\to(\pi/2)^-} \dfrac{-5\sin 5x}{-3\sin 3x} = \dfrac{-5(+1)}{(-3)(-1)} = -\dfrac{5}{3}$

24. $\lim_{x\to\pi} (x-\pi)\cot x = \lim_{x\to\pi} \dfrac{x-\pi}{\tan x} = \lim_{x\to\pi} \dfrac{1}{\sec^2 x} = 1$

25. $y = (1-3/x)^x$, $\lim_{x\to+\infty} \ln y = \lim_{x\to+\infty} \dfrac{\ln(1-3/x)}{1/x} = \lim_{x\to+\infty} \dfrac{-3}{1-3/x} = -3$, $\lim_{x\to+\infty} y = e^{-3}$

26. $y = (1+2x)^{-3/x}$, $\lim_{x\to 0} \ln y = \lim_{x\to 0} -\dfrac{3\ln(1+2x)}{x} = \lim_{x\to 0} -\dfrac{6}{1+2x} = -6$, $\lim_{x\to 0} y = e^{-6}$

27. $y = (e^x + x)^{1/x}$, $\lim_{x\to 0} \ln y = \lim_{x\to 0} \dfrac{\ln(e^x+x)}{x} = \lim_{x\to 0} \dfrac{e^x+1}{e^x+x} = 2$, $\lim_{x\to 0} y = e^2$

28. $y = (1+a/x)^{bx}$, $\lim_{x\to+\infty} \ln y = \lim_{x\to+\infty} \dfrac{b\ln(1+a/x)}{1/x} = \lim_{x\to+\infty} \dfrac{ab}{1+a/x} = ab$, $\lim_{x\to+\infty} y = e^{ab}$

29. $y = (2-x)^{\tan(\pi x/2)}$, $\lim_{x\to 1} \ln y = \lim_{x\to 1} \dfrac{\ln(2-x)}{\cot(\pi x/2)} = \lim_{x\to 1} \dfrac{2\sin^2(\pi x/2)}{\pi(2-x)} = 2/\pi$, $\lim_{x\to 1} y = e^{2/\pi}$

30. $y = [\cos(2/x)]^{x^2}$, $\lim_{x\to+\infty} \ln y = \lim_{x\to+\infty} \dfrac{\ln\cos(2/x)}{1/x^2} = \lim_{x\to+\infty} \dfrac{(-2/x^2)(-\tan(2/x))}{-2/x^3}$

$= \lim_{x\to+\infty} \dfrac{-\tan(2/x)}{1/x} = \lim_{x\to+\infty} \dfrac{(2/x^2)\sec^2(2/x)}{-1/x^2} = -2$, $\lim_{x\to+\infty} y = e^{-2}$

31. $\lim_{x\to 0} \left(\dfrac{1}{\sin x} - \dfrac{1}{x}\right) = \lim_{x\to 0} \dfrac{x-\sin x}{x\sin x} = \lim_{x\to 0} \dfrac{1-\cos x}{x\cos x + \sin x} = \lim_{x\to 0} \dfrac{\sin x}{2\cos x - x\sin x} = 0$

32. $\lim_{x\to 0} \dfrac{1-\cos 3x}{x^2} = \lim_{x\to 0} \dfrac{3\sin 3x}{2x} = \lim_{x\to 0} \dfrac{9}{2}\cos 3x = \dfrac{9}{2}$

33. $\lim_{x\to+\infty} \dfrac{(x^2+x)-x^2}{\sqrt{x^2+x}+x} = \lim_{x\to+\infty} \dfrac{x}{\sqrt{x^2+x}+x} = \lim_{x\to+\infty} \dfrac{1}{\sqrt{1+1/x}+1} = 1/2$

34. $\lim_{x\to 0} \dfrac{e^x-1-x}{xe^x-x} = \lim_{x\to 0} \dfrac{e^x-1}{xe^x+e^x-1} = \lim_{x\to 0} \dfrac{e^x}{xe^x+2e^x} = 1/2$

35. $\lim_{x\to+\infty}[x-\ln(x^2+1)] = \lim_{x\to+\infty}[\ln e^x - \ln(x^2+1)] = \lim_{x\to+\infty}\ln\frac{e^x}{x^2+1}$,

$\lim_{x\to+\infty}\frac{e^x}{x^2+1} = \lim_{x\to+\infty}\frac{e^x}{2x} = \lim_{x\to+\infty}\frac{e^x}{2} = +\infty$ so $\lim_{x\to+\infty}[x-\ln(x^2+1)] = +\infty$

36. $\lim_{x\to+\infty}\ln\frac{x}{1+x} = \lim_{x\to+\infty}\ln\frac{1}{1/x+1} = \ln(1) = 0$

38. **(a)** $\lim_{x\to+\infty}\frac{\ln x}{x^n} = \lim_{x\to+\infty}\frac{1/x}{nx^{n-1}} = \lim_{x\to+\infty}\frac{1}{nx^n} = 0$

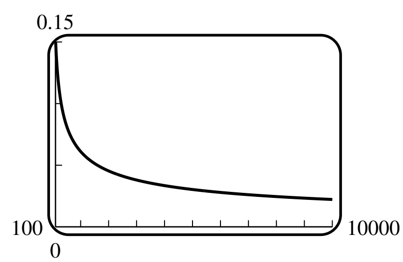
(b) $\lim_{x\to+\infty}\frac{x^n}{\ln x} = \lim_{x\to+\infty}\frac{nx^{n-1}}{1/x} = \lim_{x\to+\infty}nx^n = +\infty$

39. **(a)** L'Hôpital's Rule does not apply to the problem $\lim_{x\to1}\frac{3x^2-2x+1}{3x^2-2x}$ because it is not a $\frac{0}{0}$ form.

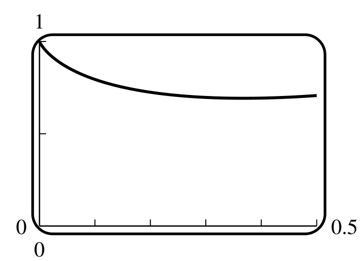
(b) $\lim_{x\to1}\frac{3x^2-2x+1}{3x^2-2x} = 2$

40. $\lim_{x\to1}\frac{4x^3-12x^2+12x-4}{4x^3-9x^2+6x-1} = \lim_{x\to1}\frac{12x^2-24x+12}{12x^2-18x+6} = \lim_{x\to1}\frac{24x-24}{24x-18} = 0$

41. $\lim_{x\to+\infty}\frac{1/(x\ln x)}{1/(2\sqrt{x})} = \lim_{x\to+\infty}\frac{2}{\sqrt{x}\ln x} = 0$



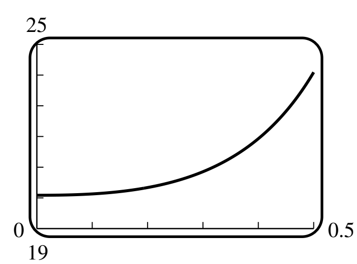
42. $y = x^x$, $\lim_{x\to0^+}\ln y = \lim_{x\to0^+}\frac{\ln x}{1/x} = \lim_{x\to0^+}-x = 0$, $\lim_{x\to0^+}y = 1$



43. $y = (\sin x)^{3/\ln x}$,

$\lim_{x\to0^+}\ln y = \lim_{x\to0^+}\frac{3\ln\sin x}{\ln x} = \lim_{x\to0^+}(3\cos x)\frac{x}{\sin x} = 3$,

$\lim_{x\to0^+}y = e^3$

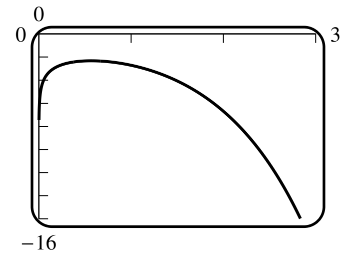


44. $\lim_{x\to\pi/2^-}\frac{4\sec^2 x}{\sec x\tan x} = \lim_{x\to\pi/2^-}\frac{4}{\sin x} = 4$

45. $\ln x - e^x = \ln x - \dfrac{1}{e^{-x}} = \dfrac{e^{-x}\ln x - 1}{e^{-x}}$;

$\lim\limits_{x\to+\infty} e^{-x}\ln x = \lim\limits_{x\to+\infty} \dfrac{\ln x}{e^x} = \lim\limits_{x\to+\infty} \dfrac{1/x}{e^x} = 0$ by L'Hôpital's Rule,

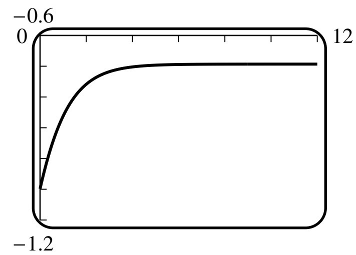
so $\lim\limits_{x\to+\infty} [\ln x - e^x] = \lim\limits_{x\to+\infty} \dfrac{e^{-x}\ln x - 1}{e^{-x}} = -\infty$



46. $\lim\limits_{x\to+\infty} [\ln e^x - \ln(1+2e^x)] = \lim\limits_{x\to+\infty} \ln \dfrac{e^x}{1+2e^x}$

$= \lim\limits_{x\to+\infty} \ln \dfrac{1}{e^{-x}+2} = \ln \dfrac{1}{2}$;

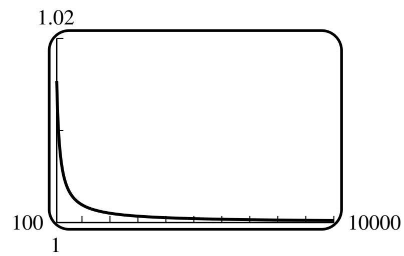
horizontal asymptote $y = -\ln 2$



47. $y = (\ln x)^{1/x}$,

$\lim\limits_{x\to+\infty} \ln y = \lim\limits_{x\to+\infty} \dfrac{\ln(\ln x)}{x} = \lim\limits_{x\to+\infty} \dfrac{1}{x\ln x} = 0$;

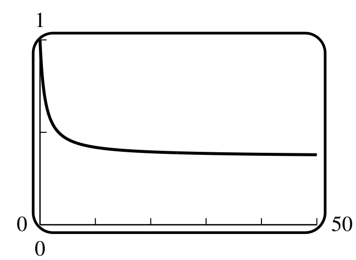
$\lim\limits_{x\to+\infty} y = 1$, $y = 1$ is the horizontal asymptote



48. $y = \left(\dfrac{x+1}{x+2}\right)^x$, $\lim\limits_{x\to+\infty} \ln y = \lim\limits_{x\to+\infty} \dfrac{\ln \dfrac{x+1}{x+2}}{1/x}$

$= \lim\limits_{x\to+\infty} \dfrac{-x^2}{(x+1)(x+2)} = -1$;

$\lim\limits_{x\to+\infty} y = e^{-1}$ is the horizontal asymptote



49. **(a)** 0 **(b)** $+\infty$ **(c)** 0 **(d)** $-\infty$ **(e)** $+\infty$ **(f)** $-\infty$

50. **(a)** Type 0^0; $y = x^{(\ln a)/(1+\ln x)}$; $\lim\limits_{x\to0^+} \ln y = \lim\limits_{x\to0^+} \dfrac{(\ln a)\ln x}{1+\ln x} = \lim\limits_{x\to0^+} \dfrac{(\ln a)/x}{1/x} = \lim\limits_{x\to0^+} \ln a = \ln a$,

$\lim\limits_{x\to0^+} y = e^{\ln a} = a$

(b) Type ∞^0; same calculation as Part (a) with $x \to +\infty$

(c) Type 1^∞; $y = (x+1)^{(\ln a)/x}$, $\lim\limits_{x\to0} \ln y = \lim\limits_{x\to0} \dfrac{(\ln a)\ln(x+1)}{x} = \lim\limits_{x\to0} \dfrac{\ln a}{x+1} = \ln a$,

$\lim\limits_{x\to0} y = e^{\ln a} = a$

51. $\lim_{x\to+\infty} \dfrac{1+2\cos 2x}{1}$ does not exist, nor is it $\pm\infty$; $\lim_{x\to+\infty} \dfrac{x+\sin 2x}{x} = \lim_{x\to+\infty}\left(1+\dfrac{\sin 2x}{x}\right) = 1$

52. $\lim_{x\to+\infty} \dfrac{2-\cos x}{3+\cos x}$ does not exist, nor is it $\pm\infty$; $\lim_{x\to+\infty} \dfrac{2x-\sin x}{3x+\sin x} = \lim_{x\to+\infty} \dfrac{2-(\sin x)/x}{3+(\sin x)/x} = \dfrac{2}{3}$

53. $\lim_{x\to+\infty}(2+x\cos 2x+\sin 2x)$ does not exist, nor is it $\pm\infty$; $\lim_{x\to+\infty}\dfrac{x(2+\sin 2x)}{x+1} = \lim_{x\to+\infty}\dfrac{2+\sin 2x}{1+1/x}$, which does not exist because $\sin 2x$ oscillates between -1 and 1 as $x\to+\infty$

54. $\lim_{x\to+\infty}\left(\dfrac{1}{x}+\dfrac{1}{2}\cos x+\dfrac{\sin x}{2x}\right)$ does not exist, nor is it $\pm\infty$;

$\lim_{x\to+\infty}\dfrac{x(2+\sin x)}{x^2+1} = \lim_{x\to+\infty}\dfrac{2+\sin x}{x+1/x} = 0$

55. $\lim_{R\to 0^+}\dfrac{\frac{Vt}{L}e^{-Rt/L}}{1} = \dfrac{Vt}{L}$

56. **(a)** $\lim_{x\to\pi/2}(\pi/2-x)\tan x = \lim_{x\to\pi/2}\dfrac{\pi/2-x}{\cot x} = \lim_{x\to\pi/2}\dfrac{-1}{-\csc^2 x} = \lim_{x\to\pi/2}\sin^2 x = 1$

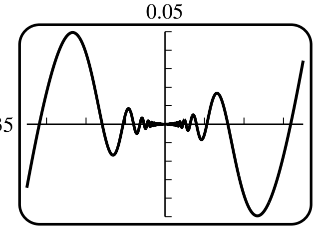
(b)
$$\lim_{x\to\pi/2}\left(\frac{1}{\pi/2-x}-\tan x\right) = \lim_{x\to\pi/2}\left(\frac{1}{\pi/2-x}-\frac{\sin x}{\cos x}\right) = \lim_{x\to\pi/2}\frac{\cos x-(\pi/2-x)\sin x}{(\pi/2-x)\cos x}$$
$$= \lim_{x\to\pi/2}\frac{-(\pi/2-x)\cos x}{-(\pi/2-x)\sin x-\cos x}$$
$$= \lim_{x\to\pi/2}\frac{(\pi/2-x)\sin x+\cos x}{-(\pi/2-x)\cos x+2\sin x} = 0$$

(c) $1/(\pi/2-1.57)\approx 1255.765849, \tan 1.57\approx 1255.765592$;
$1/(\pi/2-1.57)-\tan 1.57\approx 0.000265$

57. **(b)** $\lim_{x\to+\infty}x(k^{1/x}-1) = \lim_{t\to 0^+}\dfrac{k^t-1}{t} = \lim_{t\to 0^+}\dfrac{(\ln k)k^t}{1} = \ln k$

(c) $\ln 0.3 = -1.20397$, $1024\left(\sqrt[1024]{0.3}-1\right) = -1.20327$;
$\ln 2 = 0.69315$, $1024\left(\sqrt[1024]{2}-1\right) = 0.69338$

58. **(a)** No; $\sin(1/x)$ oscillates as $x\to 0$. **(b)**



(c) For the limit as $x\to 0^+$ use the Squeezing Theorem together with the inequalities $-x^2\le x^2\sin(1/x)\le x^2$. For $x\to 0^-$ do the same; thus $\lim_{x\to 0}f(x) = 0$.

59. If $k\ne -1$ then $\lim_{x\to 0}(k+\cos\ell x) = k+1\ne 0$, so $\lim_{x\to 0}\dfrac{k+\cos\ell x}{x^2} = \pm\infty$. Hence $k=-1$, and by the rule

$\lim_{x\to 0}\dfrac{-1+\cos\ell x}{x^2} = \lim_{x\to 0}\dfrac{-\ell\sin\ell x}{2x} = \lim_{x\to 0}\dfrac{-\ell^2\cos\ell x}{2} = -\dfrac{\ell^2}{2} = -4$ if $\ell = \pm 2\sqrt{2}$.

60. **(a)** Apply the rule to get $\lim_{x\to 0}\dfrac{-\cos(1/x)+2x\sin(1/x)}{\cos x}$ which does not exist (nor is it $\pm\infty$).

(b) Rewrite as $\lim_{x\to 0}\left[\dfrac{x}{\sin x}\right][x\sin(1/x)]$, but $\lim_{x\to 0}\dfrac{x}{\sin x}=\lim_{x\to 0}\dfrac{1}{\cos x}=1$ and $\lim_{x\to 0}x\sin(1/x)=0$, thus $\lim_{x\to 0}\left[\dfrac{x}{\sin x}\right][x\sin(1/x)]=(1)(0)=0$

61. $\lim_{x\to 0^+}\dfrac{\sin(1/x)}{(\sin x)/x}$, $\lim_{x\to 0^+}\dfrac{\sin x}{x}=1$ but $\lim_{x\to 0^+}\sin(1/x)$ does not exist because $\sin(1/x)$ oscillates between -1 and 1 as $x\to+\infty$, so $\lim_{x\to 0^+}\dfrac{x\sin(1/x)}{\sin x}$ does not exist.

EXERCISE SET 7.8

1. **(a)** $\sinh 3\approx 10.0179$
(b) $\cosh(-2)\approx 3.7622$
(c) $\tanh(\ln 4)=15/17\approx 0.8824$
(d) $\sinh^{-1}(-2)\approx -1.4436$
(e) $\cosh^{-1}3\approx 1.7627$
(f) $\tanh^{-1}\dfrac{3}{4}\approx 0.9730$

2. **(a)** $\operatorname{csch}(-1)\approx -0.8509$
(b) $\operatorname{sech}(\ln 2)=0.8$
(c) $\coth 1\approx 1.3130$
(d) $\operatorname{sech}^{-1}\dfrac{1}{2}\approx 1.3170$
(e) $\coth^{-1}3\approx 0.3466$
(f) $\operatorname{csch}^{-1}(-\sqrt{3})\approx -0.5493$

3. **(a)** $\sinh(\ln 3)=\dfrac{1}{2}(e^{\ln 3}-e^{-\ln 3})=\dfrac{1}{2}\left(3-\dfrac{1}{3}\right)=\dfrac{4}{3}$

(b) $\cosh(-\ln 2)=\dfrac{1}{2}(e^{-\ln 2}+e^{\ln 2})=\dfrac{1}{2}\left(\dfrac{1}{2}+2\right)=\dfrac{5}{4}$

(c) $\tanh(2\ln 5)=\dfrac{e^{2\ln 5}-e^{-2\ln 5}}{e^{2\ln 5}+e^{-2\ln 5}}=\dfrac{25-1/25}{25+1/25}=\dfrac{312}{313}$

(d) $\sinh(-3\ln 2)=\dfrac{1}{2}(e^{-3\ln 2}-e^{3\ln 2})=\dfrac{1}{2}\left(\dfrac{1}{8}-8\right)=-\dfrac{63}{16}$

4. **(a)** $\dfrac{1}{2}(e^{\ln x}+e^{-\ln x})=\dfrac{1}{2}\left(x+\dfrac{1}{x}\right)=\dfrac{x^2+1}{2x},\ x>0$

(b) $\dfrac{1}{2}(e^{\ln x}-e^{-\ln x})=\dfrac{1}{2}\left(x-\dfrac{1}{x}\right)=\dfrac{x^2-1}{2x},\ x>0$

(c) $\dfrac{e^{2\ln x}-e^{-2\ln x}}{e^{2\ln x}+e^{-2\ln x}}=\dfrac{x^2-1/x^2}{x^2+1/x^2}=\dfrac{x^4-1}{x^4+1},\ x>0$

(d) $\dfrac{1}{2}(e^{-\ln x}+e^{\ln x})=\dfrac{1}{2}\left(\dfrac{1}{x}+x\right)=\dfrac{1+x^2}{2x},\ x>0$

5.

	$\sinh x_0$	$\cosh x_0$	$\tanh x_0$	$\coth x_0$	$\operatorname{sech} x_0$	$\operatorname{csch} x_0$
(a)	2	$\sqrt{5}$	$2/\sqrt{5}$	$\sqrt{5}/2$	$1/\sqrt{5}$	1/2
(b)	3/4	5/4	3/5	5/3	4/5	4/3
(c)	4/3	5/3	4/5	5/4	3/5	3/4

(a) $\cosh^2 x_0 = 1 + \sinh^2 x_0 = 1 + (2)^2 = 5$, $\cosh x_0 = \sqrt{5}$

(b) $\sinh^2 x_0 = \cosh^2 x_0 - 1 = \dfrac{25}{16} - 1 = \dfrac{9}{16}$, $\sinh x_0 = \dfrac{3}{4}$ (because $x_0 > 0$)

(c) $\operatorname{sech}^2 x_0 = 1 - \tanh^2 x_0 = 1 - \left(\dfrac{4}{5}\right)^2 = 1 - \dfrac{16}{25} = \dfrac{9}{25}$, $\operatorname{sech}\, x_0 = \dfrac{3}{5}$,

$\cosh x_0 = \dfrac{1}{\operatorname{sech}\, x_0} = \dfrac{5}{3}$, from $\dfrac{\sinh x_0}{\cosh x_0} = \tanh x_0$ we get $\sinh x_0 = \left(\dfrac{5}{3}\right)\left(\dfrac{4}{5}\right) = \dfrac{4}{3}$

6. $\dfrac{d}{dx}\operatorname{csch} x = \dfrac{d}{dx}\dfrac{1}{\sinh x} = -\dfrac{\cosh x}{\sinh^2 x} = -\coth x \operatorname{csch}\, x$ for $x \neq 0$

$\dfrac{d}{dx}\operatorname{sech}\, x = \dfrac{d}{dx}\dfrac{1}{\cosh x} = -\dfrac{\sinh x}{\cosh^2 x} = -\tanh x \operatorname{sech}\, x$ for all x

$\dfrac{d}{dx}\coth x = \dfrac{d}{dx}\dfrac{\cosh x}{\sinh x} = \dfrac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = -\operatorname{csch}^2 x$ for $x \neq 0$

7. (a) $y = \sinh^{-1} x$ if and only if $x = \sinh y$; $1 = \dfrac{dy}{dx}\dfrac{dx}{dy} = \dfrac{dy}{dx}\cosh y$; so

$\dfrac{d}{dx}[\sinh^{-1} x] = \dfrac{dy}{dx} = \dfrac{1}{\cosh y} = \dfrac{1}{\sqrt{1 + \sinh^2 y}} = \dfrac{1}{\sqrt{1 + x^2}}$ for all x.

(b) Let $x \geq 1$. Then $y = \cosh^{-1} x$ if and only if $x = \cosh y$; $1 = \dfrac{dy}{dx}\dfrac{dx}{dy} = \dfrac{dy}{dx}\sinh y$, so

$\dfrac{d}{dx}[\cosh^{-1} x] = \dfrac{dy}{dx} = \dfrac{1}{\sinh y} = \dfrac{1}{\sqrt{\cosh^2 y - 1}} = \dfrac{1}{x^2 - 1}$ for $x \geq 1$.

(c) Let $-1 < x < 1$. Then $y = \tanh^{-1} x$ if and only if $x = \tanh y$; thus

$1 = \dfrac{dy}{dx}\dfrac{dx}{dy} = \dfrac{dy}{dx}\operatorname{sech}^2 y = \dfrac{dy}{dx}(1 - \tanh^2 y) = 1 - x^2$, so $\dfrac{d}{dx}[\tanh^{-1} x] = \dfrac{dy}{dx} = \dfrac{1}{1 - x^2}$.

9. $4\cosh(4x - 8)$

10. $4x^3 \sinh(x^4)$

11. $-\dfrac{1}{x}\operatorname{csch}^2(\ln x)$

12. $2\dfrac{\operatorname{sech}^2 2x}{\tanh 2x}$

13. $\dfrac{1}{x^2}\operatorname{csch}(1/x)\coth(1/x)$

14. $-2e^{2x}\operatorname{sech}(e^{2x})\tanh(e^{2x})$

15. $\dfrac{2 + 5\cosh(5x)\sinh(5x)}{\sqrt{4x + \cosh^2(5x)}}$

16. $6\sinh^2(2x)\cosh(2x)$

17. $x^{5/2}\tanh(\sqrt{x})\operatorname{sech}^2(\sqrt{x}) + 3x^2\tanh^2(\sqrt{x})$

18. $-3\cosh(\cos 3x)\sin 3x$

19. $\dfrac{1}{\sqrt{1 + x^2/9}}\left(\dfrac{1}{3}\right) = 1/\sqrt{9 + x^2}$

20. $\dfrac{1}{\sqrt{1 + 1/x^2}}(-1/x^2) = -\dfrac{1}{|x|\sqrt{x^2 + 1}}$

21. $1/\left[(\cosh^{-1} x)\sqrt{x^2 - 1}\right]$

22. $1/\left[\sqrt{(\sinh^{-1} x)^2 - 1}\,\sqrt{1 + x^2}\right]$

23. $-(\tanh^{-1} x)^{-2}/(1 - x^2)$

24. $2(\coth^{-1} x)/(1 - x^2)$

25. $\dfrac{\sinh x}{\sqrt{\cosh^2 x - 1}} = \dfrac{\sinh x}{|\sinh x|} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$

26. $(\operatorname{sech}^2 x)/\sqrt{1+\tanh^2 x}$

27. $-\dfrac{e^x}{2x\sqrt{1-x}} + e^x \operatorname{sech}^{-1} x$

28. $10(1 + x\operatorname{csch}^{-1} x)^9 \left(-\dfrac{x}{|x|\sqrt{1+x^2}} + \operatorname{csch}^{-1} x\right)$

31. $\dfrac{1}{7}\sinh^7 x + C$

32. $\dfrac{1}{2}\sinh(2x-3) + C$

33. $\dfrac{2}{3}(\tanh x)^{3/2} + C$

34. $-\dfrac{1}{3}\coth(3x) + C$

35. $\ln(\cosh x) + C$

36. $-\dfrac{1}{3}\coth^3 x + C$

37. $-\dfrac{1}{3}\operatorname{sech}^3 x\Big]_{\ln 2}^{\ln 3} = 37/375$

38. $\ln(\cosh x)\Big]_0^{\ln 3} = \ln 5 - \ln 3$

39. $u = 3x,\ \dfrac{1}{3}\displaystyle\int \frac{1}{\sqrt{1+u^2}}du = \frac{1}{3}\sinh^{-1} 3x + C$

40. $x = \sqrt{2}u,\ \displaystyle\int \frac{\sqrt{2}}{\sqrt{2u^2-2}}du = \int \frac{1}{\sqrt{u^2-1}}du = \cosh^{-1}(x/\sqrt{2}) + C$

41. $u = e^x,\ \displaystyle\int \frac{1}{u\sqrt{1-u^2}}du = -\operatorname{sech}^{-1}(e^x) + C$

42. $u = \cos\theta,\ -\displaystyle\int \frac{1}{\sqrt{1+u^2}}du = -\sinh^{-1}(\cos\theta) + C$

43. $u = 2x,\ \displaystyle\int \frac{du}{u\sqrt{1+u^2}} = -\operatorname{csch}^{-1}|u| + C = -\operatorname{csch}^{-1}|2x| + C$

44. $x = 5u/3,\ \displaystyle\int \frac{5/3}{\sqrt{25u^2-25}}du = \frac{1}{3}\int \frac{1}{\sqrt{u^2-1}}du = \frac{1}{3}\cosh^{-1}(3x/5) + C$

45. $\tanh^{-1} x\Big]_0^{1/2} = \tanh^{-1}(1/2) - \tanh^{-1}(0) = \dfrac{1}{2}\ln\dfrac{1+1/2}{1-1/2} = \dfrac{1}{2}\ln 3$

46. $\sinh^{-1} t\Big]_0^{\sqrt{3}} = \sinh^{-1}\sqrt{3} - \sinh^{-1} 0 = \ln(\sqrt{3}+2)$

49. $A = \displaystyle\int_0^{\ln 3} \sinh 2x\,dx = \frac{1}{2}\cosh 2x\Big]_0^{\ln 3} = \frac{1}{2}[\cosh(2\ln 3) - 1]$,

but $\cosh(2\ln 3) = \cosh(\ln 9) = \dfrac{1}{2}(e^{\ln 9} + e^{-\ln 9}) = \dfrac{1}{2}(9 + 1/9) = 41/9$ so $A = \dfrac{1}{2}[41/9 - 1] = 16/9$.

50. $V = \pi\displaystyle\int_0^{\ln 2} \operatorname{sech}^2 x\,dx = \pi\tanh x\Big]_0^{\ln 2} = \pi\tanh(\ln 2) = 3\pi/5$

51. $V = \pi\displaystyle\int_0^5 (\cosh^2 2x - \sinh^2 2x)dx = \pi\int_0^5 dx = 5\pi$

52. $\displaystyle\int_0^1 \cosh ax\,dx = 2, \frac{1}{a}\sinh ax\Big]_0^1 = 2, \frac{1}{a}\sinh a = 2, \sinh a = 2a;$

let $f(a) = \sinh a - 2a$, then $a_{n+1} = a_n - \dfrac{\sinh a_n - 2a_n}{\cosh a_n - 2}$, $a_1 = 2.2, \ldots, a_4 = a_5 = 2.177318985.$

53. $y' = \sinh x$, $1 + (y')^2 = 1 + \sinh^2 x = \cosh^2 x$

$$L = \int_0^{\ln 2} \cosh x\,dx = \sinh x\Big]_0^{\ln 2} = \sinh(\ln 2) = \frac{1}{2}(e^{\ln 2} - e^{-\ln 2}) = \frac{1}{2}\left(2 - \frac{1}{2}\right) = \frac{3}{4}$$

54. $y' = \sinh(x/a)$, $1 + (y')^2 = 1 + \sinh^2(x/a) = \cosh^2(x/a)$

$$L = \int_0^{x_1} \cosh(x/a)dx = a\sinh(x/a)\Big]_0^{x_1} = a\sinh(x_1/a)$$

55. $\sinh(-x) = \dfrac{1}{2}(e^{-x} - e^x) = -\dfrac{1}{2}(e^x - e^{-x}) = -\sinh x$

$\cosh(-x) = \dfrac{1}{2}(e^{-x} + e^x) = \dfrac{1}{2}(e^x + e^{-x}) = \cosh x$

56. **(a)** $\cosh x + \sinh x = \dfrac{1}{2}(e^x + e^{-x}) + \dfrac{1}{2}(e^x - e^{-x}) = e^x$

(b) $\cosh x - \sinh x = \dfrac{1}{2}(e^x + e^{-x}) - \dfrac{1}{2}(e^x - e^{-x}) = e^{-x}$

(c) $\sinh x\cosh y + \cosh x\sinh y = \dfrac{1}{4}(e^x - e^{-x})(e^y + e^{-y}) + \dfrac{1}{4}(e^x + e^{-x})(e^y - e^{-y})$

$$= \frac{1}{4}[(e^{x+y} - e^{-x+y} + e^{x-y} - e^{-x-y}) + (e^{x+y} + e^{-x+y} - e^{x-y} - e^{-x-y})]$$

$$= \frac{1}{2}[e^{(x+y)} - e^{-(x+y)}] = \sinh(x+y)$$

(d) Let $y = x$ in Part (c).

(e) The proof is similar to Part (c), or: treat x as variable and y as constant, and differentiate the result in Part (c) with respect to x.

(f) Let $y = x$ in Part (e).

(g) Use $\cosh^2 x = 1 + \sinh^2 x$ together with Part (f).

(h) Use $\sinh^2 x = \cosh^2 x - 1$ together with Part (f).

57. **(a)** Divide $\cosh^2 x - \sinh^2 x = 1$ by $\cosh^2 x$.

(b) $\tanh(x+y) = \dfrac{\sinh x\cosh y + \cosh x\sinh y}{\cosh x\cosh y + \sinh x\sinh y} = \dfrac{\dfrac{\sinh x}{\cosh x} + \dfrac{\sinh y}{\cosh y}}{1 + \dfrac{\sinh x\sinh y}{\cosh x\cosh y}} = \dfrac{\tanh x + \tanh y}{1 + \tanh x\tanh y}$

(c) Let $y = x$ in Part (b).

58. **(a)** Let $y = \cosh^{-1} x$; then $x = \cosh y = \dfrac{1}{2}(e^y + e^{-y})$, $e^y - 2x + e^{-y} = 0$, $e^{2y} - 2xe^y + 1 = 0$, $e^y = \dfrac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}$. To determine which sign to take, note that $y \geq 0$ so $e^{-y} \leq e^y$, $x = (e^y + e^{-y})/2 \leq (e^y + e^y)/2 = e^y$, hence $e^y \geq x$ thus $e^y = x + \sqrt{x^2 - 1}$, $y = \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$.

(b) Let $y = \tanh^{-1} x$; then $x = \tanh y = \dfrac{e^y - e^{-y}}{e^y + e^{-y}} = \dfrac{e^{2y} - 1}{e^{2y} + 1}$, $xe^{2y} + x = e^{2y} - 1$,

$1 + x = e^{2y}(1 - x)$, $e^{2y} = (1 + x)/(1 - x)$, $2y = \ln \dfrac{1 + x}{1 - x}$, $y = \dfrac{1}{2} \ln \dfrac{1 + x}{1 - x}$.

59. **(a)** $\dfrac{d}{dx}(\cosh^{-1} x) = \dfrac{1 + x/\sqrt{x^2 - 1}}{x + \sqrt{x^2 - 1}} = 1/\sqrt{x^2 - 1}$

(b) $\dfrac{d}{dx}(\tanh^{-1} x) = \dfrac{d}{dx}\left[\dfrac{1}{2}(\ln(1 + x) - \ln(1 - x))\right] = \dfrac{1}{2}\left(\dfrac{1}{1 + x} + \dfrac{1}{1 - x}\right) = 1/(1 - x^2)$

60. Let $y = \text{sech}^{-1} x$ then $x = \text{sech}\, y = 1/\cosh y$, $\cosh y = 1/x$, $y = \cosh^{-1}(1/x)$; the proofs for the remaining two are similar.

61. If $|u| < 1$ then, by Theorem 8.8.6, $\displaystyle\int \frac{du}{1 - u^2} = \tanh^{-1} u + C$.

For $|u| > 1$, $\displaystyle\int \frac{du}{1 - u^2} = \coth^{-1} u + C = \tanh^{-1}(1/u) + C$.

62. **(a)** $\dfrac{d}{dx}(\text{sech}^{-1}|x|) = \dfrac{d}{dx}(\text{sech}^{-1}\sqrt{x^2}) = -\dfrac{1}{\sqrt{x^2}\sqrt{1 - x^2}} \dfrac{x}{\sqrt{x^2}} = -\dfrac{1}{x\sqrt{1 - x^2}}$

(b) Similar to solution of Part (a)

63. **(a)** $\displaystyle\lim_{x\to+\infty} \sinh x = \lim_{x\to+\infty} \frac{1}{2}(e^x - e^{-x}) = +\infty - 0 = +\infty$

(b) $\displaystyle\lim_{x\to-\infty} \sinh x = \lim_{x\to-\infty} \frac{1}{2}(e^x - e^{-x}) = 0 - \infty = -\infty$

(c) $\displaystyle\lim_{x\to+\infty} \tanh x = \lim_{x\to+\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = 1$

(d) $\displaystyle\lim_{x\to-\infty} \tanh x = \lim_{x\to-\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = -1$

(e) $\displaystyle\lim_{x\to+\infty} \sinh^{-1} x = \lim_{x\to+\infty} \ln(x + \sqrt{x^2 + 1}) = +\infty$

(f) $\displaystyle\lim_{x\to1^-} \tanh^{-1} x = \lim_{x\to1^-} \frac{1}{2}[\ln(1 + x) - \ln(1 - x)] = +\infty$

64. **(a)** $\displaystyle\lim_{x\to+\infty} (\cosh^{-1} x - \ln x) = \lim_{x\to+\infty} [\ln(x + \sqrt{x^2 - 1}) - \ln x]$

$$= \lim_{x\to+\infty} \ln \frac{x + \sqrt{x^2 - 1}}{x} = \lim_{x\to+\infty} \ln(1 + \sqrt{1 - 1/x^2}) = \ln 2$$

(b) $\displaystyle\lim_{x\to+\infty} \frac{\cosh x}{e^x} = \lim_{x\to+\infty} \frac{e^x + e^{-x}}{2e^x} = \lim_{x\to+\infty} \frac{1}{2}(1 + e^{-2x}) = 1/2$

65. For $|x| < 1$, $y = \tanh^{-1} x$ is defined and $dy/dx = 1/(1 - x^2) > 0$; $y'' = 2x/(1 - x^2)^2$ changes sign at $x = 0$, so there is a point of inflection there.

66. Let $x = -u/a$, $\displaystyle\int \frac{1}{\sqrt{u^2 - a^2}}\, du = -\int \frac{a}{a\sqrt{x^2 - 1}}\, dx = -\cosh^{-1} x + C = -\cosh^{-1}(-u/a) + C$.

$$-\cosh^{-1}(-u/a) = -\ln(-u/a + \sqrt{u^2/a^2 - 1}) = \ln\left[\frac{a}{-u + \sqrt{u^2 - a^2}} \frac{u + \sqrt{u^2 - a^2}}{u + \sqrt{u^2 - a^2}}\right]$$

$$= \ln\left|u + \sqrt{u^2 - a^2}\right| - \ln a = \ln|u + \sqrt{u^2 - a^2}| + C_1$$

so $\displaystyle\int \frac{1}{\sqrt{u^2 - a^2}}\, du = \ln\left|u + \sqrt{u^2 - a^2}\right| + C_2$.

67. Using $\sinh x + \cosh x = e^x$ (Exercise 56a), $(\sinh x + \cosh x)^n = (e^x)^n = e^{nx} = \sinh nx + \cosh nx$.

68. $\displaystyle\int_{-a}^{a} e^{tx}dx = \frac{1}{t}e^{tx}\Big]_{-a}^{a} = \frac{1}{t}(e^{at} - e^{-at}) = \frac{2\sinh at}{t}$ for $t \neq 0$.

69. **(a)** $y' = \sinh(x/a), 1 + (y')^2 = 1 + \sinh^2(x/a) = \cosh^2(x/a)$

$$L = 2\int_0^b \cosh(x/a)\,dx = 2a\sinh(x/a)\Big]_0^b = 2a\sinh(b/a)$$

(b) The highest point is at $x = b$, the lowest at $x = 0$, so $S = a\cosh(b/a) - a\cosh(0) = a\cosh(b/a) - a$.

70. From Part (a) of Exercise 69, $L = 2a\sinh(b/a)$ so $120 = 2a\sinh(50/a), a\sinh(50/a) = 60$. Let $u = 50/a$, then $a = 50/u$ so $(50/u)\sinh u = 60, \sinh u = 1.2u$. If $f(u) = \sinh u - 1.2u$, then $u_{n+1} = u_n - \dfrac{\sinh u_n - 1.2u_n}{\cosh u_n - 1.2}; u_1 = 1, \ldots, u_5 = u_6 = 1.064868548 \approx 50/a$ so $a \approx 46.95415231$. From Part (b), $S = a\cosh(b/a) - a \approx 46.95415231[\cosh(1.064868548) - 1] \approx 29.2\,\text{ft}$.

71. From Part (b) of Exercise 69, $S = a\cosh(b/a) - a$ so $30 = a\cosh(200/a) - a$. Let $u = 200/a$, then $a = 200/u$ so $30 = (200/u)[\cosh u - 1], \cosh u - 1 = 0.15u$. If $f(u) = \cosh u - 0.15u - 1$, then $u_{n+1} = u_n - \dfrac{\cosh u_n - 0.15u_n - 1}{\sinh u_n - 0.15}; u_1 = 0.3, \ldots, u_4 = u_5 = 0.297792782 \approx 200/a$ so $a \approx 671.6079505$. From Part (a), $L = 2a\sinh(b/a) \approx 2(671.6079505)\sinh(0.297792782) \approx 405.9\,\text{ft}$.

72. **(a)** When the bow of the boat is at the point (x, y) and the person has walked a distance D, then the person is located at the point $(0, D)$, the line segment connecting $(0, D)$ and (x, y) has length a; thus $a^2 = x^2 + (D - y)^2$, $D = y + \sqrt{a^2 - x^2} = a\,\text{sech}^{-1}(x/a)$.

(b) Find D when $a = 15$, $x = 10$: $D = 15\,\text{sech}^{-1}(10/15) = 15\ln\left(\dfrac{1 + \sqrt{5/9}}{2/3}\right) \approx 14.44$ m.

(c) $dy/dx = -\dfrac{a^2}{x\sqrt{a^2 - x^2}} + \dfrac{x}{\sqrt{a^2 - x^2}} = \dfrac{1}{\sqrt{a^2 - x^2}}\left[-\dfrac{a^2}{x} + x\right] = -\dfrac{1}{x}\sqrt{a^2 - x^2}$,

$1 + [y']^2 = 1 + \dfrac{a^2 - x^2}{x^2} = \dfrac{a^2}{x^2}$; with $a = 15$ and $x = 5$, $L = \displaystyle\int_5^{15} \frac{225}{x^2}\,dx = -\frac{225}{x}\Big]_5^{15} = 30$ m.

CHAPTER 7 SUPPLEMENTARY EXERCISES

1. **(a)** $f(g(x)) = x$ for all x in the domain of g, and $g(f(x)) = x$ for all x in the domain of f.

(b) They are reflections of each other through the line $y = x$.

(c) The domain of one is the range of the other and vice versa.

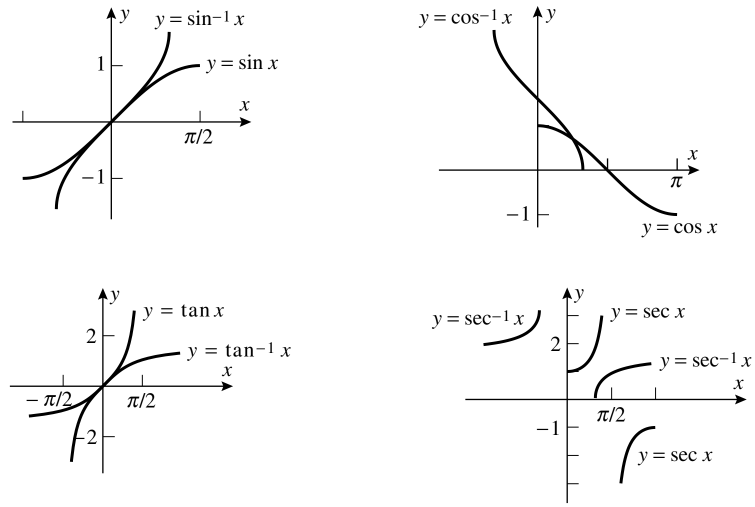
(d) The equation $y = f(x)$ can always be solved for x as a function of y. Functions with no inverses include $y = x^2$, $y = \sin x$.

(e) Yes, g is continuous; this is evident from the statement about the graphs in Part (b) above.

(f) Yes, g must be differentiable (where $f' \neq 0$); this can be inferred from the graphs. Note that if $f' = 0$ at a point then g' cannot exist (infinite slope).

2. **(a)** For $\sin x$, $-\pi/2 \le x \le \pi/2$; for $\cos x$, $0 \le x \le \pi$; for $\tan x$, $-\pi/2 < x < \pi/2$; for $\sec x$, $0 \le x < \pi/2$ or $\pi/2 < x \le \pi$.

(b)



3. **(a)** $x = f(y) = 8y^3 - 1$; $y = f^{-1}(x) = \left(\dfrac{x+1}{8}\right)^{1/3} = \dfrac{1}{2}(x+1)^{1/3}$

(b) $f(x) = (x-1)^2$; f does not have an inverse because f is not one-to-one, for example $f(0) = f(2) = 1$.

(c) $x = f(y) = (e^y)^2 + 1$; $y = f^{-1}(x) = \ln\sqrt{x-1} = \frac{1}{2}\ln(x-1)$

(d) $x = f(y) = \dfrac{y+2}{y-1}$; $y = f^{-1}(x) = \dfrac{x+2}{x-1}$

4. $f'(x) = \dfrac{ad-bc}{(cx+d)^2}$; if $ad - bc = 0$ then the function represents a horizontal line, no inverse. If $ad - bc \neq 0$ then $f'(x) > 0$ or $f'(x) < 0$ so f is invertible. If $x = f(y) = \dfrac{ay+b}{cy+d}$ then $y = f^{-1}(x) = \dfrac{b - xd}{xc - a}$.

5. $3\ln\left(e^{2x}(e^x)^3\right) + 2\exp(\ln 1) = 3\ln e^{2x} + 3\ln(e^x)^3 + 2\cdot 1 = 3(2x) + (3\cdot 3)x + 2 = 15x + 2$

6. Draw equilateral triangles of sides 5, 12, 13, and 3, 4, 5. Then $\sin[\cos^{-1}(4/5)] = 3/5$, $\sin[\cos^{-1}(5/13)] = 12/13$, $\cos[\sin^{-1}(4/5)] = 3/5$, $\cos[\sin^{-1}(5/13)] = 12/13$

(a)
$$\begin{aligned}\cos[\cos^{-1}(4/5) + \sin^{-1}(5/13)] &= \cos(\cos^{-1}(4/5))\cos(\sin^{-1}(5/13)) \\ &\quad - \sin(\cos^{-1}(4/5))\sin(\sin^{-1}(5/13)) \\ &= \frac{4}{5}\frac{12}{13} - \frac{3}{5}\frac{5}{13} = \frac{33}{65}.\end{aligned}$$

(b)
$$\begin{aligned}\sin[\sin^{-1}(4/5) + \cos^{-1}(5/13)] &= \sin(\sin^{-1}(4/5))\cos(\cos^{-1}(5/13)) \\ &\quad + \cos(\sin^{-1}(4/5))\sin(\cos^{-1}(5/13)) \\ &= \frac{4}{5}\frac{5}{13} + \frac{3}{5}\frac{12}{13} = \frac{56}{65}.\end{aligned}$$

7. **(a)**
$$\begin{aligned}\cosh 3x &= \cosh(2x + x) = \cosh 2x\cosh x + \sinh 2x\sinh x \\ &= (2\cosh^2 x - 1)\cosh x + (2\sinh x\cosh x)\sinh x \\ &= 2\cosh^3 x - \cosh x + 2\sinh^2 x\cosh x \\ &= 2\cosh^3 x - \cosh x + 2(\cosh^2 x - 1)\cosh x = 4\cosh^3 x - 3\cosh x\end{aligned}$$

(b) from Theorem 7.8.2 with x replaced by $\dfrac{x}{2}$: $\cosh x = 2\cosh^2\dfrac{x}{2} - 1$,

$2\cosh^2\dfrac{x}{2} = \cosh x + 1$, $\cosh^2\dfrac{x}{2} = \dfrac{1}{2}(\cosh x + 1)$,

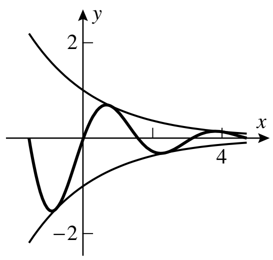
$\cosh\dfrac{x}{2} = \sqrt{\dfrac{1}{2}(\cosh x + 1)}$ (because $\cosh\dfrac{x}{2} > 0$)

(c) from Theorem 7.8.2 with x replaced by $\dfrac{x}{2}$: $\cosh x = 2\sinh^2\dfrac{x}{2} + 1$,

$2\sinh^2\dfrac{x}{2} = \cosh x - 1$, $\sinh^2\dfrac{x}{2} = \dfrac{1}{2}(\cosh x - 1)$, $\sinh\dfrac{x}{2} = \pm\sqrt{\dfrac{1}{2}(\cosh x - 1)}$

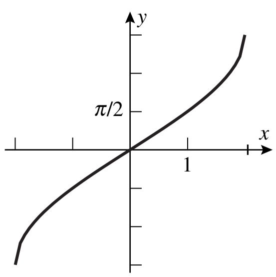
8. $Y = \ln(Ce^{kt}) = \ln C + \ln e^{kt} = \ln C + kt$, a line with slope k and Y-intercept $\ln C$

9. (a)

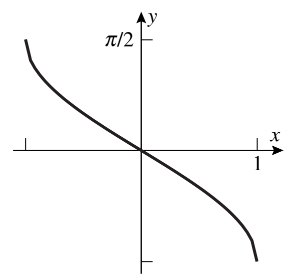


(b) The curve $y = e^{-x/2}\sin 2x$ has x-intercepts at $x = -\pi/2, 0, \pi/2, \pi, 3\pi/2$. It intersects the curve $y = e^{-x/2}$ at $x = \pi/4, 5\pi/4$, and it intersects the curve $y = -e^{-x/2}$ at $x = -\pi/4, 3\pi/4$.

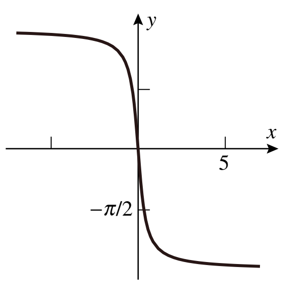
10. (a)



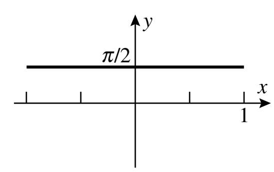
(b)



(c)

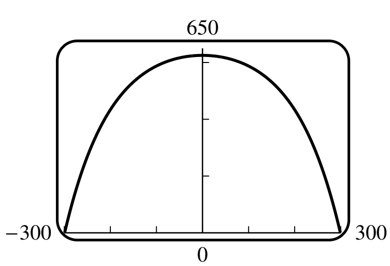


(d)



11. Set $a = 68.7672$, $b = 0.0100333$, $c = 693.8597$, $d = 299.2239$.

(a)

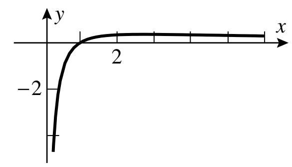


(b) $L = 2\displaystyle\int_0^d \sqrt{1 + a^2b^2\sinh^2 bx}\,dx$

$= 1480.2798$ ft

(c) $x = 283.6249$ ft **(d)** $82°$

12. **(a)** If $x^k = e^x$ then $k \ln x = x$, or $\dfrac{\ln x}{x} = \dfrac{1}{k}$. The steps are reversible.



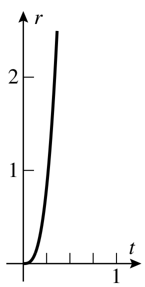
(b) By zooming it is seen that the maximum value of y is approximately 0.368 (actually, $1/e$), so there are two distinct solutions of $x^k = e^x$ whenever $k > 1/0.368 \approx 2.717$.

(c) $x \approx 1.155$

13. **(a)** The function $\ln x - x^{0.2}$ is negative at $x = 1$ and positive at $x = 4$, so it must be zero in between (IVT).

(b) $x = 3.654$

14. **(a)**



(b) $r = 1$ when $t \approx 0.673080$ s.

(c) $dr/dt = 4.48$ m/s.

15. **(a)** $y = x^3 + 1$ so $y' = 3x^2$. **(b)** $y' = \dfrac{abe^{-x}}{(1 + be^{-x})^2}$

(c) $y = \dfrac{1}{2}\ln x + \dfrac{1}{3}\ln(x+1) - \ln \sin x + \ln \cos x$, so

$$y' = \frac{1}{2x} + \frac{1}{3(x+1)} - \frac{\cos x}{\sin x} - \frac{\sin x}{\cos x} = \frac{5x+3}{6x(x+1)} - \cot x - \tan x.$$

(d) $\ln y = \dfrac{\ln(1+x)}{x}$, $\dfrac{y'}{y} = \dfrac{x/(1+x) - \ln(1+x)}{x^2} = \dfrac{1}{x(1+x)} - \dfrac{\ln(1+x)}{x^2}$,

$$\frac{dy}{dx} = \frac{1}{x}(1+x)^{(1/x)-1} - \frac{(1+x)^{(1/x)}}{x^2}\ln(1+x)$$

(e) $\ln y = e^x \ln x$, $\dfrac{y'}{y} = e^x\left(\dfrac{1}{x} + \ln x\right)$, $\dfrac{dy}{dx} = x^{e^x} e^x \left(\dfrac{1}{x} + \ln x\right) = e^x\left[x^{e^x - 1} + x^{e^x}\ln x\right]$

(f) $y = \sinh^{-1}\dfrac{1}{x^2}$, $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 + (1/x^2)^2}}\dfrac{-2}{x^3} = -\dfrac{2}{x\sqrt{x^4+1}}$

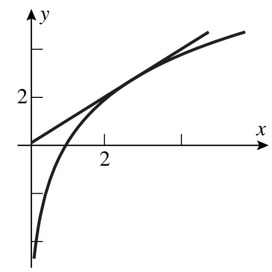
16. $y' = ae^{ax}\sin bx + be^{ax}\cos bx$ and $y'' = (a^2 - b^2)e^{ax}\sin bx + 2abe^{ax}\cos bx$, so $y'' - 2ay' + (a^2+b^2)y$ $= (a^2 - b^2)e^{ax}\sin bx + 2abe^{ax}\cos bx - 2a(ae^{ax}\sin bx + be^{ax}\cos bx) + (a^2+b^2)e^{ax}\sin bx = 0$.

17. $\sin(\tan^{-1} x) = x/\sqrt{1+x^2}$ and $\cos(\tan^{-1} x) = 1/\sqrt{1+x^2}$, and $y' = \dfrac{1}{1+x^2}$, $y'' = \dfrac{-2x}{(1+x^2)^2}$, hence

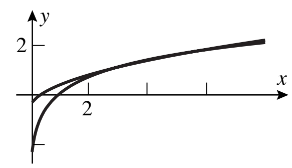
$$y'' + 2\sin y \cos^3 y = \frac{-2x}{(1+x^2)^2} + 2\frac{x}{\sqrt{1+x^2}}\frac{1}{(1+x^2)^{3/2}} = 0.$$

18. $y' = a\cosh ax, y'' = a^2\sinh ax = a^2 y$

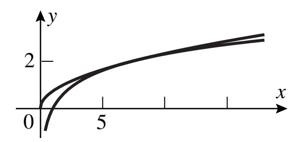
19. Set $y = \log_b x$ and solve $y' = 1$: $y' = \dfrac{1}{x\ln b} = 1$ so $x = \dfrac{1}{\ln b}$. The curves intersect when (x, x) lies on the graph of $y = \log_b x$, so $x = \log_b x$. From Formula (9), Section 7.2, $\log_b x = \dfrac{\ln x}{\ln b}$ from which $\ln x = 1$, $x = e$, $\ln b = 1/e$, $b = e^{1/e} \approx 1.4447$.



20. **(a)** Find the point of intersection: $f(x) = \sqrt{x} + k = \ln x$. The slopes are equal, so $m_1 = \dfrac{1}{x} = m_2 = \dfrac{1}{2\sqrt{x}}$, $\sqrt{x} = 2$, $x = 4$. Then $\ln 4 = \sqrt{4} + k$, $k = \ln 4 - 2$.



(b) Since the slopes are equal $m_1 = \dfrac{k}{2\sqrt{x}} = m_2 = \dfrac{1}{x}$, so $k\sqrt{x} = 2$. At the point of intersection $k\sqrt{x} = \ln x$, $2 = \ln x$, $x = e^2$, $k = 2/e$.



21. $\lim_{x\to 0^+} f(x) = \lim_{x\to +\infty} f(x) = +\infty$ and $f'(x) = \dfrac{e^x(x-2)}{x^3}$, stationary point at $x = 2$. We know $f(x)$ has no maximum and an absolute minimum; by Theorem 4.5.5 $f(x)$ has an absolute minimum at $x = 2$, and $m = e^2/4$.

22. $f'(x) = (1 + \ln x)x^x$, critical point at $x = 1/e$; $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{x\ln x} = 1$, $\lim_{x\to +\infty} f(x) = +\infty$; no absolute maximum, absolute minimum $m = e^{-1/e}$ at $x = 1/e$

23. $f_{\text{ave}} = \dfrac{1}{e-1}\displaystyle\int_1^e \frac{1}{x}dx = \frac{1}{e-1}\ln x\Big]_1^e = \frac{1}{e-1}$; $\dfrac{1}{x^*} = \dfrac{1}{e-1}$, $x^* = e - 1$

24. Find t so that $N'(t)$ is maximum. The size of the population is increasing most rapidly when $t = 8.4$ years.

25. $u = \ln x$, $du = (1/x)dx$; $\displaystyle\int_1^2 \frac{1}{u}du = \ln u\Big]_1^2 = \ln 2$

26. $\displaystyle\int_0^1 e^{-x/2}dx = 2(1 - 1/\sqrt{e})$

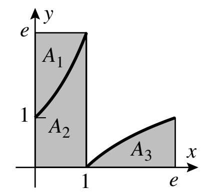
27. $u = e^{-2x}$, $du = -2e^{-2x}dx$; $-\dfrac{1}{2}\displaystyle\int_1^{1/4}(1+\cos u)du = \frac{3}{8} + \frac{1}{2}\left(\sin 1 - \sin\frac{1}{4}\right)$

28. Divide $e^x + 3$ into e^{2x} to get $\dfrac{e^{2x}}{e^x+3} = e^x - \dfrac{3e^x}{e^x+3}$ so

$$\int \frac{e^{2x}}{e^x+3}dx = \int e^x dx - 3\int \frac{e^x}{e^x+3}dx = e^x - 3\ln(e^x+3) + C$$

29. Since $y = e^x$ and $y = \ln x$ are inverse functions, their graphs are symmetric with respect to the line $y = x$; consequently the areas A_1 and A_3 are equal (see figure). But $A_1 + A_2 = e$, so

$$\int_1^e \ln x dx + \int_0^1 e^x dx = A_2 + A_3 = A_2 + A_1 = e$$



30. $\displaystyle\sum_{k=1}^{n} \frac{e^{k/n}}{n} = \sum_{k=1}^{n} f(x_k^*)\Delta x$ where $f(x) = e^x, x_k^* = k/n$, and $\Delta x = 1/n$ for $0 \le x \le 1$. Thus

$$\lim_{n\to+\infty} \sum_{k=1}^{n} \frac{e^{k/n}}{n} = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \int_0^1 e^x dx = e - 1.$$

31. Since $f(x) = \dfrac{1}{x}$ is positive and increasing on the interval $[1, 2]$, the left endpoint approximation overestimates the integral of $\dfrac{1}{x}$ and the right endpoint approximation underestimates it.

(a) For $n = 5$ this becomes

$$0.2\left[\frac{1}{1.2} + \frac{1}{1.4} + \frac{1}{1.6} + \frac{1}{1.8} + \frac{1}{2.0}\right] < \int_1^2 \frac{1}{x} dx < 0.2\left[\frac{1}{1.0} + \frac{1}{1.2} + \frac{1}{1.4} + \frac{1}{1.6} + \frac{1}{1.8}\right]$$

(b) For general n the left endpoint approximation to $\displaystyle\int_1^2 \frac{1}{x} dx = \ln 2$ is

$\displaystyle\frac{1}{n}\sum_{k=1}^{n} \frac{1}{1 + (k-1)/n} = \sum_{k=1}^{n} \frac{1}{n + k - 1} = \sum_{k=0}^{n-1} \frac{1}{n + k}$ and the right endpoint approximation is $\displaystyle\sum_{k=1}^{n} \frac{1}{n + k}$. This yields $\displaystyle\sum_{k=1}^{n} \frac{1}{n + k} < \int_1^2 \frac{1}{x} dx < \sum_{k=0}^{n-1} \frac{1}{n + k}$ which is the desired inequality.

(c) By telescoping, the difference is $\dfrac{1}{n} - \dfrac{1}{2n} = \dfrac{1}{2n}$ so $\dfrac{1}{2n} \le 0.1$, $n \ge 5$

(d) $n \ge 1000$

32. **(a)** $x_k^* = 0, 1, 2, 3, 4$

$$\sum_{k=1}^{4} f(x_k^*)\Delta x = \left(e^0 + e^1 + e^2 + e^3 + e^4\right)(1) = (1 - e^5)/(1 - e) = 85.791$$

(b) $x_k^* = 1, 2, 3, 4, 5$

$$\sum_{k=1}^{4} f(x_k^*)\Delta x = \left(e^1 + e^2 + e^3 + e^4 + e^5\right)(1) = e(1 - e^5)/(1 - e) = 233.204$$

(c) $x_k^* = 1/2, 3/2, 5/2, 7/2, 9/2$

$$\sum_{k=1}^{4} f(x_k^*)\Delta x = \left(e^{1/2} + e^{3/2} + e^{5/2} + e^{7/2} + e^{9/2}\right)(1) = e^{1/2}(1 - e^5)/(1 - e) = 141.446$$

33. $0.351220577, 0.420535296, 0.386502483$

34. $1.63379940, 1.805627583, 1.717566087$

35. $f(x) = e^x, [a, b] = [0, 1], \Delta x = \frac{1}{n}; \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*) = \int_0^1 e^x\,dx = e - 1$

36. In the case $+\infty - (-\infty)$ the limit is $+\infty$; in the case $-\infty - (+\infty)$ the limit is $-\infty$, because large positive (negative) quantities are added to large positive (negative) quantities. The cases $+\infty - (+\infty)$ and $-\infty - (-\infty)$ are indeterminate; large numbers of opposite sign are subtracted, and more information about the sizes is needed.

37. **(a)** when the limit takes the form $0/0$ or ∞/∞

(b) Not necessarily; only if $\lim_{x\to a} f(x) = 0$. Consider $g(x) = x$; $\lim_{x\to 0} g(x) = 0$. For $f(x)$ choose $\cos x$, x^2, and $|x|^{1/2}$. Then: $\lim_{x\to 0} \frac{\cos x}{x}$ does not exist, $\lim_{x\to 0} \frac{x^2}{x} = 0$, and $\lim_{x\to 0} \frac{|x|^{1/2}}{x^2} = +\infty$.

38. **(a)** $\lim_{x\to+\infty}(e^x - x^2) = \lim_{x\to+\infty} x^2(e^x/x^2 - 1)$, but $\lim_{x\to+\infty} \frac{e^x}{x^2} = \lim_{x\to+\infty} \frac{e^x}{2x} = \lim_{x\to+\infty} \frac{e^x}{2} = +\infty$

so $\lim_{x\to+\infty}(e^x/x^2 - 1) = +\infty$ and thus $\lim_{x\to+\infty} x^2(e^x/x^2 - 1) = +\infty$

(b) $\lim_{x\to 1} \frac{\ln x}{x^4 - 1} = \lim_{x\to 1} \frac{1/x}{4x^3} = \frac{1}{4}$; $\lim_{x\to 1} \sqrt{\frac{\ln x}{x^4 - 1}} = \sqrt{\lim_{x\to 1} \frac{\ln x}{x^4 - 1}} = \frac{1}{2}$

(c) $\lim_{x\to 0} a^x \ln a = \ln a$