CHAPTER 6

Applications of the Definite Integral in Geometry, Science, and Engineering

EXERCISE SET 6.1

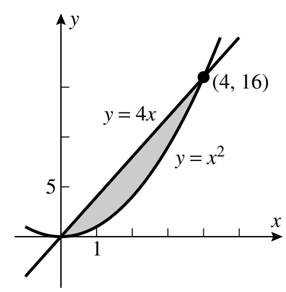
1. $A = \int_{-1}^{2}(x^2+1-x)dx = (x^3/3+x-x^2/2)\Big]_{-1}^{2} = 9/2$

2. $A = \int_{0}^{4}(\sqrt{x}+x/4)dx = (2x^{3/2}/3+x^2/8)\Big]_{0}^{4} = 22/3$

3. $A = \int_{1}^{2}(y-1/y^2)dy = (y^2/2+1/y)\Big]_{1}^{2} = 1$

4. $A = \int_{0}^{2}(2-y^2+y)dy = (2y-y^3/3+y^2/2)\Big]_{0}^{2} = 10/3$

5. **(a)** $A = \int_{0}^{4}(4x-x^2)dx = 32/3$ **(b)** $A = \int_{0}^{16}(\sqrt{y}-y/4)dy = 32/3$

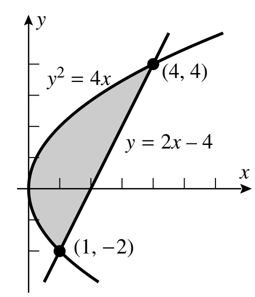


6. Eliminate x to get $y^2 = 4(y+4)/2$, $y^2-2y-8=0$, $(y-4)(y+2)=0$; $y=-2,4$ with corresponding values of $x=1,4$.

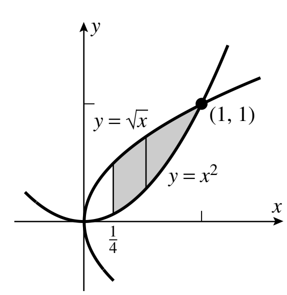
(a) $A = \int_{0}^{1}[2\sqrt{x}-(-2\sqrt{x})]dx + \int_{1}^{4}[2\sqrt{x}-(2x-4)]dx$

$= \int_{0}^{1}4\sqrt{x}dx + \int_{1}^{4}(2\sqrt{x}-2x+4)dx = 8/3+19/3 = 9$

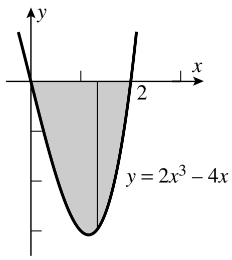
(b) $A = \int_{-2}^{4}[(y/2+2)-y^2/4]dy = 9$



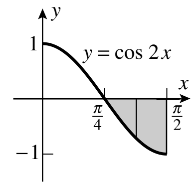
7. $A = \int_{1/4}^{1}(\sqrt{x}-x^2)dx = 49/192$



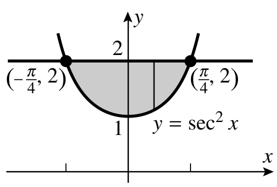
8. $A = \int_0^2 [0 - (x^3 - 4x)]dx$
$= \int_0^2 (4x - x^3)dx = 4$



9. $A = \int_{\pi/4}^{\pi/2} (0 - \cos 2x)dx$
$= -\int_{\pi/4}^{\pi/2} \cos 2x\, dx = 1/2$

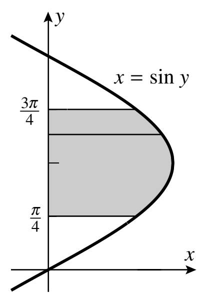


10. Equate $\sec^2 x$ and 2 to get $\sec^2 x = 2$,

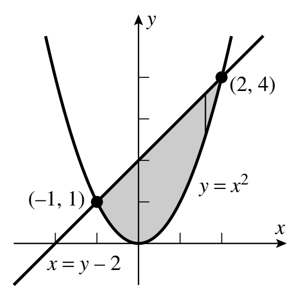


$\sec x = \pm\sqrt{2}$, $x = \pm\pi/4$
$A = \int_{-\pi/4}^{\pi/4} (2 - \sec^2 x)dx = \pi - 2$

11. $A = \int_{\pi/4}^{3\pi/4} \sin y\, dy = \sqrt{2}$

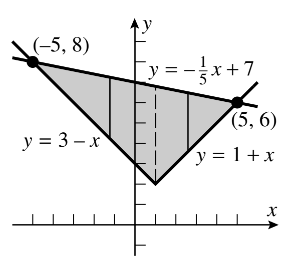


12. $A = \int_{-1}^{2} [(x + 2) - x^2]dx = 9/2$

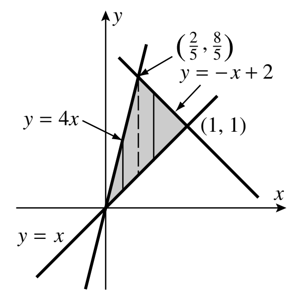


13. $y = 2 + |x - 1| = \begin{cases} 3 - x, & x \le 1 \\ 1 + x, & x \ge 1 \end{cases}$,

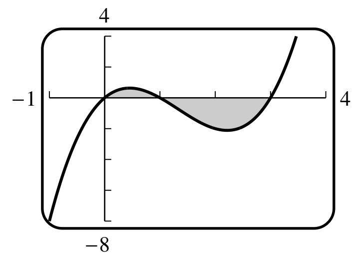
$A = \int_{-5}^{1} \left[\left(-\frac{1}{5}x + 7\right) - (3 - x)\right] dx$
$+ \int_{1}^{5} \left[\left(-\frac{1}{5}x + 7\right) - (1 + x)\right] dx$
$= \int_{-5}^{1} \left(\frac{4}{5}x + 4\right) dx + \int_{1}^{5} \left(6 - \frac{6}{5}x\right) dx$
$= 72/5 + 48/5 = 24$



14. $A = \int_0^{2/5}(4x - x)dx$
$+ \int_{2/5}^{1}(-x + 2 - x)dx$
$= \int_0^{2/5} 3x\,dx + \int_{2/5}^{1}(2 - 2x)dx = 3/5$

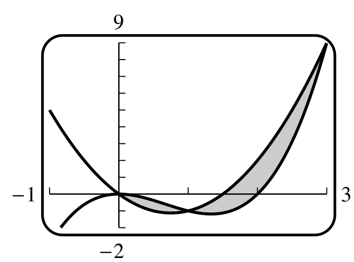


15. $A = \int_0^1 (x^3 - 4x^2 + 3x)dx$
$+ \int_1^3 [-(x^3 - 4x^2 + 3x)]dx$
$= 5/12 + 32/12 = 37/12$



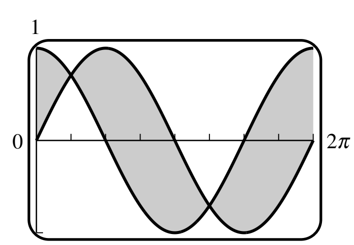
16. Equate $y = x^3 - 2x^2$ and $y = 2x^2 - 3x$ to get $x^3 - 4x^2 + 3x = 0$,
$x(x-1)(x-3) = 0; x = 0, 1, 3$
with corresponding values of $y = 0, -1.9$.

$A = \int_0^1 [(x^3 - 2x^2) - (2x^2 - 3x)]dx$
$+ \int_1^3 [(2x^3 - 3x) - (x^3 - 2x^2)]dx$
$= \int_0^1 (x^3 - 4x^2 + 3x)dx + \int_1^3 (-x^3 + 4x^2 - 3x)dx$
$= \frac{5}{12} + \frac{8}{3} = \frac{37}{12}$

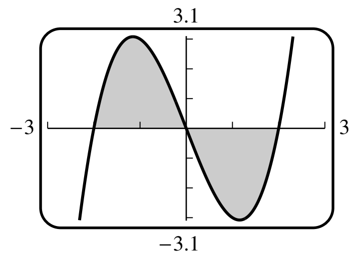


17. From the symmetry of the region

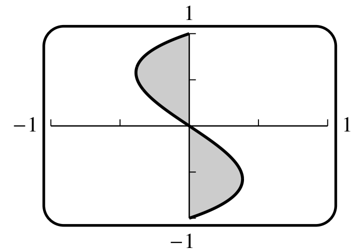
$$A = 2\int_{\pi/4}^{5\pi/4}(\sin x - \cos x)dx = 4\sqrt{2}$$



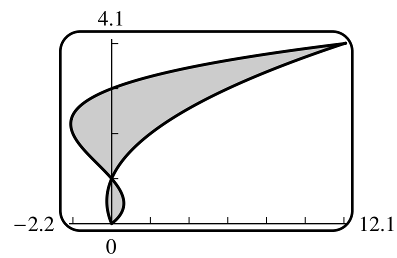
18. The region is symmetric about the origin so
$A = 2\int_0^2 |x^3 - 4x|dx = 8$



19. $A = \int_{-1}^0 (y^3 - y)dy + \int_0^1 -(y^3 - y)dy$
$= 1/2$



20. $A = \int_0^1 \left[y^3 - 4y^2 + 3y - (y^2 - y)\right]\, dy$
$+ \int_1^4 \left[y^2 - y - (y^3 - 4y^2 + 3y)\right]\, dy$
$= 7/12 + 45/4 = 71/6$



21. Solve $3-2x = x^6+2x^5-3x^4+x^2$ to find the real roots $x = -3, 1$; from a plot it is seen that the line is above the polynomial when $-3 < x < 1$, so $A = \int_{-3}^1 (3-2x-(x^6+2x^5-3x^4+x^2))\, dx = 9152/105$

22. Solve $x^5 - 2x^3 - 3x = x^3$ to find the roots $x = 0, \pm\frac{1}{2}\sqrt{6 + 2\sqrt{21}}$. Thus, by symmetry,
$A = 2\int_0^{\sqrt{(6+2\sqrt{21})/2}} (x^3 - (x^5 - 2x^3 - 3x))\, dx = \frac{27}{4} + \frac{7}{4}\sqrt{21}$

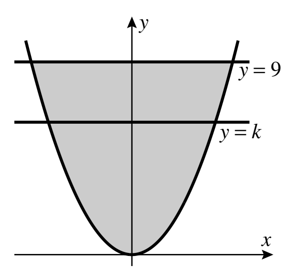
23. $\int_0^k 2\sqrt{y}dy = \int_k^9 2\sqrt{y}dy$

$\int_0^k y^{1/2}dy = \int_k^9 y^{1/2}dy$

$\frac{2}{3}k^{3/2} = \frac{2}{3}(27 - k^{3/2})$

$k^{3/2} = 27/2$

$k = (27/2)^{2/3} = 9/\sqrt[3]{4}$

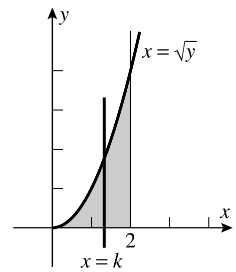


24. $\int_0^k x^2 dx = \int_k^2 x^2 dx$

$\frac{1}{3}k^3 = \frac{1}{3}(8 - k^3)$

$k^3 = 4$

$k = \sqrt[3]{4}$



25. **(a)** $A = \int_0^2 (2x - x^2)dx = 4/3$

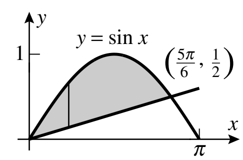
(b) $y = mx$ intersects $y = 2x - x^2$ where $mx = 2x - x^2, x^2 + (m-2)x = 0, x(x + m - 2) = 0$ so $x = 0$ or $x = 2 - m$. The area below the curve and above the line is

$$\int_0^{2-m} (2x - x^2 - mx)dx = \int_0^{2-m} [(2-m)x - x^2]dx = \left[\frac{1}{2}(2-m)x^2 - \frac{1}{3}x^3\right]_0^{2-m} = \frac{1}{6}(2-m)^3$$

so $(2-m)^3/6 = (1/2)(4/3) = 2/3, (2-m)^3 = 4, m = 2 - \sqrt[3]{4}$.

26. The line through $(0, 0)$ and $(5\pi/6, 1/2)$ is $y = \frac{3}{5\pi}x$;

$$A = \int_0^{5\pi/6} \left(\sin x - \frac{3}{5\pi}x\right) dx = \frac{\sqrt{3}}{2} - \frac{5}{24}\pi + 1$$



27. **(a)** It gives the area of the region that is between f and g when $f(x) > g(x)$ <u>minus</u> the area of the region between f and g when $f(x) < g(x)$, for $a \le x \le b$.

(b) It gives the area of the region that is between f and g for $a \le x \le b$.

28. **(b)** $\lim_{n\to+\infty} \int_0^1 (x^{1/n} - x)\, dx = \lim_{n\to+\infty} \left[\frac{n}{n+1}x^{(n+1)/n} - \frac{x^2}{2}\right]_0^1 = \lim_{n\to+\infty} \left(\frac{n}{n+1} - \frac{1}{2}\right) = 1/2$

29. The curves intersect at $x = 0$ and, by Newton's Method, at $x \approx 2.595739080 = b$, so $A \approx \int_0^b (\sin x - 0.2x)dx = -\left[\cos x + 0.1x^2\right]_0^b \approx 1.180898334$

30. By Newton's Method, the points of intersection are at $x \approx \pm 0.824132312$, so with $b = 0.824132312$ we have $A \approx 2\int_0^b (\cos x - x^2)dx = 2(\sin x - x^3/3)\Big]_0^b \approx 1.094753609$

31. distance $= \int |v|\, dt$, so

(a) distance $= \int_0^{60} (3t - t^2/20)\, dt = 1800$ ft.

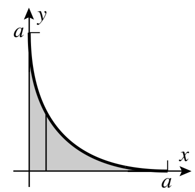
(b) If $T \le 60$ then distance $= \int_0^T (3t - t^2/20)\, dt = \frac{3}{2}T^2 - \frac{1}{60}T^3$ ft.

32. Since $a_1(0) = a_2(0) = 0$, $A = \int_0^T (a_2(t) - a_1(t))\,dt = v_2(T) - v_1(T)$ is the difference in the velocities of the two cars at time T.

33. Solve $x^{1/2} + y^{1/2} = a^{1/2}$ for y to get

$$y = (a^{1/2} - x^{1/2})^2 = a - 2a^{1/2}x^{1/2} + x$$

$$A = \int_0^a (a - 2a^{1/2}x^{1/2} + x)dx = a^2/6$$



34. Solve for y to get $y = (b/a)\sqrt{a^2 - x^2}$ for the upper half of the ellipse; make use of symmetry to get $A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}dx = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}dx = \frac{4b}{a} \cdot \frac{1}{4}\pi a^2 = \pi ab$.

35. Let A be the area between the curve and the x-axis and A_R the area of the rectangle, then $A = \int_0^b kx^m dx = \frac{k}{m+1}x^{m+1}\Big]_0^b = \frac{kb^{m+1}}{m+1}$, $A_R = b(kb^m) = kb^{m+1}$, so $A/A_R = 1/(m+1)$.

EXERCISE SET 6.2

1. $V = \pi \int_{-1}^3 (3 - x)dx = 8\pi$

2. $V = \pi \int_0^1 [(2 - x^2)^2 - x^2]dx$

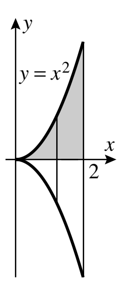
$= \pi \int_0^1 (4 - 5x^2 + x^4)dx$

$= 38\pi/15$

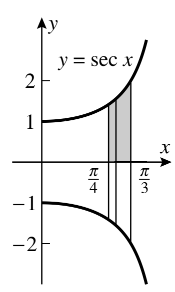
3. $V = \pi \int_0^2 \frac{1}{4}(3 - y)^2 dy = 13\pi/6$

4. $V = \pi \int_{1/2}^2 (4 - 1/y^2)dy = 9\pi/2$

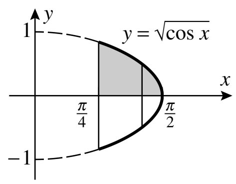
5. $V = \pi \int_0^2 x^4 dx = 32\pi/5$



6. $V = \pi \int_{\pi/4}^{\pi/3} \sec^2 x\,dx = \pi(\sqrt{3} - 1)$

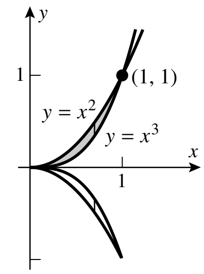


7. $V = \pi \int_{\pi/4}^{\pi/2} \cos x \, dx = (1 - \sqrt{2}/2)\pi$



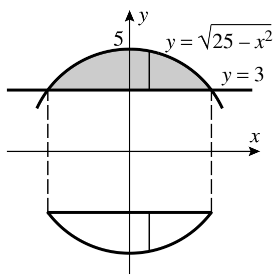
8. $V = \pi \int_0^1 [(x^2)^2 - (x^3)^2] dx$

$= \pi \int_0^1 (x^4 - x^6) dx = 2\pi/35$



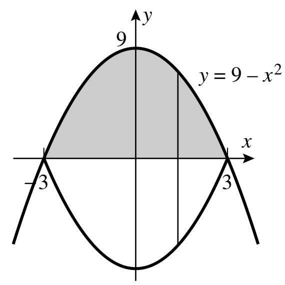
9. $V = \pi \int_{-4}^{4} [(25 - x^2) - 9] dx$

$= 2\pi \int_0^4 (16 - x^2) dx = 256\pi/3$



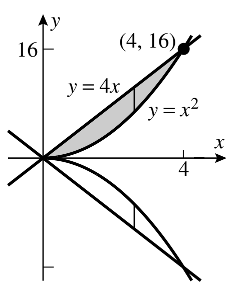
10. $V = \pi \int_{-3}^{3} (9 - x^2)^2 dx$

$= \pi \int_{-3}^{3} (81 - 18x^2 + x^4) dx = 1296\pi/5$



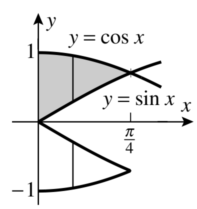
11. $V = \pi \int_0^4 [(4x)^2 - (x^2)^2] dx$

$= \pi \int_0^4 (16x^2 - x^4) dx = 2048\pi/15$

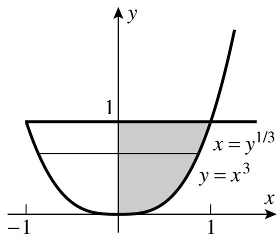


12. $V = \pi \int_0^{\pi/4} (\cos^2 x - \sin^2 x) dx$

$= \pi \int_0^{\pi/4} \cos 2x \, dx = \pi/2$

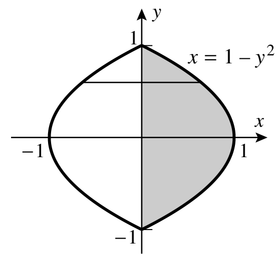


13. $V = \pi \int_0^1 y^{2/3} dy = 3\pi/5$

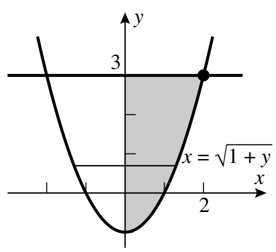


14. $V = \pi \int_{-1}^1 (1 - y^2)^2 dy$

$= \pi \int_{-1}^1 (1 - 2y^2 + y^4) dy = 16\pi/15$

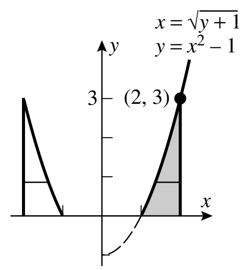


15. $V = \pi \int_{-1}^3 (1 + y) dy = 8\pi$

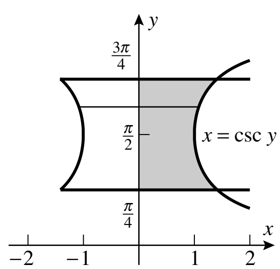


16. $V = \pi \int_0^3 [2^2 - (y + 1)] dy$

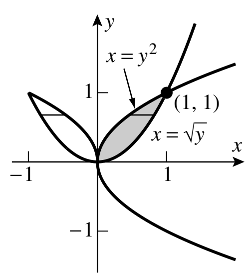
$= \pi \int_0^3 (3 - y) dy = 9\pi/2$



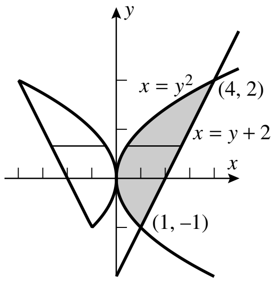
17. $V = \pi \int_{\pi/4}^{3\pi/4} \csc^2 y \, dy = 2\pi$



18. $V = \pi \int_0^1 (y - y^4) dy = 3\pi/10$

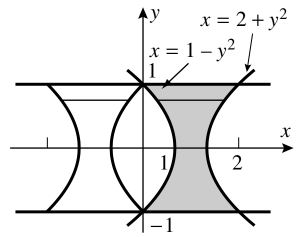


19. $V = \pi \int_{-1}^{2} [(y+2)^2 - y^4]dy = 72\pi/5$

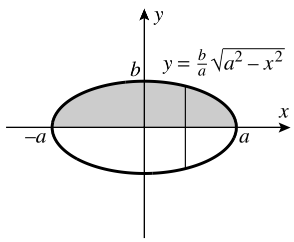


20. $V = \pi \int_{-1}^{1} \left[(2+y^2)^2 - (1-y^2)^2\right] dy$

$= \pi \int_{-1}^{1} (3 + 6y^2)dy = 10\pi$



21. $V = \pi \int_{-a}^{a} \frac{b^2}{a^2}(a^2 - x^2)dx = 4\pi ab^2/3$



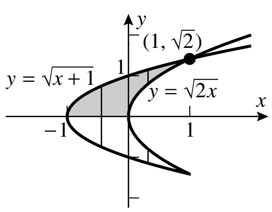
22. $V = \pi \int_{b}^{2} \frac{1}{x^2} dx = \pi(1/b - 1/2)$;

$\pi(1/b - 1/2) = 3, b = 2\pi/(\pi + 6)$

23. $V = \pi \int_{-1}^{0} (x+1)dx$

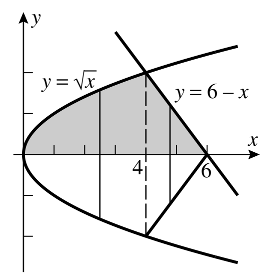
$+ \pi \int_{0}^{1} [(x+1) - 2x]dx$

$= \pi/2 + \pi/2 = \pi$



24. $V = \pi \int_{0}^{4} x\, dx + \pi \int_{4}^{6} (6-x)^2 dx$

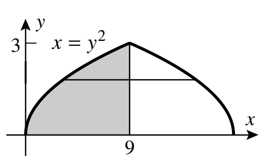
$= 8\pi + 8\pi/3 = 32\pi/3$



25. $V = \pi \int_{0}^{3} (9 - y^2)^2 dy$

$= \pi \int_{0}^{3} (81 - 18y^2 + y^4)dy$

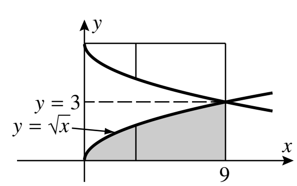
$= 648\pi/5$



26. $V = \pi \int_{0}^{9} [3^2 - (3 - \sqrt{x})^2]dx$

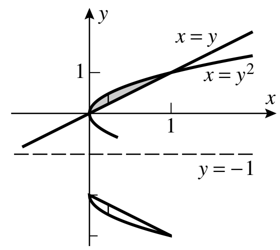
$= \pi \int_{0}^{9} (6\sqrt{x} - x)dx$

$= 135\pi/2$



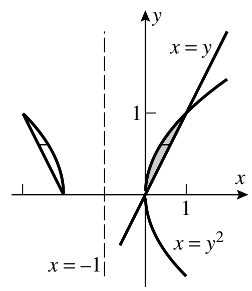
27. $V = \pi \int_0^1 [(\sqrt{x}+1)^2 - (x+1)^2]dx$

$= \pi \int_0^1 (2\sqrt{x} - x - x^2)dx = \pi/2$



28. $V = \pi \int_0^1 [(y+1)^2 - (y^2+1)^2]dy$

$= \pi \int_0^1 (2y - y^2 - y^4)dy = 7\pi/15$



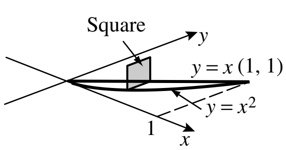
29. $A(x) = \pi(x^2/4)^2 = \pi x^4/16$,

$V = \int_0^{20} (\pi x^4/16)dx = 40,000\pi \text{ ft}^3$

30. $V = \pi \int_0^1 (x - x^4)dx = 3\pi/10$

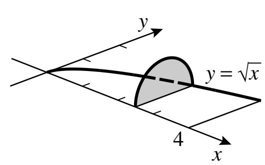
31. $V = \int_0^1 (x - x^2)^2 dx$

$= \int_0^1 (x^2 - 2x^3 + x^4)dx = 1/30$



32. $A(x) = \frac{1}{2}\pi\left(\frac{1}{2}\sqrt{x}\right)^2 = \frac{1}{8}\pi x$,

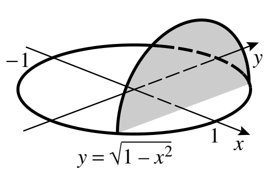
$V = \int_0^4 \frac{1}{8}\pi x \; dx = \pi$

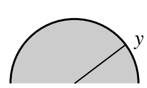


33. On the upper half of the circle, $y = \sqrt{1 - x^2}$, so:

(a) $A(x)$ is the area of a semicircle of radius y, so

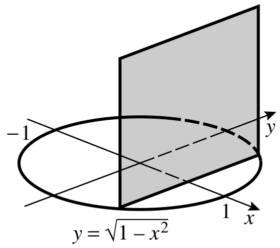
$A(x) = \pi y^2/2 = \pi(1 - x^2)/2$; $V = \frac{\pi}{2}\int_{-1}^1 (1 - x^2)\,dx = \pi \int_0^1 (1 - x^2)\,dx = 2\pi/3$

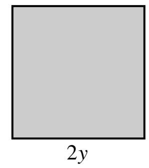




(b) $A(x)$ is the area of a square of side $2y$, so

$$A(x) = 4y^2 = 4(1 - x^2);\ V = 4\int_{-1}^{1}(1 - x^2)\,dx = 8\int_{0}^{1}(1 - x^2)\,dx = 16/3$$

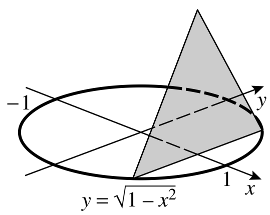


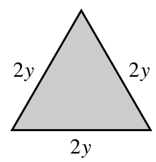


(c) $A(x)$ is the area of an equilateral triangle with sides $2y$, so

$$A(x) = \frac{\sqrt{3}}{4}(2y)^2 = \sqrt{3}y^2 = \sqrt{3}(1 - x^2);$$

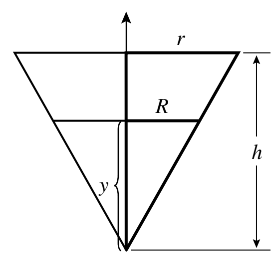
$$V = \int_{-1}^{1}\sqrt{3}(1 - x^2)\,dx = 2\sqrt{3}\int_{0}^{1}(1 - x^2)\,dx = 4\sqrt{3}/3$$





34. By similar triangles, $R/r = y/h$ so $R = ry/h$ and $A(y) = \pi r^2 y^2/h^2$.

$$V = (\pi r^2/h^2)\int_{0}^{h} y^2 dy = \pi r^2 h/3$$



35. The two curves cross at $x = b \approx 1.403288534$, so

$$V = \pi\int_{0}^{b}((2x/\pi)^2 - \sin^{16} x)\,dx + \pi\int_{b}^{\pi/2}(\sin^{16} x - (2x/\pi)^2)\,dx \approx 0.710172176.$$

36. Note that $\pi^2 \sin x \cos^3 x = 4x^2$ for $x = \pi/4$. From the graph it is apparent that this is the first positive solution, thus the curves don't cross on $(0, \pi/4)$ and

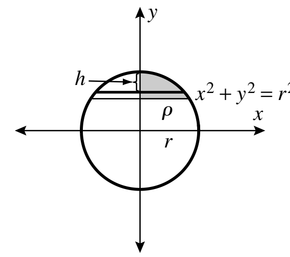
$$V = \pi\int_{0}^{\pi/4}[(\pi^2 \sin x \cos^3 x)^2 - (4x^2)^2]\,dx = \frac{1}{48}\pi^5 + \frac{17}{2560}\pi^6$$

37. **(a)** $V = \pi \int_{r-h}^{r} (r^2 - y^2)\,dy = \pi(rh^2 - h^3/3) = \frac{1}{3}\pi h^2(3r - h)$

(b) By the Pythagorean Theorem,

$r^2 = (r-h)^2 + \rho^2$, $2hr = h^2 + \rho^2$; from Part (a),

$$V = \frac{\pi h}{3}(3hr - h^2) = \frac{\pi h}{3}\left(\frac{3}{2}(h^2 + \rho^2) - h^2\right)$$
$$= \frac{1}{6}\pi h(h^2 + 3\rho^2).$$

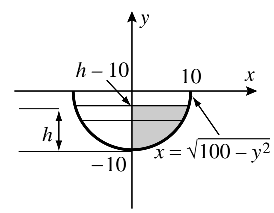


38. Find the volume generated by revolving the shaded region about the y-axis.

$$V = \pi \int_{-10}^{-10+h} (100 - y^2)dy = \frac{\pi}{3}h^2(30 - h)$$

Find dh/dt when $h = 5$ given that $dV/dt = 1/2$.

$$V = \frac{\pi}{3}(30h^2 - h^3),\ \frac{dV}{dt} = \frac{\pi}{3}(60h - 3h^2)\frac{dh}{dt},$$
$$\frac{1}{2} = \frac{\pi}{3}(300 - 75)\frac{dh}{dt},\ \frac{dh}{dt} = 1/(150\pi) \text{ ft/min}$$



39. **(b)** $\Delta x = \frac{5}{10} = 0.5$; $\{y_0, y_1, \cdots, y_{10}\} = \{0, 2.00, 2.45, 2.45, 2.00, 1.46, 1.26, 1.25, 1.25, 1.25, 1.25\}$;

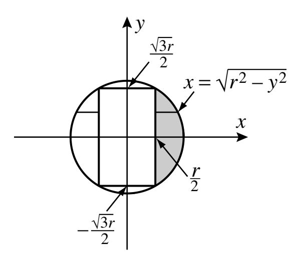
$$\text{left} = \pi \sum_{i=0}^{9} \left(\frac{y_i}{2}\right)^2 \Delta x \approx 11.157;$$

$$\text{right} = \pi \sum_{i=1}^{10} \left(\frac{y_i}{2}\right)^2 \Delta x \approx 11.771;\ V \approx \text{average} = 11.464 \text{ cm}^3$$

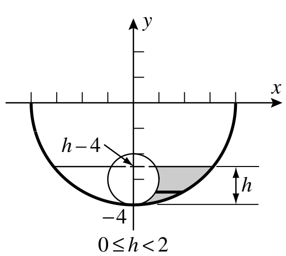
40. If $x = r/2$ then from $y^2 = r^2 - x^2$ we get $y = \pm\sqrt{3}r/2$ as limits of integration; for $-\sqrt{3} \le y \le \sqrt{3}$,

$A(y) = \pi[(r^2 - y^2) - r^2/4] = \pi(3r^2/4 - y^2)$, thus

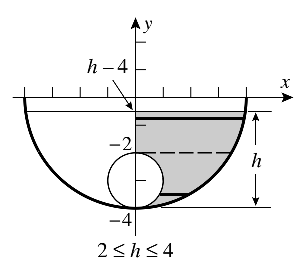
$$V = \pi \int_{-\sqrt{3}r/2}^{\sqrt{3}r/2} (3r^2/4 - y^2)dy$$
$$= 2\pi \int_{0}^{\sqrt{3}r/2} (3r^2/4 - y^2)dy = \sqrt{3}\pi r^3/2.$$



41. **(a)**



(b)



If the cherry is partially submerged then $0 \le h < 2$ as shown in Figure (a); if it is totally submerged then $2 \le h \le 4$ as shown in Figure (b). The radius of the glass is 4 cm and that of the cherry is 1 cm so points on the sections shown in the figures satisfy the equations $x^2 + y^2 = 16$ and $x^2 + (y+3)^2 = 1$. We will find the volumes of the solids that are generated when the shaded regions are revolved about the y-axis.

For $0 \le h < 2$,

$$V = \pi \int_{-4}^{h-4} [(16 - y^2) - (1 - (y+3)^2)]dy = 6\pi \int_{-4}^{h-4} (y+4)dy = 3\pi h^2;$$

for $2 \le h \le 4$,

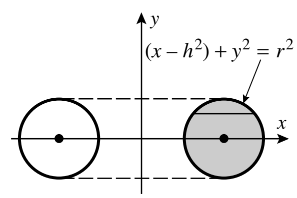
$$\begin{aligned} V &= \pi \int_{-4}^{-2} [(16 - y^2) - (1 - (y+3)^2)]dy + \pi \int_{-2}^{h-4} (16 - y^2)dy \\ &= 6\pi \int_{-4}^{-2} (y+4)dy + \pi \int_{-2}^{h-4} (16 - y^2)dy = 12\pi + \frac{1}{3}\pi(12h^2 - h^3 - 40) \\ &= \frac{1}{3}\pi(12h^2 - h^3 - 4) \end{aligned}$$

so

$$V = \begin{cases} 3\pi h^2 & \text{if } 0 \le h < 2 \\ \dfrac{1}{3}\pi(12h^2 - h^3 - 4) & \text{if } 2 \le h \le 4 \end{cases}$$

42. $x = h \pm \sqrt{r^2 - y^2}$,

$$\begin{aligned} V &= \pi \int_{-r}^{r} \left[(h + \sqrt{r^2 - y^2})^2 - (h - \sqrt{r^2 - y^2})^2\right] dy \\ &= 4\pi h \int_{-r}^{r} \sqrt{r^2 - y^2} dy \\ &= 4\pi h \left(\frac{1}{2}\pi r^2\right) = 2\pi^2 r^2 h \end{aligned}$$

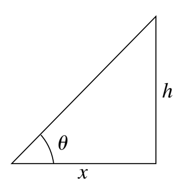


43. $\tan\theta = h/x$ so $h = x\tan\theta$,

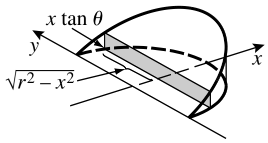
$$A(y) = \frac{1}{2}hx = \frac{1}{2}x^2\tan\theta = \frac{1}{2}(r^2 - y^2)\tan\theta$$

because $x^2 = r^2 - y^2$,

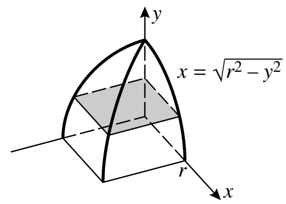
$$\begin{aligned} V &= \frac{1}{2}\tan\theta \int_{-r}^{r} (r^2 - y^2)dy \\ &= \tan\theta \int_{0}^{r} (r^2 - y^2)dy = \frac{2}{3}r^3\tan\theta \end{aligned}$$



44. $A(x) = (x\tan\theta)(2\sqrt{r^2 - x^2})$
$= 2(\tan\theta)x\sqrt{r^2 - x^2},$
$V = 2\tan\theta \int_0^r x\sqrt{r^2 - x^2}dx$
$= \frac{2}{3}r^3\tan\theta$



45. Each cross section perpendicular to the y-axis is a square so
$A(y) = x^2 = r^2 - y^2,$
$\frac{1}{8}V = \int_0^r (r^2 - y^2)dy$
$V = 8(2r^3/3) = 16r^3/3$



46. The regular cylinder of radius r and height h has the same circular cross sections as do those of the oblique clinder, so by Cavalieri's Principle, they have the same volume: $\pi r^2 h$.

EXERCISE SET 6.3

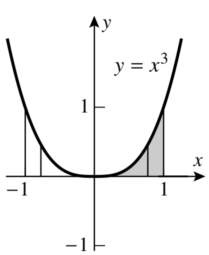
1. $V = \int_1^2 2\pi x(x^2)dx = 2\pi \int_1^2 x^3 dx = 15\pi/2$

2. $V = \int_0^{\sqrt{2}} 2\pi x(\sqrt{4 - x^2} - x)dx = 2\pi \int_0^{\sqrt{2}} (x\sqrt{4 - x^2} - x^2)dx = \frac{8\pi}{3}(2 - \sqrt{2})$

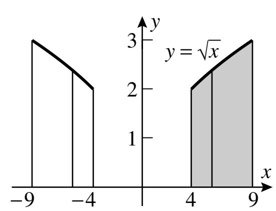
3. $V = \int_0^1 2\pi y(2y - 2y^2)dy = 4\pi \int_0^1 (y^2 - y^3)dy = \pi/3$

4. $V = \int_0^2 2\pi y[y - (y^2 - 2)]dy = 2\pi \int_0^2 (y^2 - y^3 + 2y)dy = 16\pi/3$

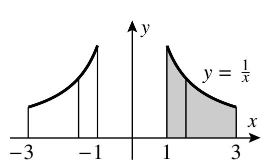
5. $V = \int_0^1 2\pi(x)(x^3)dx$
$= 2\pi \int_0^1 x^4 dx = 2\pi/5$



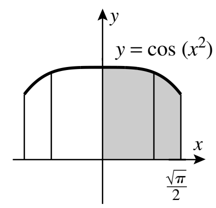
6. $V = \int_4^9 2\pi x(\sqrt{x})dx$
$= 2\pi \int_4^9 x^{3/2}dx = 844\pi/5$



7. $V = \int_1^3 2\pi x(1/x)dx = 2\pi \int_1^3 dx = 4\pi$

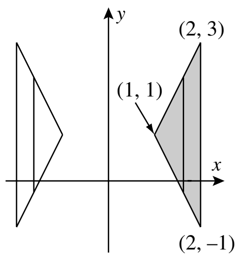


8. $V = \int_0^{\sqrt{\pi}/2} 2\pi x\cos(x^2)dx = \pi/\sqrt{2}$



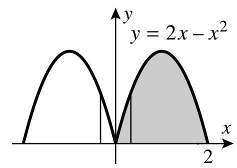
9. $V = \int_1^2 2\pi x[(2x-1)-(-2x+3)]dx$

$= 8\pi\int_1^2 (x^2-x)dx = 20\pi/3$

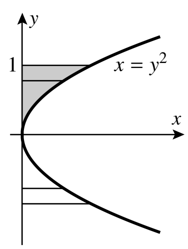


10. $V = \int_0^2 2\pi x(2x-x^2)dx$

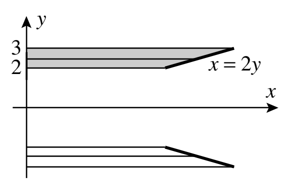
$= 2\pi\int_0^2 (2x^2-x^3)dx = \frac{8}{3}\pi$



11. $V = \int_0^1 2\pi y^3 dy = \pi/2$

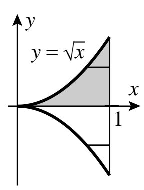


12. $V = \int_2^3 2\pi y(2y)dy = 4\pi\int_2^3 y^2 dy = 76\pi/3$



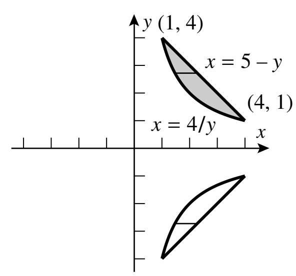
13. $V = \int_0^1 2\pi y(1-\sqrt{y})dy$

$= 2\pi\int_0^1 (y-y^{3/2})dy = \pi/5$



14. $V = \int_1^4 2\pi y(5-y-4/y)dy$

$= 2\pi\int_1^4 (5y-y^2-4)dy = 9\pi$

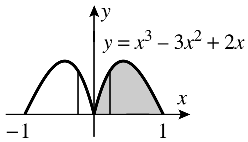


15. $V = 2\pi\int_0^\pi x\sin x dx = 2\pi^2$

16. $V = 2\pi\int_0^{\pi/2} x\cos x dx = \pi^2 - 2\pi$

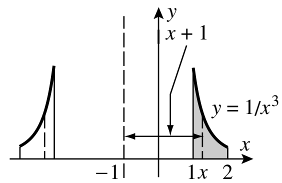
17. **(a)** $V = \int_0^1 2\pi x(x^3 - 3x^2 + 2x)dx = 7\pi/30$

(b) much easier; the method of slicing would require that x be expressed in terms of y.



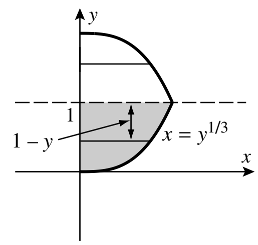
18. $V = \int_1^2 2\pi(x+1)(1/x^3)dx$

$= 2\pi \int_1^2 (x^{-2} + x^{-3})dx = 7\pi/4$



19. $V = \int_0^1 2\pi(1-y)y^{1/3}dy$

$= 2\pi \int_0^1 (y^{1/3} - y^{4/3})dy = 9\pi/14$

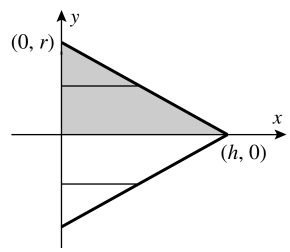


20. **(a)** $\int_a^b 2\pi x[f(x) - g(x)]dx$ **(b)** $\int_c^d 2\pi y[f(y) - g(y)]dy$

21. $x = \dfrac{h}{r}(r - y)$ is an equation of the line through $(0, r)$ and $(h, 0)$ so

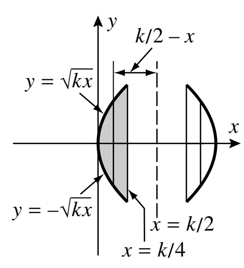
$V = \int_0^r 2\pi y \left[\dfrac{h}{r}(r-y)\right] dy$

$= \dfrac{2\pi h}{r} \int_0^r (ry - y^2)dy = \pi r^2 h/3$



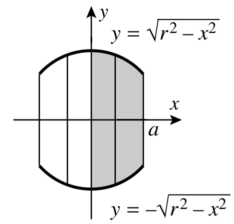
22. $V = \int_0^{k/4} 2\pi(k/2 - x)2\sqrt{kx}dx$

$= 2\pi\sqrt{k} \int_0^{k/4} (kx^{1/2} - 2x^{3/2})dx = 7\pi k^3/60$



23. $V = \int_0^a 2\pi x(2\sqrt{r^2 - x^2})dx = 4\pi \int_0^a x(r^2 - x^2)^{1/2}dx$

$= -\frac{4\pi}{3}(r^2 - x^2)^{3/2}\Big]_0^a = \frac{4\pi}{3}\left[r^3 - (r^2 - a^2)^{3/2}\right]$

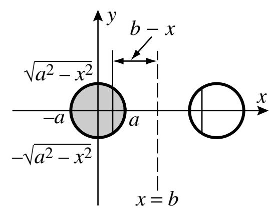


24. $V = \int_{-a}^{a} 2\pi(b - x)(2\sqrt{a^2 - x^2})dx$

$= 4\pi b \int_{-a}^{a} \sqrt{a^2 - x^2}dx - 4\pi \int_{-a}^{a} x\sqrt{a^2 - x^2}dx$

$= 4\pi b \cdot$ (area of a semicircle of radius a) $- 4\pi(0)$

$= 2\pi^2 a^2 b$



25. $V_x = \pi \int_{1/2}^{b} \frac{1}{x^2}dx = \pi(2 - 1/b)$, $V_y = 2\pi \int_{1/2}^{b} dx = \pi(2b - 1)$;

$V_x = V_y$ if $2 - 1/b = 2b - 1$, $2b^2 - 3b + 1 = 0$, solve to get $b = 1/2$ (reject) or $b = 1$.

EXERCISE SET 6.4

1. **(a)** $\frac{dy}{dx} = 2$, $L = \int_1^2 \sqrt{1 + 4}dx = \sqrt{5}$

(b) $\frac{dx}{dy} = \frac{1}{2}$, $L = \int_2^4 \sqrt{1 + 1/4}\,dy = 2\sqrt{5}/2 = \sqrt{5}$

2. $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = 5$, $L = \int_0^1 \sqrt{1^2 + 5^2}\,dt = \sqrt{26}$

3. $f'(x) = \frac{9}{2}x^{1/2}$, $1 + [f'(x)]^2 = 1 + \frac{81}{4}x$,

$L = \int_0^1 \sqrt{1 + 81x/4}\,dx = \frac{8}{243}\left(1 + \frac{81}{4}x\right)^{3/2}\Big]_0^1 = (85\sqrt{85} - 8)/243$

4. $g'(y) = y(y^2 + 2)^{1/2}$, $1 + [g'(y)]^2 = 1 + y^2(y^2 + 2) = y^4 + 2y^2 + 1 = (y^2 + 1)^2$,

$L = \int_0^1 \sqrt{(y^2 + 1)^2}dy = \int_0^1 (y^2 + 1)dy = 4/3$

5. $\dfrac{dy}{dx} = \dfrac{2}{3}x^{-1/3}$, $1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \dfrac{4}{9}x^{-2/3} = \dfrac{9x^{2/3}+4}{9x^{2/3}}$,

$$L = \int_1^8 \frac{\sqrt{9x^{2/3}+4}}{3x^{1/3}}dx = \frac{1}{18}\int_{13}^{40} u^{1/2}du,\ u = 9x^{2/3}+4$$

$$= \frac{1}{27}u^{3/2}\bigg]_{13}^{40} = \frac{1}{27}(40\sqrt{40} - 13\sqrt{13}) = \frac{1}{27}(80\sqrt{10} - 13\sqrt{13})$$

or (alternate solution)

$$x = y^{3/2},\ \frac{dx}{dy} = \frac{3}{2}y^{1/2},\ 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{9}{4}y = \frac{4+9y}{4},$$

$$L = \frac{1}{2}\int_1^4 \sqrt{4+9y}\,dy = \frac{1}{18}\int_{13}^{40} u^{1/2}du = \frac{1}{27}(80\sqrt{10} - 13\sqrt{13})$$

6. $f'(x) = \dfrac{1}{4}x^3 - x^{-3}$, $1 + [f'(x)]^2 = 1 + \left(\dfrac{1}{16}x^6 - \dfrac{1}{2} + x^{-6}\right) = \dfrac{1}{16}x^6 + \dfrac{1}{2} + x^{-6} = \left(\dfrac{1}{4}x^3 + x^{-3}\right)^2$,

$$L = \int_2^3 \sqrt{\left(\frac{1}{4}x^3 + x^{-3}\right)^2}dx = \int_2^3 \left(\frac{1}{4}x^3 + x^{-3}\right)dx = 595/144$$

7. $x = g(y) = \dfrac{1}{24}y^3 + 2y^{-1}$, $g'(y) = \dfrac{1}{8}y^2 - 2y^{-2}$,

$$1 + [g'(y)]^2 = 1 + \left(\frac{1}{64}y^4 - \frac{1}{2} + 4y^{-4}\right) = \frac{1}{64}y^4 + \frac{1}{2} + 4y^{-4} = \left(\frac{1}{8}y^2 + 2y^{-2}\right)^2,$$

$$L = \int_2^4 \left(\frac{1}{8}y^2 + 2y^{-2}\right)dy = 17/6$$

8. $g'(y) = \dfrac{1}{2}y^3 - \dfrac{1}{2}y^{-3}$, $1 + [g'(y)]^2 = 1 + \left(\dfrac{1}{4}y^6 - \dfrac{1}{2} + \dfrac{1}{4}y^{-6}\right) = \left(\dfrac{1}{2}y^3 + \dfrac{1}{2}y^{-3}\right)^2$,

$$L = \int_1^4 \left(\frac{1}{2}y^3 + \frac{1}{2}y^{-3}\right)dy = 2055/64$$

9. $(dx/dt)^2 + (dy/dt)^2 = (t^2)^2 + (t)^2 = t^2(t^2+1)$, $L = \displaystyle\int_0^1 t(t^2+1)^{1/2}dt = (2\sqrt{2} - 1)/3$

10. $(dx/dt)^2 + (dy/dt)^2 = [2(1+t)]^2 + [3(1+t)^2]^2 = (1+t)^2[4 + 9(1+t)^2]$,

$$L = \int_0^1 (1+t)[4 + 9(1+t)^2]^{1/2}dt = (80\sqrt{10} - 13\sqrt{13})/27$$

11. $(dx/dt)^2 + (dy/dt)^2 = (-2\sin 2t)^2 + (2\cos 2t)^2 = 4$, $L = \displaystyle\int_0^{\pi/2} 2\,dt = \pi$

12. $(dx/dt)^2 + (dy/dt)^2 = (-\sin t + \sin t + t\cos t)^2 + (\cos t - \cos t + t\sin t)^2 = t^2$,

$$L = \int_0^{\pi} t\,dt = \pi^2/2$$

13. **(a)** $(dx/d\theta)^2 + (dy/d\theta)^2 = (a(1-\cos\theta))^2 + (a\sin\theta)^2 = a^2(2-2\cos\theta)$, so

$$L = \int_0^{2\pi} \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\, d\theta = a\int_0^{2\pi} \sqrt{2(1-\cos\theta)}\, d\theta$$

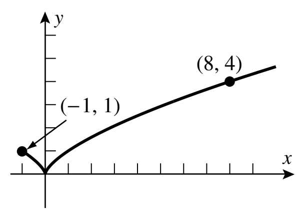
14. **(a)** Use the interval $0 \le \phi < 2\pi$.

(b) $(dx/d\phi)^2 + (dy/d\phi)^2 = (-3a\cos^2\phi\sin\phi)^2 + (3a\sin^2\phi\cos\phi)^2$

$$= 9a^2\cos^2\phi\sin^2\phi(\cos^2\phi + \sin^2\phi) = (9a^2/4)\sin^2 2\phi, \text{ so}$$

$$L = (3a/2)\int_0^{2\pi} |\sin 2\phi|\, d\phi = 6a\int_0^{\pi/2} \sin 2\phi\, d\phi = -3a\cos 2\phi\Big]_0^{\pi/2} = 6a$$

15. **(a)**



(b) dy/dx does not exist at $x = 0$.

(c) $x = g(y) = y^{3/2},\ g'(y) = \dfrac{3}{2}\, y^{1/2}$,

$$L = \int_0^1 \sqrt{1 + 9y/4}\, dy \quad (\text{portion for } -1 \le x \le 0)$$

$$+ \int_0^4 \sqrt{1 + 9y/4}\, dy \quad (\text{portion for } 0 \le x \le 8)$$

$$= \frac{8}{27}\left(\frac{13}{8}\sqrt{13} - 1\right) + \frac{8}{27}(10\sqrt{10} - 1) = (13\sqrt{13} + 80\sqrt{10} - 16)/27$$

16. For (4), express the curve $y = f(x)$ in the parametric form $x = t, y = f(t)$ so $dx/dt = 1$ and $dy/dt = f'(t) = f'(x) = dy/dx$. For (5), express $x = g(y)$ as $x = g(t), y = t$ so $dx/dt = g'(t) = g'(y) = dx/dy$ and $dy/dt = 1$.

17. $L = \displaystyle\int_0^2 \sqrt{1 + 4x^2}\, dx \approx 4.645975301$

18. $L = \displaystyle\int_0^{\pi} \sqrt{1 + \cos^2 y}\, dy \approx 3.820197789$

19. Numerical integration yields: in Exercise 17, $L \approx 4.646783762$; in Exercise 18, $L \approx 3.820197788$.

20. $0 \le m \le f'(x) \le M$, so $m^2 \le [f'(x)]^2 \le M^2$, and $1 + m^2 \le 1 + [f'(x)]^2 \le 1 + M^2$; thus

$\sqrt{1+m^2} \le \sqrt{1 + [f'(x)]^2} \le \sqrt{1 + M^2}$,

$\displaystyle\int_a^b \sqrt{1+m^2}dx \le \int_a^b \sqrt{1+[f'(x)]^2}dx \le \int_a^b \sqrt{1+M^2}\, dx$, and

$(b-a)\sqrt{1+m^2} \le L \le (b-a)\sqrt{1+M^2}$

21. $f'(x) = \cos x$, $\sqrt{2}/2 \le \cos x \le 1$ for $0 \le x \le \pi/4$ so

$(\pi/4)\sqrt{1 + 1/2} \le L \le (\pi/4)\sqrt{1+1}$, $\dfrac{\pi}{4}\sqrt{3/2} \le L \le \dfrac{\pi}{4}\sqrt{2}$.

22. $(dx/dt)^2 + (dy/dt)^2 = (-a\sin t)^2 + (b\cos t)^2 = a^2\sin^2 t + b^2\cos^2 t$

$$= a^2(1-\cos^2 t) + b^2\cos^2 t = a^2 - (a^2-b^2)\cos^2 t$$

$$= a^2\left[1 - \frac{a^2-b^2}{a^2}\cos^2 t\right] = a^2[1-k^2\cos^2 t],$$

$$L = \int_0^{2\pi} a\sqrt{1-k^2\cos^2 t}\,dt = 4a\int_0^{\pi/2}\sqrt{1-k^2\cos^2 t}\,dt$$

23. **(a)** $(dx/dt)^2 + (dy/dt)^2 = 4\sin^2 t + \cos^2 t = 4\sin^2 t + (1-\sin^2 t) = 1 + 3\sin^2 t$,

$$L = \int_0^{2\pi}\sqrt{1+3\sin^2 t}\,dt = 4\int_0^{\pi/2}\sqrt{1+3\sin^2 t}\,dt$$

(b) 9.69

(c) distance traveled $= \displaystyle\int_{1.5}^{4.8}\sqrt{1+3\sin^2 t}\,dt \approx 5.16$ cm

24. The distance is $\displaystyle\int_0^{4.6}\sqrt{1+(2.09-0.82x)^2}\,dx \approx 6.65$ m

25. $L = \displaystyle\int_0^{\pi}\sqrt{1+(k\cos x)^2}\,dx$

k	1	2	1.84	1.83	1.832
L	3.8202	5.2704	5.0135	4.9977	5.0008

Experimentation yields the values in the table, which by the Intermediate-Value Theorem show that the true solution k to $L = 5$ lies between $k = 1.83$ and $k = 1.832$, so $k = 1.83$ to two decimal places.

EXERCISE SET 6.5

1. $S = \displaystyle\int_0^1 2\pi(7x)\sqrt{1+49}dx = 70\pi\sqrt{2}\int_0^1 x\ dx = 35\pi\sqrt{2}$

2. $f'(x) = \dfrac{1}{2\sqrt{x}},\ 1 + [f'(x)]^2 = 1 + \dfrac{1}{4x}$

$$S = \int_1^4 2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}dx = 2\pi\int_1^4\sqrt{x+1/4}dx = \pi(17\sqrt{17}-5\sqrt{5})/6$$

3. $f'(x) = -x/\sqrt{4-x^2},\ 1 + [f'(x)]^2 = 1 + \dfrac{x^2}{4-x^2} = \dfrac{4}{4-x^2},$

$$S = \int_{-1}^1 2\pi\sqrt{4-x^2}(2/\sqrt{4-x^2})dx = 4\pi\int_{-1}^1 dx = 8\pi$$

4. $y = f(x) = x^3$ for $1 \le x \le 2$, $f'(x) = 3x^2$,

$$S = \int_1^2 2\pi x^3\sqrt{1+9x^4}dx = \frac{\pi}{27}(1+9x^4)^{3/2}\Big]_1^2 = 5\pi(29\sqrt{145}-2\sqrt{10})/27$$

5. $S = \int_0^2 2\pi(9y+1)\sqrt{82}dy = 2\pi\sqrt{82}\int_0^2 (9y+1)dy = 40\pi\sqrt{82}$

6. $g'(y) = 3y^2,\ S = \int_0^1 2\pi y^3\sqrt{1+9y^4}dy = \pi(10\sqrt{10}-1)/27$

7. $g'(y) = -y/\sqrt{9-y^2},\ 1+[g'(y)]^2 = \dfrac{9}{9-y^2},\ S = \int_{-2}^{2} 2\pi\sqrt{9-y^2}\cdot\dfrac{3}{\sqrt{9-y^2}}dy = 6\pi\int_{-2}^{2} dy = 24\pi$

8. $g'(y) = -(1-y)^{-1/2},\ 1+[g'(y)]^2 = \dfrac{2-y}{1-y},$

$$S = \int_{-1}^{0} 2\pi(2\sqrt{1-y})\frac{\sqrt{2-y}}{\sqrt{1-y}}dy = 4\pi\int_{-1}^{0}\sqrt{2-y}\,dy = 8\pi(3\sqrt{3}-2\sqrt{2})/3$$

9. $f'(x) = \dfrac{1}{2}x^{-1/2} - \dfrac{1}{2}x^{1/2},\ 1+[f'(x)]^2 = 1+\dfrac{1}{4}x^{-1}-\dfrac{1}{2}+\dfrac{1}{4}x = \left(\dfrac{1}{2}x^{-1/2}+\dfrac{1}{2}x^{1/2}\right)^2,$

$$S = \int_1^3 2\pi\left(x^{1/2}-\frac{1}{3}x^{3/2}\right)\left(\frac{1}{2}x^{-1/2}+\frac{1}{2}x^{1/2}\right)dx = \frac{\pi}{3}\int_1^3 (3+2x-x^2)dx = 16\pi/9$$

10. $f'(x) = x^2 - \dfrac{1}{4}x^{-2},\ 1+[f'(x)]^2 = 1+\left(x^4-\dfrac{1}{2}+\dfrac{1}{16}x^{-4}\right) = \left(x^2+\dfrac{1}{4}x^{-2}\right)^2,$

$$S = \int_1^2 2\pi\left(\frac{1}{3}x^3+\frac{1}{4}x^{-1}\right)\left(x^2+\frac{1}{4}x^{-2}\right)dx = 2\pi\int_1^2\left(\frac{1}{3}x^5+\frac{1}{3}x+\frac{1}{16}x^{-3}\right)dx = 515\pi/64$$

11. $x = g(y) = \dfrac{1}{4}y^4+\dfrac{1}{8}y^{-2},\ g'(y) = y^3-\dfrac{1}{4}y^{-3},$

$$1+[g'(y)]^2 = 1+\left(y^6-\frac{1}{2}+\frac{1}{16}y^{-6}\right) = \left(y^3+\frac{1}{4}y^{-3}\right)^2,$$

$$S = \int_1^2 2\pi\left(\frac{1}{4}y^4+\frac{1}{8}y^{-2}\right)\left(y^3+\frac{1}{4}y^{-3}\right)dy = \frac{\pi}{16}\int_1^2 (8y^7+6y+y^{-5})dy = 16{,}911\pi/1024$$

12. $x = g(y) = \sqrt{16-y};\ g'(y) = -\dfrac{1}{2\sqrt{16-y}},\ 1+[g'(y)]^2 = \dfrac{65-4y}{4(16-y)},$

$$S = \int_0^{15} 2\pi\sqrt{16-y}\sqrt{\frac{65-4y}{4(16-y)}}\,dy = \pi\int_0^{15}\sqrt{65-4y}\,dy = (65\sqrt{65}-5\sqrt{5})\frac{\pi}{6}$$

13. $f'(x) = \cos x,\ 1+[f'(x)]^2 = 1+\cos^2 x,\ S = \int_0^{\pi} 2\pi\sin x\sqrt{1+\cos^2 x}\,dx = 2\pi(\sqrt{2}+\ln(\sqrt{2}+1))$

14. $x = g(y) = \tan y,\ g'(y) = \sec^2 y,\ 1+[g'(y)]^2 = 1+\sec^4 y;$

$$S = \int_0^{\pi/4} 2\pi\tan y\sqrt{1+\sec^4 y}\,dy \approx 3.84$$

15. Revolve the line segment joining the points $(0,0)$ and (h,r) about the x-axis. An equation of the line segment is $y = (r/h)x$ for $0 \le x \le h$ so

$$S = \int_0^h 2\pi(r/h)x\sqrt{1+r^2/h^2}dx = \frac{2\pi r}{h^2}\sqrt{r^2+h^2}\int_0^h x\,dx = \pi r\sqrt{r^2+h^2}$$

16. $f(x)=\sqrt{r^2-x^2}$, $f'(x)=-x/\sqrt{r^2-x^2}$, $1+[f'(x)]^2=r^2/(r^2-x^2)$,

$$S=\int_{-r}^{r}2\pi\sqrt{r^2-x^2}(r/\sqrt{r^2-x^2})dx=2\pi r\int_{-r}^{r}dx=4\pi r^2$$

17. $g(y)=\sqrt{r^2-y^2}$, $g'(y)=-y/\sqrt{r^2-y^2}$, $1+[g'(y)]^2=r^2/(r^2-y^2)$,

(a) $S=\displaystyle\int_{r-h}^{r}2\pi\sqrt{r^2-y^2}\sqrt{r^2/(r^2-y^2)}\,dy=2\pi r\int_{r-h}^{r}dy=2\pi rh$

(b) From Part (a), the surface area common to two polar caps of height $h_1>h_2$ is $2\pi rh_1-2\pi rh_2=2\pi r(h_1-h_2)$.

18. For (4), express the curve $y=f(x)$ in the parametric form $x=t,y=f(t)$ so $dx/dt=1$ and $dy/dt=f'(t)=f'(x)=dy/dx$. For (5), express $x=g(y)$ as $x=g(t),y=t$ so $dx/dt=g'(t)=g'(y)=dx/dy$ and $dy/dt=1$.

19. $x'=2t,y'=2,(x')^2+(y')^2=4t^2+4$

$$S=2\pi\int_0^4(2t)\sqrt{4t^2+4}dt=8\pi\int_0^4 t\sqrt{t^2+1}dt=\frac{8\pi}{3}(17\sqrt{17}-1)$$

20. $x'=-2\cos t\sin t,y'=5\cos t,(x')^2+(y')^2=4\cos^2 t\sin^2 t+25\cos^2 t$,

$$S=2\pi\int_0^{\pi/2}5\sin t\sqrt{4\cos^2 t\sin^2 t+25\cos^2 t}\,dt=\frac{\pi}{6}(145\sqrt{29}-625)$$

21. $x'=1$, $y'=4t$, $(x')^2+(y')^2=1+16t^2$, $S=2\pi\displaystyle\int_0^1 t\sqrt{1+16t^2}\,dt=\frac{\pi}{24}(17\sqrt{17}-1)$

22. $x'=-2\sin t\cos t,y'=2\sin t\cos t,(x')^2+(y')^2=8\sin^2 t\cos^2 t$

$$S=2\pi\int_0^{\pi/2}\cos^2 t\sqrt{8\sin^2 t\cos^2 t}\,dt=4\sqrt{2}\pi\int_0^{\pi/2}\cos^3 t\sin t\,dt=\sqrt{2}\pi$$

23. $x'=-r\sin t$, $y'=r\cos t$, $(x')^2+(y')^2=r^2$,

$$S=2\pi\int_0^{\pi}r\sin t\sqrt{r^2}\,dt=2\pi r^2\int_0^{\pi}\sin t\,dt=4\pi r^2$$

24. $\dfrac{dx}{d\phi}=a(1-\cos\phi)$, $\dfrac{dy}{d\phi}=a\sin\phi$, $\left(\dfrac{dx}{d\phi}\right)^2+\left(\dfrac{dy}{d\phi}\right)^2=2a^2(1-\cos\phi)$

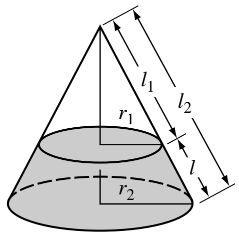
$$S=2\pi\int_0^{2\pi}a(1-\cos\phi)\sqrt{2a^2(1-\cos\phi)}\,d\phi=2\sqrt{2}\pi a^2\int_0^{2\pi}(1-\cos\phi)^{3/2}d\phi,$$

but $1-\cos\phi=2\sin^2\dfrac{\phi}{2}$ so $(1-\cos\phi)^{3/2}=2\sqrt{2}\sin^3\dfrac{\phi}{2}$ for $0\le\phi\le\pi$ and, taking advantage of the symmetry of the cycloid, $S=16\pi a^2\displaystyle\int_0^{\pi}\sin^3\frac{\phi}{2}d\phi=64\pi a^2/3$.

25. **(a)** length of arc of sector = circumference of base of cone,

$\ell\theta=2\pi r,\theta=2\pi r/\ell$; $S=$ area of sector $=\dfrac{1}{2}\ell^2(2\pi r/\ell)=\pi r\ell$

(b) $S = \pi r_2\ell_2 - \pi r_1\ell_1 = \pi r_2(\ell_1 + \ell) - \pi r_1\ell_1 = \pi[(r_2 - r_1)\ell_1 + r_2\ell]$;
Using similar triangles $\ell_2/r_2 = \ell_1/r_1, r_1\ell_2 = r_2\ell_1, r_1(\ell_1 + \ell) = r_2\ell_1, (r_2 - r_1)\ell_1 = r_1\ell$
so $S = \pi\,(r_1\ell + r_2\ell) = \pi\,(r_1 + r_2)\,\ell$.



26. $S = \int_a^b 2\pi[f(x) + k]\sqrt{1 + [f'(x)]^2}dx$

27. $2\pi k\sqrt{1 + [f'(x)]^2} \le 2\pi f(x)\sqrt{1 + [f'(x)]^2} \le 2\pi K\sqrt{1 + [f'(x)]^2}$, so

$$\int_a^b 2\pi k\sqrt{1 + [f'(x)]^2}dx \le \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}dx \le \int_a^b 2\pi K\sqrt{1 + [f'(x)]^2}dx,$$

$$2\pi k\int_a^b \sqrt{1 + [f'(x)]^2}dx \le S \le 2\pi K\int_a^b \sqrt{1 + [f'(x)]^2}dx,\ 2\pi kL \le S \le 2\pi KL$$

28. **(a)** $1 \le \sqrt{1 + [f'(x)]^2}$ so $2\pi f(x) \le 2\pi f(x)\sqrt{1 + [f'(x)]^2}$,

$$\int_a^b 2\pi f(x)dx \le \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}dx,\ 2\pi\int_a^b f(x)dx \le S, 2\pi A \le S$$

(b) $2\pi A = S$ if $f'(x) = 0$ for all x in $[a, b]$ so $f(x)$ is constant on $[a, b]$.

EXERCISE SET 6.6

1. **(a)** $W = F \cdot d = 30(7) = 210$ ft·lb

(b) $W = \int_1^6 F(x)\,dx = \int_1^6 x^{-2}\,dx = -\frac{1}{x}\Big]_1^6 = 5/6$ ft·lb

2. $W = \int_0^5 F(x)\,dx = \int_0^2 40\,dx - \int_2^5 \frac{40}{3}(x - 5)\,dx = 80 + 60 = 140$ J

3. distance traveled $= \int_0^5 v(t)\,dt = \int_0^5 \frac{4t}{5}\,dt = \frac{2}{5}t^2\Big]_0^5 = 10$ ft. The force is a constant 10 lb, so the work done is $10 \cdot 10 = 100$ ft·lb.

4. **(a)** $F(x) = kx, F(0.05) = 0.05k = 45, k = 900$ N/m

(b) $W = \int_0^{0.03} 900x\ dx = 0.405$ J **(c)** $W = \int_{0.05}^{0.10} 900x\,dx = 3.375$ J

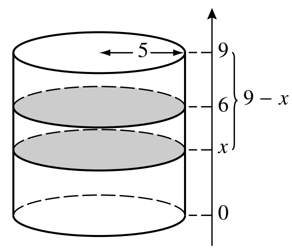
5. $F(x) = kx,\ F(0.2) = 0.2k = 100,\ k = 500\ \text{N/m},\ W = \int_0^{0.8} 500x\,dx = 160\,\text{J}$

6. $F(x) = kx,\ F(1/2) = k/2 = 6,\ k = 12\,\text{N/m},\ W = \int_0^2 12x\,dx = 24\,\text{J}$

7. $W = \int_0^1 kx\,dx = k/2 = 10,\ k = 20\,\text{lb/ft}$

8. $W = \int_0^6 (9-x)62.4(25\pi)dx$

$= 1560\pi \int_0^6 (9-x)dx = 56,160\pi\ \text{ft·lb}$



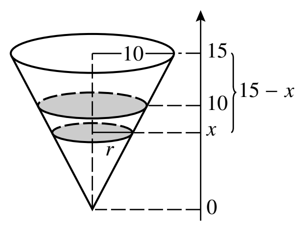
9. $W = \int_0^6 (9-x)\rho(25\pi)dx = 900\pi\rho\ \text{ft·lb}$

10. $r/10 = x/15,\ r = 2x/3,$

$W = \int_0^{10} (15-x)62.4(4\pi x^2/9)dx$

$= \frac{83.2}{3}\pi \int_0^{10} (15x^2 - x^3)dx$

$= 208,000\pi/3\ \text{ft·lb}$

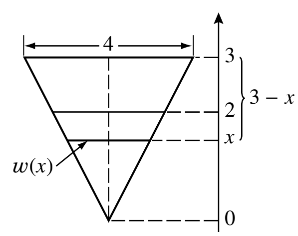


11. $w/4 = x/3, w = 4x/3,$

$W = \int_0^2 (3-x)(9810)(4x/3)(6)dx$

$= 78480 \int_0^2 (3x - x^2)dx$

$= 261,600\,\text{J}$

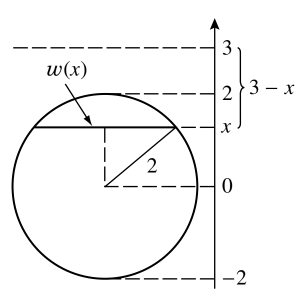


12. $w = 2\sqrt{4-x^2}$

$W = \int_{-2}^2 (3-x)(50)(2\sqrt{4-x^2})(10)dx$

$= 3000\int_{-2}^2 \sqrt{4-x^2}dx - 1000\int_{-2}^2 x\sqrt{4-x^2}dx$

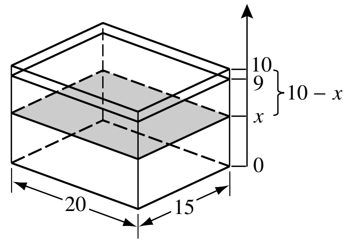
$= 3000[\pi(2)^2/2] - 0 = 6000\pi\ \text{ft·lb}$



13. **(a)** $W = \int_0^9 (10 - x)62.4(300)dx$

$= 18{,}720 \int_0^9 (10 - x)dx$

$= 926{,}640$ ft·lb

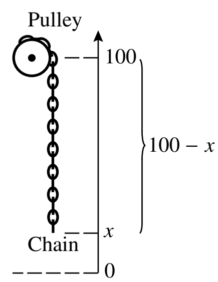


(b) to empty the pool in one hour would require
926,640/3600 = 257.4 ft·lb of work per second
so hp of motor = 257.4/550 = 0.468

14. $W = \int_0^9 x(62.4)(300)\,dx = 18{,}720\int_0^9 x\,dx = (81/2)18{,}720 = 758{,}160$ ft·lb

15. $W = \int_0^{100} 15(100 - x)dx$

$= 75{,}000$ ft·lb



16. The total time of winding the rope is (20 ft)/(2 ft/s) = 10 s. During the time interval from time t to time $t + \Delta t$ the work done is $\Delta W = F(t) \cdot \Delta x$.
The distance $\Delta x = 2\Delta t$, and the force $F(t)$ is given by the weight $w(t)$ of the bucket, rope and water at time t. The bucket and its remaining water together weigh $(3+20) - t/2$ lb, and the rope is $20 - 2t$ ft long and weighs $4(20 - 2t)$ oz or $5 - t/2$ lb. Thus at time t the bucket, water and rope together weigh $w(t) = 23 - t/2 + 5 - t/2 = 28 - t$ lb.
The amount of work done in the time interval from time t to time $t + \Delta t$ is thus
$\Delta W = (28 - t)2\Delta t$, and the total work done is
$W = \lim_{n\to+\infty} \sum (28 - t)2\Delta t = \int_0^{10} (28 - t)2\,dt = 2(28t - t^2/2)\Big|_0^{10} = 460$ ft·lb.

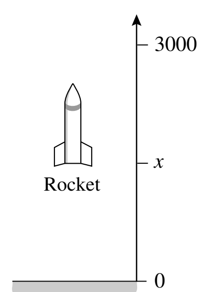
17. When the rocket is x ft above the ground

total weight = weight of rocket + weight of fuel

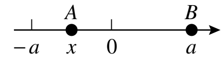
$= 3 + [40 - 2(x/1000)]$

$= 43 - x/500$ tons,

$W = \int_0^{3000} (43 - x/500)dx = 120{,}000$ ft·tons



18. Let $F(x)$ be the force needed to hold charge A at position x, then



$F(x) = \dfrac{c}{(a-x)^2}$, $F(-a) = \dfrac{c}{4a^2} = k$,

so $c = 4a^2k$.

$W = \displaystyle\int_{-a}^{0} 4a^2k(a-x)^{-2}dx = 2ak\,\text{J}$

19. **(a)** $150 = k/(4000)^2$, $k = 2.4 \times 10^9$, $w(x) = k/x^2 = 2{,}400{,}000{,}000/x^2$ lb

(b) $6000 = k/(4000)^2$, $k = 9.6 \times 10^{10}$, $w(x) = \left(9.6 \times 10^{10}\right)/(x+4000)^2$ lb

(c) $W = \displaystyle\int_{4000}^{5000} 9.6(10^{10})x^{-2}dx = 4{,}800{,}000$ mi·lb $= 2.5344 \times 10^{10}$ ft·lb

20. **(a)** $20 = k/(1080)^2$, $k = 2.3328 \times 10^7$, weight $= w(x+1080) = 2.3328 \cdot 10^7/(x+1080)^2$ lb

(b) $W = \displaystyle\int_{0}^{10.8} [2.3328 \cdot 10^7/(x+1080)^2]\,dx = 213.86$ mi·lb $= 1{,}129{,}188$ ft·lb

21. $W = F \cdot d = (6.40 \times 10^5)(3.00 \times 10^3) = 1.92 \times 10^9$ J; from the Work-Energy Relationship (5), $v_f^2 = 2W/m + v_i^2 = 2(1.92 \cdot 10^9)/(4 \cdot 10^5) + 20^2 = 10{,}000$, $v_f = 100$ m/s

22. $W = F \cdot d = (2.00 \times 10^5)(2.00 \times 10^5) = 4 \times 10^{10}$ J; from the Work-Energy Relationship (5), $v_f^2 = 2W/m + v_i^2 = 8 \cdot 10^{10}/(2 \cdot 10^3) + 10^8 \approx 11.832$ m/s.

23. **(a)** The kinetic energy would have decreased by $\dfrac{1}{2}mv^2 = \dfrac{1}{2}4 \cdot 10^6(15000)^2 = 4.5 \times 10^{14}$ J

(b) $(4.5 \times 10^{14})/(4.2 \times 10^{15}) \approx 0.107$

(c) $\dfrac{1000}{13}(0.107) \approx 8.24$ bombs

EXERCISE SET 6.7

1. **(a)** $F = \rho h A = 62.4(5)(100) = 31{,}200$ lb
$P = \rho h = 62.4(5) = 312$ lb/ft^2

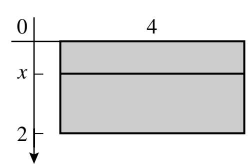
(b) $F = \rho h A = 9810(10)(25) = 2{,}452{,}500$ N
$P = \rho h = 9810(10) = 98.1$ kPa

2. **(a)** $F = PA = 6 \cdot 10^5(160) = 9.6 \times 10^7$ N

(b) $F = PA = 100(60) = 6000$ lb

3. $F = \displaystyle\int_0^2 62.4x(4)dx$

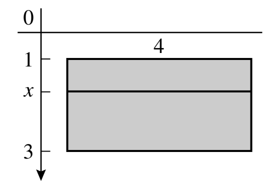
$= 249.6 \displaystyle\int_0^2 x\,dx = 499.2\,\text{lb}$



4. $F = \int_1^3 9810x(4)dx$

$= 39{,}240 \int_1^3 x\,dx$

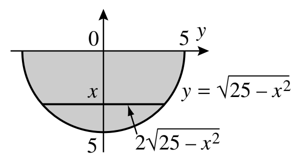
$= 156{,}960\,\text{N}$



5. $F = \int_0^5 9810x(2\sqrt{25 - x^2})dx$

$= 19{,}620 \int_0^5 x(25 - x^2)^{1/2}dx$

$= 8.175 \times 10^5\,\text{N}$

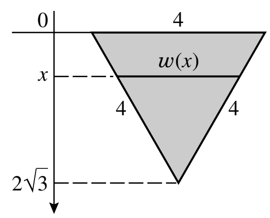


6. By similar triangles

$\frac{w(x)}{4} = \frac{2\sqrt{3} - x}{2\sqrt{3}}$, $w(x) = \frac{2}{\sqrt{3}}(2\sqrt{3} - x)$,

$F = \int_0^{2\sqrt{3}} 62.4x \left[\frac{2}{\sqrt{3}}(2\sqrt{3} - x)\right] dx$

$= \frac{124.8}{\sqrt{3}} \int_0^{2\sqrt{3}} (2\sqrt{3}x - x^2)dx = 499.2\,\text{lb}$



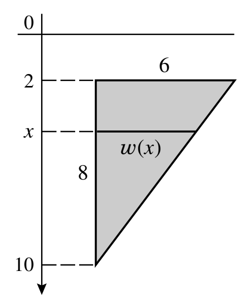
7. By similar triangles

$\frac{w(x)}{6} = \frac{10 - x}{8}$

$w(x) = \frac{3}{4}(10 - x)$,

$F = \int_2^{10} 9810x \left[\frac{3}{4}(10 - x)\right] dx$

$= 7357.5 \int_2^{10} (10x - x^2)dx = 1{,}098{,}720\ \text{N}$



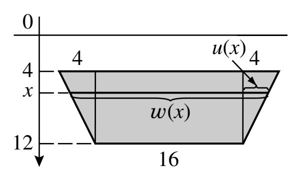
8. $w(x) = 16 + 2u(x)$, but

$\frac{u(x)}{4} = \frac{12 - x}{8}$ so $u(x) = \frac{1}{2}(12 - x)$,

$w(x) = 16 + (12 - x) = 28 - x$,

$F = \int_4^{12} 62.4x(28 - x)dx$

$= 62.4 \int_4^{12} (28x - x^2)dx = 77{,}209.6\ \text{lb}.$

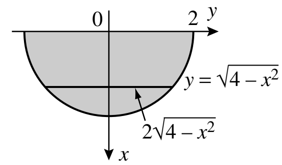


9. Yes: if $\rho_2 = 2\rho_1$ then $F_2 = \int_a^b \rho_2 h(x)w(x)\,dx = \int_a^b 2\rho_1 h(x)w(x)\,dx = 2\int_a^b \rho_1 h(x)w(x)\,dx = 2F_1$.

10. $F = \int_0^2 50x(2\sqrt{4-x^2})dx$

$= 100\int_0^2 x(4-x^2)^{1/2}dx$

$= 800/3\,\text{lb}$



11. Find the forces on the upper and lower halves and add them:

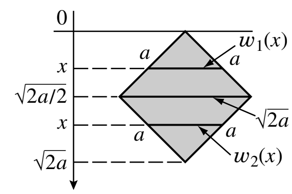
$\dfrac{w_1(x)}{\sqrt{2}a} = \dfrac{x}{\sqrt{2}a/2}$, $w_1(x) = 2x$

$F_1 = \int_0^{\sqrt{2}a/2} \rho x(2x)dx = 2\rho \int_0^{\sqrt{2}a/2} x^2 dx = \sqrt{2}\rho a^3/6$,

$\dfrac{w_2(x)}{\sqrt{2}a} = \dfrac{\sqrt{2}a - x}{\sqrt{2}a/2}$, $w_2(x) = 2(\sqrt{2}a - x)$

$F_2 = \int_{\sqrt{2}a/2}^{\sqrt{2}a} \rho x[2(\sqrt{2}a - x)]dx = 2\rho \int_{\sqrt{2}a/2}^{\sqrt{2}a} (\sqrt{2}ax - x^2)dx = \sqrt{2}\rho a^3/3$,

$F = F_1 + F_2 = \sqrt{2}\rho a^3/6 + \sqrt{2}\rho a^3/3 = \rho a^3/\sqrt{2}$ lb



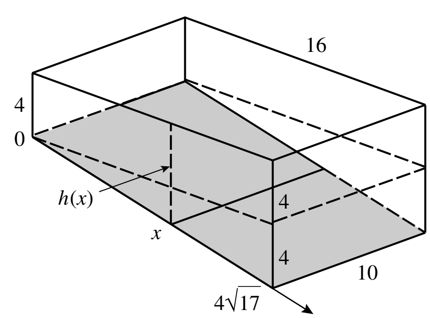
12. If a constant vertical force is applied to a flat plate which is horizontal and the magnitude of the force is F, then, if the plate is tilted so as to form an angle θ with the vertical, the magnitude of the force on the plate decreases to $F\cos\theta$.
Suppose that a flat surface is immersed, at an angle θ with the vertical, in a fluid of weight density ρ, and that the submerged portion of the surface extends from $x = a$ to $x = b$ along an x-axis whose positive diretion is not necessarily down, but is slanted.
Following the derivation of equation (8), we divide the interval $[a, b]$ into n subintervals
$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$. Then the magnitude F_k of the force on the plate satisfies the inequalities $\rho h(x_{k-1})A_k \cos\theta \le F_k \le \rho h(x_k)A_k\cos\theta$, or equivalently that
$h(x_{k-1}) \le \dfrac{F_k \sec\theta}{\rho A_k} \le h(x_k)$. Following the argument in the text we arrive at the desired equation
$F = \int_a^b \rho h(x)w(x)\sec\theta\,dx$.

13. $\sqrt{16^2 + 4^2} = \sqrt{272} = 4\sqrt{17}$ is the other dimension of the bottom.

$(h(x) - 4)/4 = x/(4\sqrt{17})$

$h(x) = x/\sqrt{17} + 4$,

$\sec\theta = 4\sqrt{17}/16 = \sqrt{17}/4$



$$F = \int_0^{4\sqrt{17}} 62.4(x/\sqrt{17}+4)10(\sqrt{17}/4)\,dx$$
$$= 156\sqrt{17}\int_0^{4\sqrt{17}} (x/\sqrt{17}+4)dx$$
$$= 63{,}648\,\text{lb}$$

14. If we lower the water level by y ft then the force F_1 is computed as in Exercise 13, but with $h(x)$ replaced by $h_1(x) = x/\sqrt{17}+4-y$, and we obtain

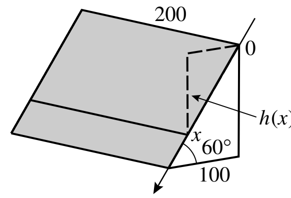
$$F_1 = F - y\int_0^{4\sqrt{17}} 62.4(10)\sqrt{17}/4\,dx = F - 624(17)y = 63{,}648 - 10{,}608y.$$

If $F_1 = F/2$ then $63{,}648/2 = 63{,}648 - 10{,}608y$, $y = 63{,}648/(2\cdot 10{,}608) = 3$,

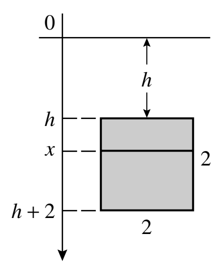
so the water level should be reduced by 3 ft.

15. $h(x) = x\sin 60^\circ = \sqrt{3}x/2$,
$\theta = 30^\circ$, $\sec\theta = 2/\sqrt{3}$,

$$F = \int_0^{100} 9810(\sqrt{3}x/2)(200)(2/\sqrt{3})\,dx$$
$$= 1{,}962{,}000\int_0^{100} x\,dx$$
$$= 9.81\times 10^9\ \text{N}$$



16. $$F = \int_h^{h+2} \rho_0 x(2)dx$$
$$= 2\rho_0\int_h^{h+2} x\,dx$$
$$= 4\rho_0(h+1)$$



17. **(a)** From Exercise 16, $F = 4\rho_0(h+1)$ so (assuming that ρ_0 is constant) $dF/dt = 4\rho_0(dh/dt)$ which is a positive constant if dh/dt is a positive constant.

(b) If $dh/dt = 20$ then $dF/dt = 80\rho_0$ lb/min from Part (a).

18. **(a)** Let h_1 and h_2 be the maximum and minimum depths of the disk D_r. The pressure $P(r)$ on one side of the disk satisfies inequality (5):
$\rho h_1 \le P(r) \le \rho h_2$. But
$\lim_{r\to 0^+} h_1 = \lim_{r\to 0^+} h_2 = h$, and hence
$\rho h = \lim_{r\to 0^+} \rho h_1 \le \lim_{r\to 0^+} P(r) \le \lim_{r\to 0^+} \rho h_2 = \rho h$, so $\lim_{r\to 0^+} P(r) = \rho h$.

(b) The disks D_r in Part (a) have no particular direction (the axes of the disks have arbitrary direction). Thus P, the limiting value of $P(r)$, is independent of direction.

CHAPTER 6 SUPPLEMENTARY EXERCISES

6. **(a)** $A = \int_0^2 (2 + x - x^2)\,dx$

(b) $A = \int_0^2 \sqrt{y}\,dy + \int_2^4 [(\sqrt{y} - (y - 2)]\,dy$

(c) $V = \pi \int_0^2 [(2 + x)^2 - x^4]\,dx$

(d) $V = 2\pi \int_0^2 y\sqrt{y}\,dy + 2\pi \int_2^4 y[\sqrt{y} - (y - 2)]\,dy$

(e) $V = 2\pi \int_0^2 x(2 + x - x^2)\,dx$

(f) $V = \pi \int_0^2 y\,dy + \int_2^4 \pi(y - (y - 2)^2)\,dy$

7. **(a)** $A = \int_a^b (f(x) - g(x))\,dx + \int_b^c (g(x) - f(x))\,dx + \int_c^d (f(x) - g(x))\,dx$

(b) $A = \int_{-1}^0 (x^3 - x)\,dx + \int_0^1 (x - x^3)\,dx + \int_1^2 (x^3 - x)\,dx = \frac{1}{4} + \frac{1}{4} + \frac{9}{4} = \frac{11}{4}$

8. **(a)** $S = \int_0^{8/27} 2\pi x\sqrt{1 + x^{-4/3}}\,dx$ **(b)** $S = \int_0^2 2\pi \frac{y^3}{27}\sqrt{1 + y^4/81}\,dy$

(c) $S = \int_0^2 2\pi(y + 2)\sqrt{1 + y^4/81}\,dy$

9. By implicit differentiation $\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3}$, so $1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{y}{x}\right)^{2/3} = \frac{x^{2/3} + y^{2/3}}{x^{2/3}} = \frac{a^{2/3}}{x^{2/3}}$,

$L = \int_{-a}^{-a/8} \frac{a^{1/3}}{(-x^{1/3})} dx = -a^{1/3} \int_{-a}^{-a/8} x^{-1/3} dx = 9a/8.$

10. The base of the dome is a hexagon of side r. An equation of the circle of radius r that lies in a vertical x-y plane and passes through two opposite vertices of the base hexagon is $x^2 + y^2 = r^2$. A horizontal, hexagonal cross section at height y above the base has area

$A(y) = \frac{3\sqrt{3}}{2} x^2 = \frac{3\sqrt{3}}{2}(r^2 - y^2)$, hence the volume is $V = \int_0^r \frac{3\sqrt{3}}{2}(r^2 - y^2)\,dy = \sqrt{3}r^3$.

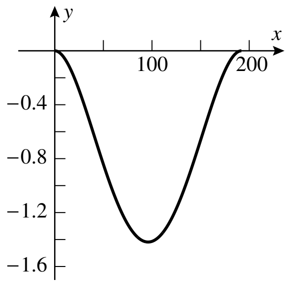
11. Let the sphere have radius R, the hole radius r. By the Pythagorean Theorem, $r^2 + (L/2)^2 = R^2$. Use cylindrical shells to calculate the volume of the solid obtained by rotating about the y-axis the region $r < x < R$, $-\sqrt{R^2 - x^2} < y < \sqrt{R^2 - x^2}$:

$V = \int_r^R (2\pi x)2\sqrt{R^2 - x^2}\,dx = -\frac{4}{3}\pi(R^2 - x^2)^{3/2}\Big]_r^R = \frac{4}{3}\pi(L/2)^3,$

so the volume is independent of R.

12. $V = 2\int_0^{L/2} \pi \frac{16R^2}{L^4}(x^2 - L^2/4)^2 = \frac{4\pi}{15}LR^2$

13. **(a)**



(b) The maximum deflection occurs at $x = 96$ inches (the midpoint of the beam) and is about 1.42 in.

(c) The length of the centerline is $\int_0^{192} \sqrt{1 + (dy/dx)^2}\, dx = 192.026$ in.

14. $y = 0$ at $x = b = 30.585$; distance $= \int_0^{b} \sqrt{1 + (12.54 - 0.82x)^2}\, dx = 196.306$ yd

15. **(a)** $F = kx$, $\frac{1}{2} = k\frac{1}{4}$, $k = 2$, $W = \int_0^{1/4} kx\, dx = 1/16$ J

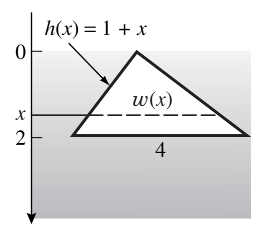
(b) $25 = \int_0^{L} kx\, dx = kL^2/2$, $L = 5$ m

16. $F = 30x + 2000$, $W = \int_0^{150} (30x + 2000)\, dx = 15 \cdot 150^2 + 2000 \cdot 150 = 637{,}500$ lb·ft

17. **(a)** $F = \int_0^{1} \rho x 3\, dx$ N

(b) By similar triangles $\frac{w(x)}{4} = \frac{x}{2}$, $w(x) = 2x$, so

$F = \int_1^{4} \rho(1 + x)2x\, dx$ lb/ft^2.



(c) A formula for the parabola is $y = \frac{8}{125}x^2 - 10$, so $F = \int_{-10}^{0} 9810|y|2\sqrt{\frac{125}{8}(y + 10)}\, dy$ N.

18. The x-coordinates of the points of intersection are $a \approx -0.423028$ and $b \approx 1.725171$; the area is $\int_a^b (2\sin x - x^2 + 1)dx \approx 2.542696$.

19. Let (a, k), where $\pi/2 < a < \pi$, be the coordinates of the point of intersection of $y = k$ with $y = \sin x$. Thus $k = \sin a$ and if the shaded areas are equal,

$$\int_0^a (k - \sin x)dx = \int_0^a (\sin a - \sin x)\,dx = a\sin a + \cos a - 1 = 0$$

Solve for a to get $a \approx 2.331122$, so $k = \sin a \approx 0.724611$.

20. The volume is given by $2\pi \displaystyle\int_0^k x\sin x\,dx = 2\pi(\sin k - k\cos k) = 8$; solve for k to get $k = 1.736796$.