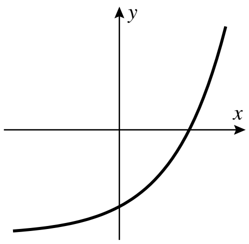
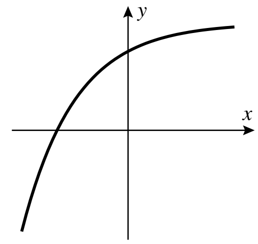
CHAPTER 4
The Derivative in Graphing and Applications

EXERCISE SET 4.1

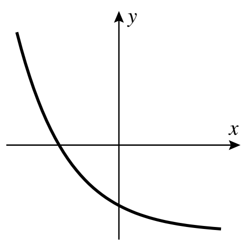
1. **(a)** $f' > 0$ and $f'' > 0$



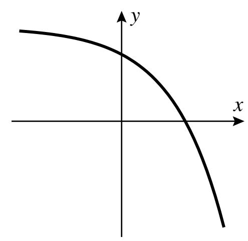
(b) $f' > 0$ and $f'' < 0$



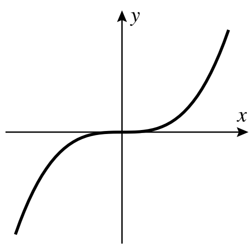
(c) $f' < 0$ and $f'' > 0$



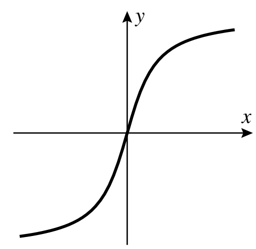
(d) $f' < 0$ and $f'' < 0$



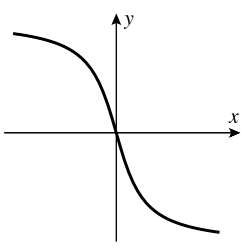
2. **(a)**



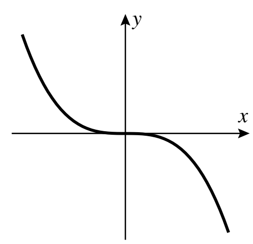
(b)



(c)



(d)



3. A: $dy/dx < 0,\ d^2y/dx^2 > 0$
B: $dy/dx > 0,\ d^2y/dx^2 < 0$
C: $dy/dx < 0,\ d^2y/dx^2 < 0$

4. A: $dy/dx < 0,\ d^2y/dx^2 < 0$
B: $dy/dx < 0,\ d^2y/dx^2 > 0$
C: $dy/dx > 0,\ d^2y/dx^2 < 0$

5. An inflection point occurs when f'' changes sign: at $x = -1, 0, 1$ and 2.

6. **(a)** $f(0) < f(1)$ since $f' > 0$ on $(0, 1)$.
(b) $f(1) > f(2)$ since $f' < 0$ on $(1, 2)$.
(c) $f'(0) > 0$ by inspection.
(d) $f'(1) = 0$ by inspection.
(e) $f''(0) < 0$ since f' is decreasing there.
(f) $f''(2) = 0$ since f' has a minimum there.

7. **(a)** $[4, 6]$
(b) $[1, 4]$ and $[6, 7]$
(c) $(1, 2)$ and $(3, 5)$
(d) $(2, 3)$ and $(5, 7)$
(e) $x = 2, 3, 5$

8.

	$(1,2)$	$(2,3)$	$(3,4)$	$(4,5)$	$(5,6)$	$(6,7)$
f'	$-$	$-$	$-$	$+$	$+$	$-$
f''	$+$	$-$	$+$	$+$	$-$	$-$

9. **(a)** f is increasing on $[1,3]$ **(b)** f is decreasing on $(-\infty,1],[3,+\infty]$
(c) f is concave up on $(-\infty,2),(4,+\infty)$ **(d)** f is concave down on $(2,4)$
(e) points of inflection at $x=2,4$

10. **(a)** f is increasing on $(-\infty,+\infty)$ **(b)** f is nowhere decreasing
(c) f is concave up on $(-\infty,1),(3,+\infty)$ **(d)** f is concave down on $(1,3)$
(e) f has points of inflection at $x=1,3$

11. $f'(x)=2x-5$, $f''(x)=2$
(a) $[5/2,+\infty)$ **(b)** $(-\infty,5/2]$ **(c)** $(-\infty,+\infty)$ **(d)** none **(e)** none

12. $f'(x)=-2(x+3/2)$, $f''(x)=-2$
(a) $(-\infty,-3/2]$ **(b)** $[-3/2,+\infty)$ **(c)** none **(d)** $(-\infty,+\infty)$ **(e)** none

13. $f'(x)=3(x+2)^2$, $f''(x)=6(x+2)$
(a) $(-\infty,+\infty)$ **(b)** none **(c)** $(-2,+\infty)$ **(d)** $(-\infty,-2)$ **(e)** -2

14. $f'(x)=3(4-x^2)$, $f''(x)=-6x$
(a) $[-2,2]$ **(b)** $(-\infty,-2]$, $[2,+\infty)$ **(c)** $(-\infty,0)$ **(d)** $(0,+\infty)$ **(e)** 0

15. $f'(x)=12x^2(x-1)$, $f''(x)=36x(x-2/3)$
(a) $[1,+\infty)$ **(b)** $(-\infty,1]$ **(c)** $(-\infty,0)$, $(2/3,+\infty)$ **(d)** $(0,2/3)$ **(e)** $0,2/3$

16. $f'(x)=4x(x^2-4)$, $f''(x)=12(x^2-4/3)$
(a) $[-2,0]$, $[2,+\infty)$ **(b)** $(-\infty,-2]$, $[0,2]$ **(c)** $(-\infty,-2/\sqrt{3})$, $(2/\sqrt{3},+\infty)$ **(d)** $(-2/\sqrt{3},2/\sqrt{3})$ **(e)** $-2/\sqrt{3}$, $2/\sqrt{3}$

17. $f'(x)=\dfrac{4x}{(x^2+2)^2}$ $\quad f''(x)=-4\dfrac{3x^2-2}{(x^2+2)^3}$

(a) $[0,+\infty)$ **(b)** $(-\infty,0]$ **(c)** $(-\sqrt{2/3},+\sqrt{2/3})$
(d) $(-\infty,-\sqrt{2/3})$, $(+\sqrt{2/3},+\infty)$ **(e)** $-\sqrt{2/3}$, $\sqrt{2/3}$

18. $f'(x)=\dfrac{2-x^2}{(x^2+2)^2}$ $\quad f''(x)=\dfrac{2x(x^2-6)}{(x^2+2)^3}$

(a) $[-\sqrt{2},\sqrt{2}]$ **(b)** $(-\infty,-\sqrt{2}]$, $[\sqrt{2},+\infty)$ **(c)** $(-\sqrt{6},0)$, $(\sqrt{6},+\infty)$
(d) $(-\infty,-\sqrt{6})$, $(0,\sqrt{6})$ **(e)** $-\sqrt{6},0,\sqrt{6}$

19. $f'(x)=\frac{1}{3}(x+2)^{-2/3}$, $f''(x)=-\frac{2}{9}(x+2)^{-5/3}$
(a) $(-\infty,+\infty)$ **(b)** none **(c)** $(-\infty,-2)$ **(d)** $(-2,+\infty)$ **(e)** -2

20. $f'(x) = \frac{2}{3}x^{-1/3}$
$f''(x) = -\frac{2}{9}x^{-4/3}$

(a) $[0, +\infty)$ **(b)** $(-\infty, 0]$
(c) none **(d)** $(-\infty, 0)$, $(0, +\infty)$
(e) none

21. $f'(x) = \dfrac{4(x+1)}{3x^{2/3}}$
$f''(x) = \dfrac{4(x-2)}{9x^{5/3}}$

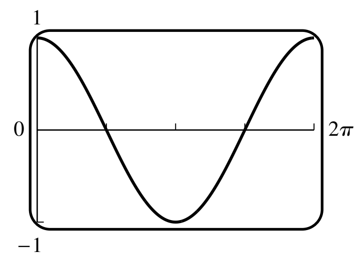
(a) $[-1, +\infty)$ **(b)** $(-\infty, -1]$
(c) $(-\infty, 0)$, $(2, +\infty)$ **(d)** $(0, 2)$
(e) $0, 2$

22. $f'(x) = \dfrac{4(x-1/4)}{3x^{2/3}}$
$f''(x) = \dfrac{4(x+1/2)}{9x^{5/3}}$

(a) $[1/4, +\infty)$ **(b)** $(-\infty, 1/4]$
(c) $(-\infty, -1/2)$, $(0, +\infty)$ **(d)** $(-1/2, 0)$
(e) $-1/2, 0$

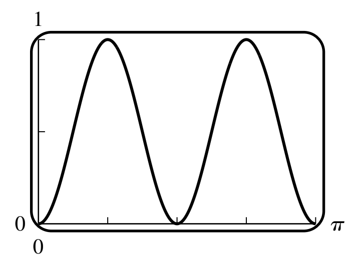
23. $f'(x) = -\sin x$
$f''(x) = -\cos x$

(a) $[\pi, 2\pi]$ **(b)** $[0, \pi]$
(c) $(\pi/2, 3\pi/2)$ **(d)** $(0, \pi/2)$, $(3\pi/2, 2\pi)$
(e) $\pi/2$, $3\pi/2$



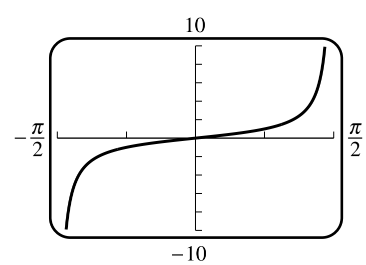
24. $f'(x) = 2\sin 4x$
$f''(x) = 8\cos 4x$

(a) $(0, \pi/4]$, $[\pi/2, 3\pi/4]$
(b) $[\pi/4, \pi/2]$, $[3\pi/4, \pi]$
(c) $(0, \pi/8)$, $(3\pi/8, 5\pi/8)$, $(7\pi/8, \pi)$
(d) $(\pi/8, 3\pi/8)$, $(5\pi/8, 7\pi/8)$
(e) $\pi/8$, $3\pi/8$, $5\pi/8$, $7\pi/8$



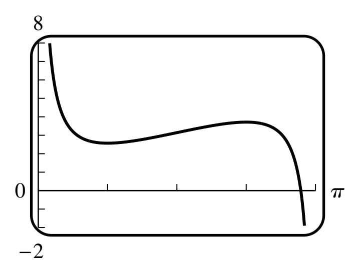
25. $f'(x) = \sec^2 x$
$f''(x) = 2\sec^2 x \tan x$

(a) $(-\pi/2, \pi/2)$ **(b)** none
(c) $(0, \pi/2)$ **(d)** $(-\pi/2, 0)$
(e) 0



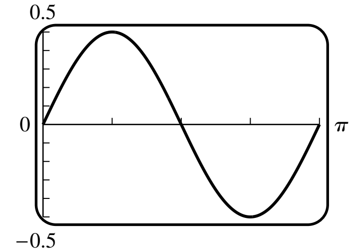
26. $f'(x) = 2 - \csc^2 x$
$f''(x) = 2\csc^2 x \cot x = 2\dfrac{\cos x}{\sin^3 x}$

(a) $[\pi/4, 3\pi/4]$ **(b)** $(0, \pi/4], [3\pi/4, \pi)$
(c) $(0, \pi/2)$ **(d)** $(\pi/2, \pi)$
(e) $\pi/2$



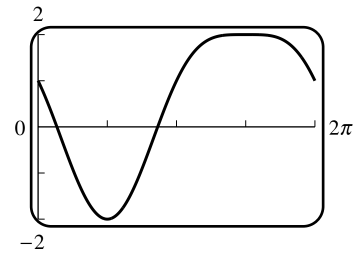
27. $f'(x) = \cos 2x$
$f''(x) = -2\sin 2x$

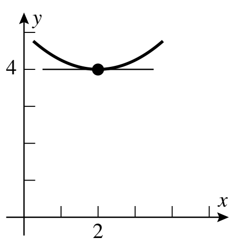
(a) $[0, \pi/4]$, $[3\pi/4, \pi]$ **(b)** $[\pi/4, 3\pi/4]$
(c) $(\pi/2, \pi)$ **(d)** $(0, \pi/2)$
(e) $\pi/2$

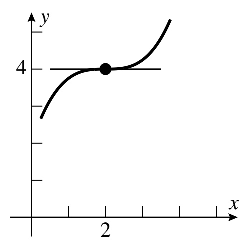


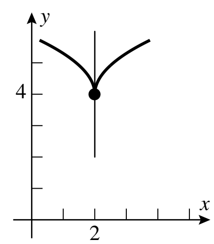
28. $f'(x) = -2\cos x \sin x - 2\cos x = -2\cos x(1 + \sin x)$
$f''(x) = 2\sin x\,(\sin x + 1) - 2\cos^2 x = 2\sin x(\sin x + 1) - 2 + 2\sin^2 x = 4(1 + \sin x)(\sin x - 1/2)$
Note: $1 + \sin x \geq 0$

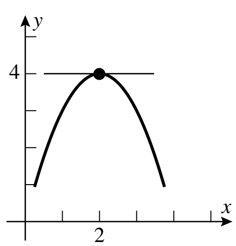
(a) $[\pi/2, 3\pi/2]$ **(b)** $[0, \pi/2]$, $[3\pi/2, 2\pi]$
(c) $(\pi/6, 5\pi/6)$ **(d)** $(0, \pi/6)$, $(5\pi/6, 2\pi)$
(e) $\pi/6$, $5\pi/6$

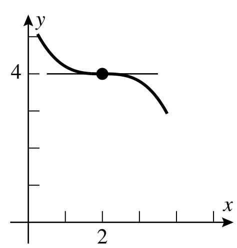


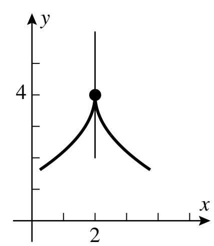
29. **(a)**


(b)


(c)


30. **(a)**


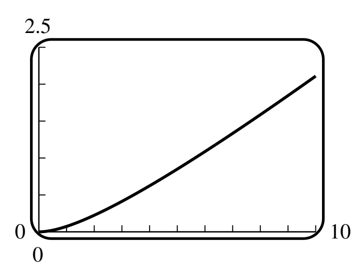
(b)


(c)


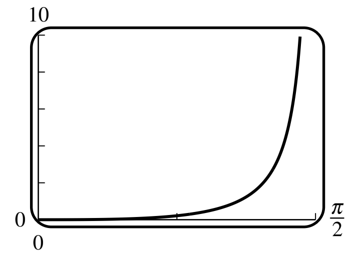
31. **(a)** $f'(x) = 3(x - a)^2$, $f''(x) = 6(x - a)$; inflection point is $(a, 0)$
(b) $f'(x) = 4(x - a)^3$, $f''(x) = 12(x - a)^2$; no inflection points

32. For $n \geq 2$, $f''(x) = n(n-1)(x - a)^{n-2}$; there is a sign change of f'' (point of inflection) at $(a, 0)$ if and only if n is odd. For $n = 1$, $y = x - a$, so there is no point of inflection.

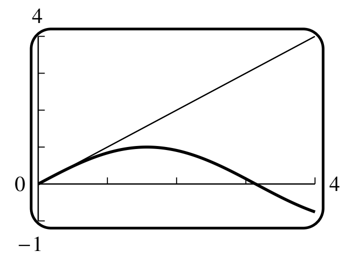
33. $f'(x) = 1/3 - 1/[3(1 + x)^{2/3}]$ so f is increasing on $[0, +\infty)$ thus if $x > 0$, then $f(x) > f(0) = 0$, $1 + x/3 - \sqrt[3]{1 + x} > 0$, $\sqrt[3]{1 + x} < 1 + x/3$.



34. $f'(x) = \sec^2 x - 1$ so f is increasing on $[0, \pi/2)$ thus if $0 < x < \pi/2$, then $f(x) > f(0) = 0$, $\tan x - x > 0$, $x < \tan x$.

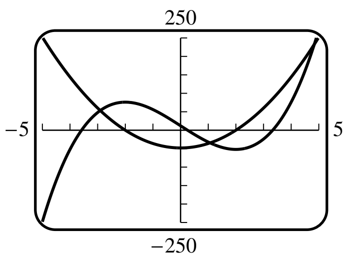


35. $x \geq \sin x$ on $[0, +\infty)$: let $f(x) = x - \sin x$. Then $f(0) = 0$ and $f'(x) = 1 - \cos x \geq 0$, so $f(x)$ is increasing on $[0, +\infty)$.

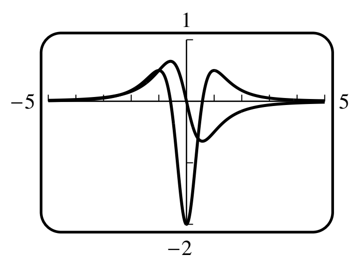


36. Let $f(x) = 1 - x^2/2 - \cos x$ for $x \geq 0$. Then $f(0) = 0$ and $f'(x) = -x + \sin x$. By Exercise 35, $f'(x) \leq 0$ for $x \geq 0$, so $f(x) \leq 0$ for all $x \geq 0$, that is, $\cos x \geq 1 - x^2/2$.

37. Points of inflection at $x = -2, +2$. Concave up on $(-5, -2)$ and $(2, 5)$; concave down on $(-2, 2)$. Increasing on $[-3.5829, 0.2513]$ and $[3.3316, 5]$, and decreasing on $[-5, -3.5829]$ and $[0.2513, 3.3316]$.



38. Points of inflection at $x = \pm 1/\sqrt{3}$. Concave up on $[-5, -1/\sqrt{3}]$ and $[1/\sqrt{3}, 5]$, and concave down on $[-1/\sqrt{3}, 1/\sqrt{3}]$. Increasing on $[-5, 0]$ and decreasing on $[0, 5]$.



39. $f''(x) = 2\dfrac{90x^3 - 81x^2 - 585x + 397}{(3x^2 - 5x + 8)^3}$. The denominator has complex roots, so is always positive; hence the x-coordinates of the points of inflection of $f(x)$ are the roots of the numerator (if it changes sign). A plot of the numerator over $[-5, 5]$ shows roots lying in $[-3, -2]$, $[0, 1]$, and $[2, 3]$. To six decimal places the roots are $x = -2.464202, 0.662597, 2.701605$.

40. $f''(x) = \dfrac{2x^5 + 5x^3 + 14x^2 + 30x - 7}{(x^2 + 1)^{5/2}}$. Points of inflection will occur when the numerator changes sign, since the denominator is always positive. A plot of $y = 2x^5 + 5x^3 + 14x^2 + 30x - 7$ shows that there is only one root and it lies in $[0, 1]$. To six decimal place the point of inflection is located at $x = 0.210970$.

41. $f(x_1) - f(x_2) = x_1^2 - x_2^2 = (x_1 + x_2)(x_1 - x_2) < 0$ if $x_1 < x_2$ for x_1, x_2 in $[0, +\infty)$, so $f(x_1) < f(x_2)$ and f is thus increasing.

42. $f(x_1) - f(x_2) = \dfrac{1}{x_1} - \dfrac{1}{x_2} = \dfrac{x_2 - x_1}{x_1 x_2} > 0$ if $x_1 < x_2$ for x_1, x_2 in $(0, +\infty)$, so $f(x_1) > f(x_2)$ and thus f is decreasing.

43. **(a)** If $x_1 < x_2$ where x_1 and x_2 are in I, then $f(x_1) < f(x_2)$ and $g(x_1) < g(x_2)$, so $f(x_1) + g(x_1) < f(x_2) + g(x_2)$, $(f + g)(x_1) < (f + g)(x_2)$. Thus $f + g$ is increasing on I.

(b) Case I: If f and g are ≥ 0 on I, and if $x_1 < x_2$ where x_1 and x_2 are in I, then $0 < f(x_1) < f(x_2)$ and $0 < g(x_1) < g(x_2)$, so $f(x_1)g(x_1) < f(x_2)g(x_2)$, $(f \cdot g)(x_1) < (f \cdot g)(x_2)$. Thus $f \cdot g$ is increasing on I.

Case II: If f and g are not necessarily positive on I then no conclusion can be drawn: for example, $f(x) = g(x) = x$ are both increasing on $(-\infty, 0)$, but $(f \cdot g)(x) = x^2$ is decreasing there.

44. **(a)** $f(x) = x$, $g(x) = 2x$ **(b)** $f(x) = x$, $g(x) = x + 6$ **(c)** $f(x) = 2x$, $g(x) = x$

45. **(a)** $f''(x) = 6ax + 2b = 6a\left(x + \dfrac{b}{3a}\right)$, $f''(x) = 0$ when $x = -\dfrac{b}{3a}$. f changes its direction of concavity at $x = -\dfrac{b}{3a}$ so $-\dfrac{b}{3a}$ is an inflection point.

(b) If $f(x) = ax^3 + bx^2 + cx + d$ has three x-intercepts, then it has three roots, say x_1, x_2 and x_3, so we can write $f(x) = a(x - x_1)(x - x_2)(x - x_3) = ax^3 + bx^2 + cx + d$, from which it follows that $b = -a(x_1 + x_2 + x_3)$. Thus $-\dfrac{b}{3a} = \dfrac{1}{3}(x_1 + x_2 + x_3)$, which is the average.

(c) $f(x) = x(x^2 - 3x^2 + 2) = x(x - 1)(x - 2)$ so the intercepts are 0, 1, and 2 and the average is 1. $f''(x) = 6x - 6 = 6(x - 1)$ changes sign at $x = 1$.

46. $f''(x) = 6x + 2b$, so the point of inflection is at $x = -\dfrac{b}{3}$. Thus an increase in b moves the point of inflection to the left.

47. **(a)** Let $x_1 < x_2$ belong to (a, b). If both belong to $(a, c]$ or both belong to $[c, b)$ then we have $f(x_1) < f(x_2)$ by hypothesis. So assume $x_1 < c < x_2$. We know by hypothesis that $f(x_1) < f(c)$, and $f(c) < f(x_2)$. We conclude that $f(x_1) < f(x_2)$.

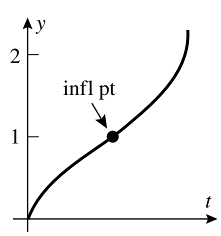
(b) Use the same argument as in Part (a), but with inequalities reversed.

48. By Theorem 4.1.2, f is increasing on any interval $[(2n-1)\pi, 2(n+1)\pi]$ $(n = 0, \pm 1, \pm 2, \cdots)$, because $f'(x) = 1 + \cos x > 0$ on $((2n - 1)\pi, (2n + 1)\pi)$. By Exercise 47 (a) we can piece these intervals together to show that $f(x)$ is increasing on $(-\infty, +\infty)$.

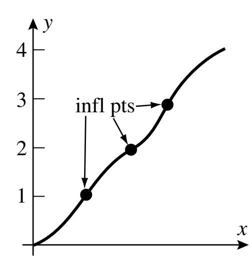
49. By Theorem 4.1.2, $f$is decreasing on any interval $[(2n\pi + \pi/2, 2(n+1)\pi + \pi/2]$ $(n = 0, \pm 1, \pm 2, \cdots)$, because $f'(x) = -\sin x + 1 < 0$ on $((2n\pi + \pi/2, 2(n + 1)\pi + \pi/2)$. By Exercise 47 (b) we can piece these intervals together to show that $f(x)$ is decreasing on $(-\infty, +\infty)$.

50. By zooming on the graph of $y'(t)$, maximum increase is at $x = -0.577$ and maximum decrease is at $x = 0.577$.

51.



52.



53. **(a)** $g(x)$ has no zeros:

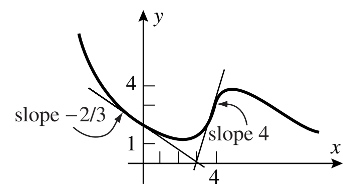
There can be no zero of $g(x)$ on the interval $-\infty < x < 0$ because if there were, say $g(x_0) = 0$ where $x_0 < 0$, then $g'(x)$ would have to be positive between $x = x_0$ and $x = 0$, say $g'(x_1) > 0$ where $x_0 < x_1 < 0$. But then $g'(x)$ cannot be concave up on the interval $(x_1, 0)$, a contradiction.

There can be no zero of $g(x)$ on $0 < x < 4$ because $g(x)$ is concave up for $0 < x < 4$ and thus the graph of $g(x)$, for $0 < x < 4$, must lie above the line $y = -\dfrac{2}{3}x + 2$, which is the tangent line to the curve at $(0, 2)$, and above the line $y = 3(x - 4) + 3 = 3x - 9$ also for $0 < x < 4$ (see figure). The first condition says that $g(x)$ could only be zero for $x > 3$ and the second condition says that $g(x)$ could only be zero for $x < 3$, thus $g(x)$ has no zeros for $0 < x < 4$.

Finally, if $4 < x < +\infty$, $g(x)$ could only have a zero if $g'(x)$ were negative somewhere for $x > 4$, and since $g'(x)$ is decreasing there we would ultimately have $g(x) < -10$, a contradiction.

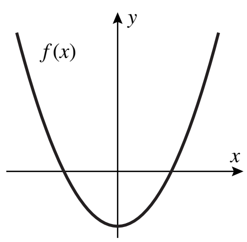
(b) one, between 0 and 4

(c) We must have $\lim_{x \to +\infty} g'(x) = 0$; if the limit were -5 then $g(x)$ would at some time cross the line $x = -10$; if the limit were 5 then, since g is concave down for $x > 4$ and $g'(4) = 3$, g' must decrease for $x > 4$ and thus the limit would be < 4.

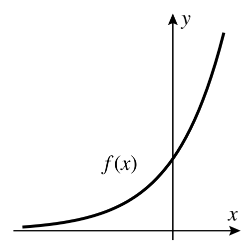


EXERCISE SET 4.2

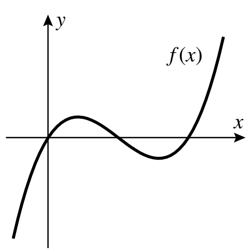
1. **(a)**



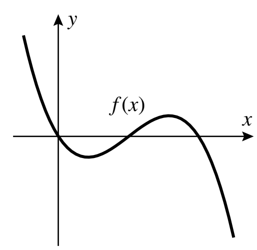
(b)



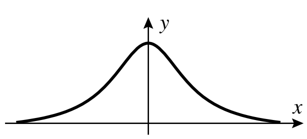
(c)



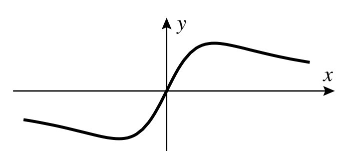
(d)



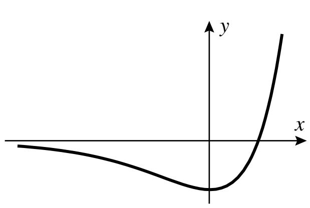
2. **(a)**



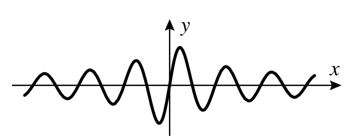
(b)



(c)



(d)



3. **(a)** $f'(x) = 6x - 6$ and $f''(x) = 6$, with $f'(1) = 0$. For the first derivative test, $f' < 0$ for $x < 1$ and $f' > 0$ for $x > 1$. For the second derivative test, $f''(1) > 0$.

(b) $f'(x) = 3x^2 - 3$ and $f''(x) = 6x$. $f'(x) = 0$ at $x = \pm 1$. First derivative test: $f' > 0$ for $x < -1$ and $x > 1$, and $f' < 0$ for $-1 < x < 1$, so there is a relative maximum at $x = -1$, and a relative minimum at $x = 1$. Second derivative test: $f'' < 0$ at $x = -1$, a relative maximum; and $f'' > 0$ at $x = 1$, a relative minimum.

4. **(a)** $f'(x) = 2\sin x \cos x = \sin 2x$ (so $f'(0) = 0$) and $f''(x) = 2\cos 2x$. First derivative test: if x is near 0 then $f' < 0$ for $x < 0$ and $f' > 0$ for $x > 0$, so a relative minimum at $x = 0$. Second derivative test: $f''(0) = 2 > 0$, so relative minimum at $x = 0$.

(b) $g'(x) = 2\tan x \sec^2 x$ (so $g'(0) = 0$) and $g''(x) = 2\sec^2 x(\sec^2 x + 2\tan^2 x)$. First derivative test: $g' < 0$ for $x < 0$ and $g' > 0$ for $x > 0$, so a relative minimum at $x = 0$. Second derivative test: $g''(0) = 2 > 0$, relative minimum at $x = 0$.

(c) Both functions are squares, and so are positive for values of x near zero; both functions are zero at $x = 0$, so that must be a relative minimum.

5. **(a)** $f'(x) = 4(x-1)^3$, $g'(x) = 3x^2 - 6x + 3$ so $f'(1) = g'(1) = 0$.

(b) $f''(x) = 12(x-1)^2$, $g''(x) = 6x - 6$, so $f''(1) = g''(1) = 0$, which yields no information.

(c) $f' < 0$ for $x < 1$ and $f' > 0$ for $x > 1$, so there is a relative minimum at $x = 1$; $g'(x) = 3(x-1)^2 > 0$ on both sides of $x = 1$, so there is no relative extremum at $x = 1$.

6. **(a)** $f'(x) = -5x^4$, $g'(x) = 12x^3 - 24x^2$ so $f'(0) = g'(0) = 0$.

(b) $f''(x) = -20x^3$, $g''(x) = 36x^2 - 48x$, so $f''(0) = g''(0) = 0$, which yields no information.

(c) $f' < 0$ on both sides of $x = 0$, so there is no relative extremum there; $g'(x) = 12x^2(x-2) < 0$ on both sides of $x = 0$ (for x near 0), so again there is no relative extremum there.

7. **(a)** $f'(x) = 3x^2 + 6x - 9 = 3(x+3)(x-1)$, $f'(x) = 0$ when $x = -3, 1$ (stationary points).

(b) $f'(x) = 4x(x^2 - 3)$, $f'(x) = 0$ when $x = 0, \pm\sqrt{3}$ (stationary points).

8. **(a)** $f'(x) = 6(x^2 - 1)$, $f'(x) = 0$ when $x = \pm 1$ (stationary points).

(b) $f'(x) = 12x^3 - 12x^2 = 12x^2(x-1)$, $f'(x) = 0$ when $x = 0, 1$ (stationary points).

9. **(a)** $f'(x) = (2 - x^2)/(x^2+2)^2$, $f'(x) = 0$ when $x = \pm\sqrt{2}$ (stationary points).

(b) $f'(x) = \frac{2}{3}x^{-1/3} = 2/(3x^{1/3})$, $f'(x)$ does not exist when $x = 0$.

10. **(a)** $f'(x) = 8x/(x^2+1)^2$, $f'(x) = 0$ when $x = 0$ (stationary point).

(b) $f'(x) = \frac{1}{3}(x+2)^{-2/3}$, $f'(x)$ does not exist when $x = -2$.

11. **(a)** $f'(x) = \dfrac{4(x+1)}{3x^{2/3}}$, $f'(x) = 0$ when $x = -1$ (stationary point), $f'(x)$ does not exist when $x = 0$.

(b) $f'(x) = -3\sin 3x$, $f'(x) = 0$ when $\sin 3x = 0, 3x = n\pi, n = 0, \pm 1, \pm 2, \ldots$
$x = n\pi/3, n = 0, \pm 1, \pm 2, \ldots$ (stationary points)

12. **(a)** $f'(x) = \dfrac{4(x - 3/2)}{3x^{2/3}}$, $f'(x) = 0$ when $x = 3/2$ (stationary point), $f'(x)$ does not exist when $x = 0$.

(b) $f(x) = |\sin x| = \begin{cases} \sin x, & \sin x \ge 0 \\ -\sin x, & \sin x < 0 \end{cases}$ so $f'(x) = \begin{cases} \cos x, & \sin x > 0 \\ -\cos x, & \sin x < 0 \end{cases}$ and $f'(x)$ does not exist when $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$ (the points where $\sin x = 0$) because $\lim_{x \to n\pi^-} f'(x) \ne \lim_{x \to n\pi^+} f'(x)$ (see Theorem preceding Exercise 75, Section 3.3). Now $f'(x) = 0$ when $\pm\cos x = 0$ provided $\sin x \ne 0$ so $x = \pi/2 + n\pi$, $n = 0, \pm 1, \pm 2, \ldots$ are stationary points.

13. **(a)** none
(b) $x = 1$ because f' changes sign from $+$ to $-$ there
(c) none because $f'' = 0$ (never changes sign)

14. **(a)** $x = 1$ because $f'(x)$ changes sign from $-$ to $+$ there
(b) $x = 3$ because $f'(x)$ changes sign from $+$ to $-$ there
(c) $x = 2$ because $f''(x)$ changes sign there

15. **(a)** $x = 2$ because $f'(x)$ changes sign from $-$ to $+$ there.
(b) $x = 0$ because $f'(x)$ changes sign from $+$ to $-$ there.
(c) $x = 1, 3$ because $f''(x)$ changes sign at these points.

16. **(a)** $x = 1$ **(b)** $x = 5$ **(c)** $x = -1, 0, 3$

17. **(a)** critical numbers $x = 0, \pm\sqrt{5}$; f': $- \; - \; 0 \; + \; + \; 0 \; - \; - \; 0 \; + \; +$ (at $-\sqrt{5}$, 0, $\sqrt{5}$)

$x = 0$: relative maximum; $x = \pm\sqrt{5}$: relative minimum

(b) critical number $x = 1, -1$; f': $+ \; + \; + \; 0 \; - \; - \; - \; 0 \; + \; + \; +$ (at -1, 1)

$x = -1$: relative maximum; $x = 1$: relative minimum

18. **(a)** critical numbers $x = 0, -1/2, 1$; f': $+ \; + \; + \; 0 \; - \; 0 \; - \; - \; - \; 0 \; +$ (at $-\frac{1}{2}$, 0, 1)

$x = 0$: neither; $x = -1/2$: relative maximum; $x = 1$: relative minimum

(b) critical numbers: $x = \pm 3/2, -1$; f': $+ \; + \; 0 \; - \; - \; ? \; + \; + \; 0 \; - \; -$ (at $-\frac{3}{2}$, -1, $\frac{3}{2}$)

$x = \pm 3/2$: relative maximum; $x = -1$: relative minimum

19. $f'(x) = -2(x + 2)$; critical number $x = -2$; $f'(x)$: $+ \; + \; + \; 0 \; - \; - \; -$ (at -2)

$f''(x) = -2$; $f''(-2) < 0$, $f(-2) = 5$; relative maximum of 5 at $x = -2$

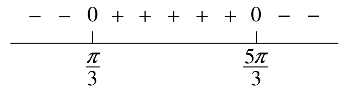
20. $f'(x) = 6(x - 2)(x - 1)$; critical numbers $x = 1, 2$; $f'(x)$: $+ \; + \; + \; 0 \; - \; - \; - \; 0 \; + \; + \; +$ (at 1, 2)

$f''(x) = 12x - 18$; $f''(1) < 0$, $f''(2) > 0$, $f(1) = 5$, $f(2) = 4$; relative minimum of 4 at $x = 2$, relative maximum of 5 at $x = 1$

21. $f'(x) = 2 \sin x \cos x = \sin 2x$;
critical numbers $x = \pi/2$, π, $3\pi/2$; $f'(x)$: $+ \; + \; 0 \; - \; - \; 0 \; + \; + \; 0 \; - \; -$ (at $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$)

$f''(x) = 2\cos 2x$; $f''(\pi/2) < 0$, $f''(\pi) > 0$, $f''(3\pi/2) < 0$, $f(\pi/2) = f(3\pi/2) = 1$,
$f(\pi) = 0$; relative minimum of 0 at $x = \pi$, relative maximum of 1 at $x = \pi/2$, $3\pi/2$

22. $f'(x) = 1/2 - \cos x$; critical numbers $x = \pi/3,\ 5\pi/3$; $f'(x)$:



$f''(x) = -\sin x$; $f''(\pi/3) < 0$, $f''(5\pi/3) > 0$
$f(\pi/3) = \pi/6 - \sqrt{3}/2$, $f(5\pi/3) = 5\pi/6 + \sqrt{3}/2$;
relative minimum of $\pi/6 - \sqrt{3}/2$ at $x = \pi/3$, relative maximum of $5\pi/6 + \sqrt{3}/2$ at $x = 5\pi/3$

23. $f'(x) = 3x^2 + 5$; no relative extrema because there are no critical numbers.

24. $f'(x) = 4x(x^2 - 1)$; critical numbers $x = 0, 1, -1$
$f''(x) = 12x^2 - 4$; $f''(0) < 0$, $f''(1) > 0$, $f''(-1) > 0$
relative minimum of 6 at $x = 1, -1$, relative maximum of 7 at $x = 0$

25. $f'(x) = (x - 1)(3x - 1)$; critical numbers $x = 1, 1/3$
$f''(x) = 6x - 4$; $f''(1) > 0$, $f''(1/3) < 0$
relative minimum of 0 at $x = 1$, relative maximum of $4/27$ at $x = 1/3$

26. $f'(x) = 2x^2(2x + 3)$; critical numbers $x = 0, -3/2$
relative minimum of $-27/16$ at $x = -3/2$ (first derivative test)

27. $f'(x) = 4x(1 - x^2)$; critical numbers $x = 0, 1, -1$
$f''(x) = 4 - 12x^2$; $f''(0) > 0$, $f''(1) < 0$, $f''(-1) < 0$
relative minimum of 0 at $x = 0$, relative maximum of 1 at $x = 1, -1$

28. $f'(x) = 10(2x - 1)^4$; critical number $x = 1/2$; no relative extrema (first derivative test)

29. $f'(x) = \frac{4}{5}x^{-1/5}$; critical number $x = 0$; relative minimum of 0 at $x = 0$ (first derivative test)

30. $f'(x) = 2 + \frac{2}{3}x^{-1/3}$; critical numbers $x = 0, -1/27$
relative minimum of 0 at $x = 0$, relative maximum of $1/27$ at $x = -1/27$

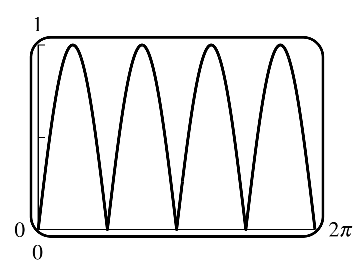
31. $f'(x) = 2x/(x^2 + 1)^2$; critical number $x = 0$; relative minimum of 0 at $x = 0$

32. $f'(x) = 2/(x + 2)^2$; no critical numbers ($x = -2$ is not in the domain of f) no relative extrema

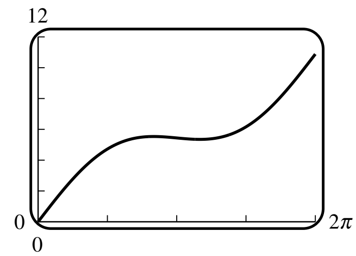
33. $f'(x) = 2x$ if $|x| > 2$, $f'(x) = -2x$ if $|x| < 2$,
$f'(x)$ does not exist when $x = \pm 2$; critical numbers $x = 0, 2, -2$
relative minimum of 0 at $x = 2, -2$, relative maximum of 4 at $x = 0$

34. $f'(x) = -1$ if $x < 3$, $f'(x) = 2x$ if $x > 3$, $f'(3)$ does not exist;
critical number $x = 3$, relative minimum of 6 at $x = 3$

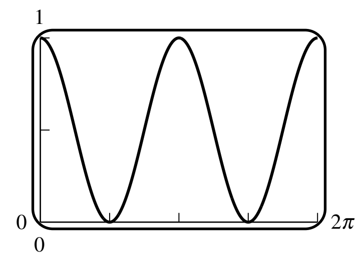
35. $f'(x) = 2\cos 2x$ if $\sin 2x > 0$,
$f'(x) = -2\cos 2x$ if $\sin 2x < 0$,
$f'(x)$ does not exist when $x = \pi/2, \pi, 3\pi/2$;
critical numbers $x = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4, \pi/2, \pi, 3\pi/2$
relative minimum of 0 at $x = \pi/2, \pi, 3\pi/2$;
relative maximum of 1 at $x = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$



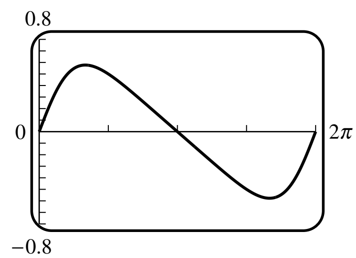
36. $f'(x) = \sqrt{3} + 2\cos x$; critical numbers $x = 5\pi/6, 7\pi/6$
relative minimum of $7\sqrt{3}\pi/6 - 1$ at $x = 7\pi/6$;
relative maximum of $5\sqrt{3}\pi/6 + 1$ at $x = 5\pi/6$



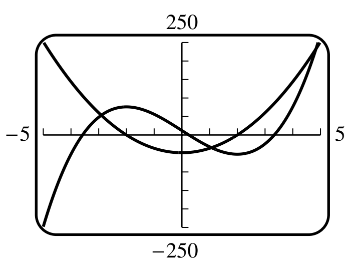
37. $f'(x) = -\sin 2x$; critical numbers $x = \pi/2, \pi, 3\pi/2$
relative minimum of 0 at $x = \pi/2, 3\pi/2$;
relative maximum of 1 at $x = \pi$



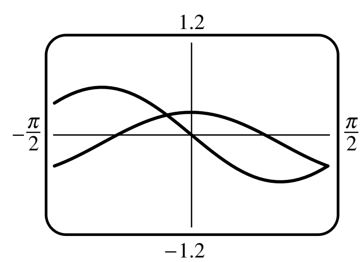
38. $f'(x) = (2\cos x - 1)/(2 - \cos x)^2$;
critical numbers $x = \pi/3, 5\pi/3$
relative maximum of $\sqrt{3}/3$ at $x = \pi/3$,
relative minimum of $-\sqrt{3}/3$ at $x = 5\pi/3$



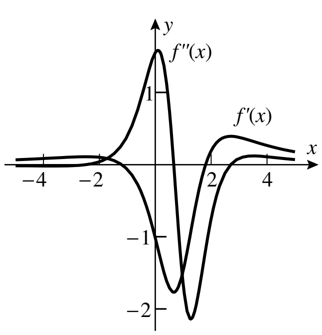
39. Relative minima at $x = -3.58, 3.33$;
relative maximum at $x = 0.25$



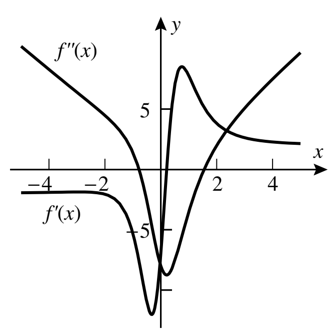
40. Relative minimum at $x = -0.84$;
relative maximum at $x = 0.84$



41. Relative minimum at $x = -1.20$ and
a relative maximum at $x = 1.80$



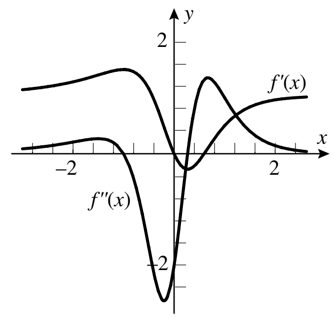
42. Relative maximum at $x = -0.78$ and
a relative minimum at $x = 1.55$



43. $f'(x) = \dfrac{x^4 + 3x^2 - 2x}{(x^2+1)^2}$

$f''(x) = -2\dfrac{x^3 - 3x^2 - 3x + 1}{(x^2+1)^3}$

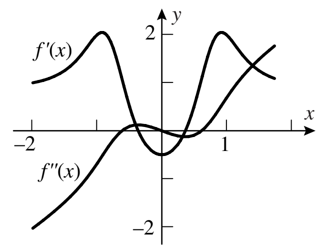
Relative maximum at $x = 0$,
relative minimum at $x \approx 0.59$



44. $f'(x) = \dfrac{4x^3 - \sin 2x}{2\sqrt{x^4 - \cos^2 x}}$,

$f''(x) = \dfrac{6x^2 - \cos 2x}{\sqrt{x^4 + \cos^2 x}} - \dfrac{(4x^3 - \sin 2x)(4x^3 - \sin 2x)}{4(x^4 + \cos^2 x)^{3/2}}$

Relative minima at $x \approx \pm 0.62$, relative maximum at $x = 0$

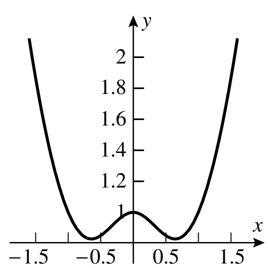


45. **(a)** Let $f(x) = x^2 + \dfrac{k}{x}$, then $f'(x) = 2x - \dfrac{k}{x^2} = \dfrac{2x^3 - k}{x^2}$. f has a relative extremum when $2x^3 - k = 0$, so $k = 2x^3 = 2(3)^3 = 54$.

(b) Let $f(x) = \dfrac{x}{x^2 + k}$, then $f'(x) = \dfrac{k - x^2}{(x^2+k)^2}$. f has a relative extremum when $k - x^2 = 0$, so $k = x^2 = 3^2 = 9$.

46. **(a)** relative minima at $x \approx \pm 0.6436$,
relative maximum at $x = 0$

(b) $x \approx \pm 0.6436, 0$



47. **(a)** $(-2.2, 4), (2, 2.2), (4.2, 3)$

(b) f' exists everywhere, so the critical numbers are when $f' = 0$, i.e. when $x = \pm 2$ or $r(x) = 0$, so $x \approx -5.1, -2, 0.2, 2$. At $x = -5.1$ f' changes sign from $-$ to $+$, so minimum; at $x = -2$ f' changes sign from $+$ to $-$, so maximum; at $x = 0.2$ f' doesn't change sign, so neither; at $x = 2$ f' changes sign from $-$ to $+$, so minimum.
Finally, $f''(1) = (1^2 - 4)r'(1) + 2r(1) \approx -3(0.6) + 2(0.3) = -1.2$.

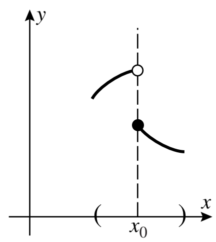
48. $g'(x)$ exists everywhere, so the critical points are the points when $g'(x) = 0$, or $r(x) = x$, so $r(x)$ crosses the line $y = x$. From the graph it appears that this happens precisely when $x = 0$.

49. $f'(x) = 3ax^2 + 2bx + c$ and $f'(x)$ has roots at $x = 0, 1$, so $f'(x)$ must be of the form $f'(x) = 3ax(x-1)$; thus $c = 0$ and $2b = -3a$, $b = -3a/2$. $f''(x) = 6ax + 2b = 6ax - 3a$, so $f''(0) > 0$ and $f''(1) < 0$ provided $a < 0$. Finally $f(0) = d$, so $d = 0$; and $f(1) = a + b + c + d = a + b = -a/2$ so $a = -2$. Thus $f(x) = -2x^3 + 3x^2$.

50. **(a)** Because h and g have relative maxima at x_0, $h(x) \le h(x_0)$ for all x in I_1 and $g(x) \le g(x_0)$ for all x in I_2, where I_1 and I_2 are open intervals containing x_0. If x is in both I_1 and I_2 then both inequalities are true and by addition so is $h(x)+g(x) \le h(x_0)+g(x_0)$ which shows that $h+g$ has a relative maximum at x_0.

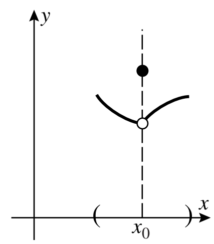
(b) By counterexample; both $h(x) = -x^2$ and $g(x) = -2x^2$ have relative maxima at $x = 0$ but $h(x) - g(x) = x^2$ has a relative minimum at $x = 0$ so in general $h - g$ does not necessarily have a relative maximum at x_0.

51. **(a)**



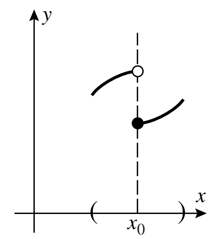
$f(x_0)$ is not an extreme value.

(b)



$f(x_0)$ is a relative maximum.

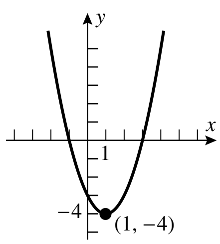
(c)



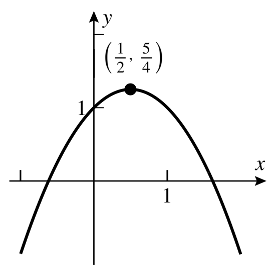
$f(x_0)$ is a relative minimum.

EXERCISE SET 4.3

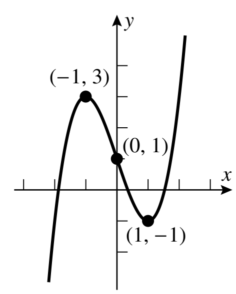
1. $y = x^2 - 2x - 3$;
$y' = 2(x - 1)$;
$y'' = 2$



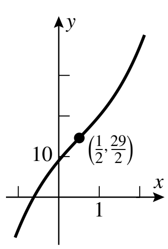
2. $y = 1 + x - x^2$;
$y' = -2(x - 1/2)$;
$y'' = -2$



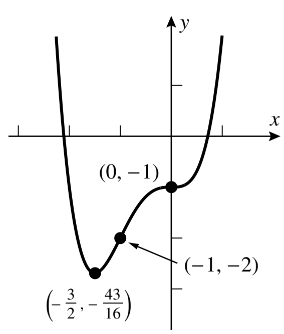
3. $y = x^3 - 3x + 1$;
$y' = 3(x^2 - 1)$;
$y'' = 6x$



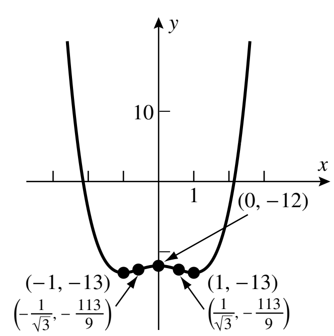
4. $y = 2x^3 - 3x^2 + 12x + 9$;
$y' = 6(x^2 - x + 2)$;
$y'' = 12(x - 1/2)$



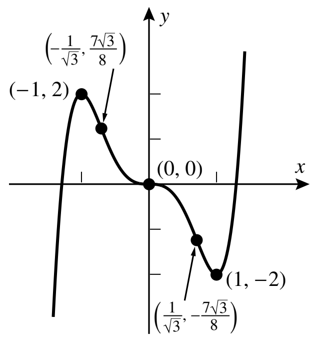
5. $y = x^4 + 2x^3 - 1$;
$y' = 4x^2(x + 3/2)$;
$y'' = 12x(x + 1)$



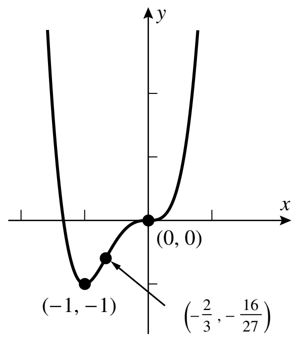
6. $y = x^4 - 2x^2 - 12$;
$y' = 4x(x^2 - 1)$;
$y'' = 12(x^2 - 1/3)$



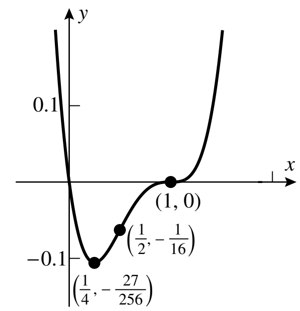
7. $y = x^3(3x^2 - 5);$
$y' = 15x^2(x^2 - 1);$
$y'' = 30x(2x^2 - 1)$



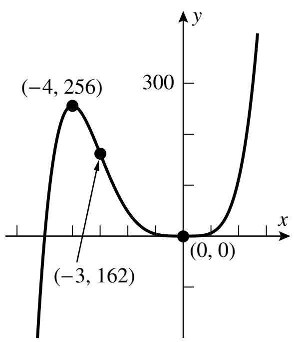
8. $y = 3x^3(x + 4/3);$
$y' = 12x^2(x + 1);$
$y'' = 36x(x + 2/3)$



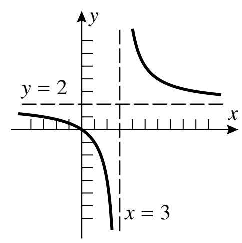
9. $y = x(x - 1)^3;$
$y' = (4x - 1)(x - 1)^2;$
$y'' = 6(2x - 1)(x - 1)$



10. $y = x^4(x + 5);$
$y' = 5x^3(x + 4);$
$y'' = 20x^2(x + 3)$



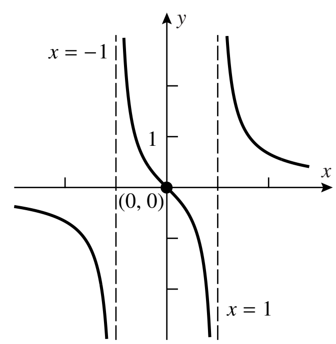
11. $y = 2x/(x - 3);$
$y' = -6/(x - 3)^2;$
$y'' = 12/(x - 3)^3$



12. $y = \dfrac{x}{x^2 - 1};$

$y' = -\dfrac{x^2 + 1}{(x^2 - 1)^2};$

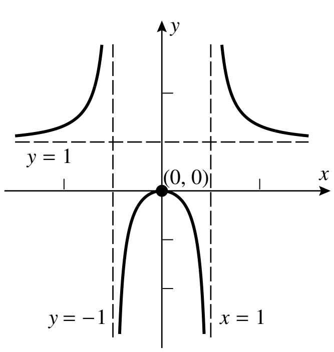
$y'' = \dfrac{2x(x^2 + 3)}{(x^2 - 1)^3}$



13. $y = \dfrac{x^2}{x^2 - 1};$

$y' = -\dfrac{2x}{(x^2 - 1)^2};$

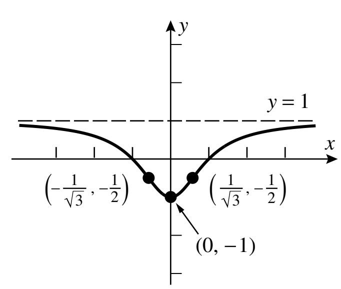
$y'' = \dfrac{2(3x^2 + 1)}{(x^2 - 1)^3}$



14. $y = \dfrac{x^2 - 1}{x^2 + 1};$

$y' = \dfrac{4x}{(x^2 + 1)^2};$

$y'' = \dfrac{4(1 - 3x^2)}{(x^2 + 1)^3}$

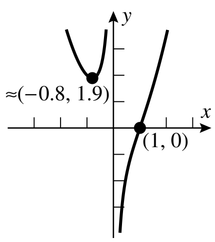


15. $y = x^2 - \dfrac{1}{x} = \dfrac{x^3 - 1}{x};$

$y' = \dfrac{2x^3 + 1}{x^2},$

$y' = 0$ when $x = -\sqrt[3]{\dfrac{1}{2}} \approx -0.8;$

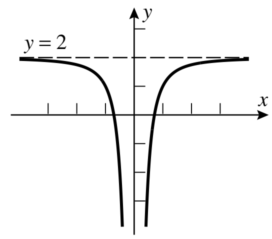
$y'' = \dfrac{2(x^3 - 1)}{x^3}$



16. $y = \dfrac{2x^2 - 1}{x^2};$

$y' = \dfrac{2}{x^3};$

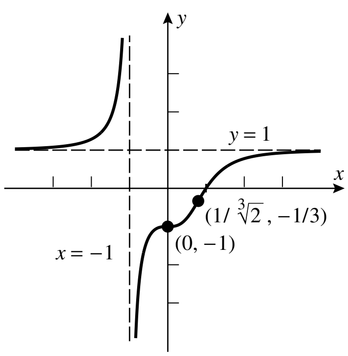
$y'' = -\dfrac{6}{x^4}$



17. $y = \dfrac{x^3 - 1}{x^3 + 1};$

$y' = \dfrac{6x^2}{(x^3 + 1)^2};$

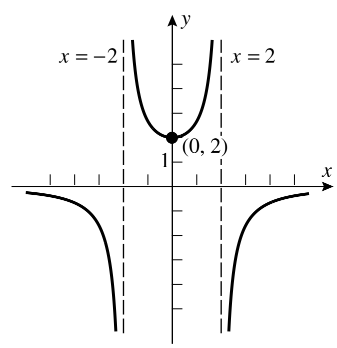
$y'' = \dfrac{12x(1 - 2x^3)}{(x^3 + 1)^3}$



18. $y = \dfrac{8}{4 - x^2};$

$y' = \dfrac{16x}{(4 - x^2)^2};$

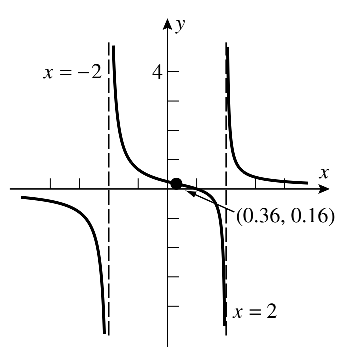
$y'' = \dfrac{16(3x^2 + 4)}{(4 - x^2)^3}$



19. $y = \dfrac{x - 1}{x^2 - 4};$

$y' = -\dfrac{x^2 - 2x + 4}{(x^2 - 4)^2}$

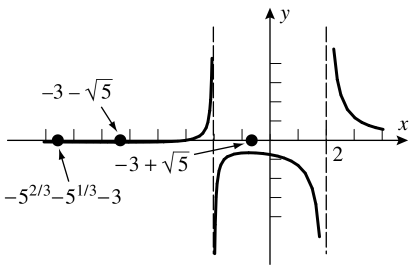
$y'' = 2\dfrac{x^3 - 3x^2 + 12x - 4}{(x^2 - 4)^3}$



20. $y = \dfrac{x+3}{x^2-4};$

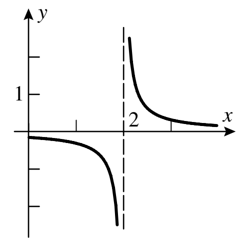
$y' = -\dfrac{(x^2+6x+4)}{(x^2-4)^2}$

$y'' = 2\dfrac{x^3+9x^2+12x+12}{(x^2-4)^3}$



21. $y = \dfrac{1}{x-2};$

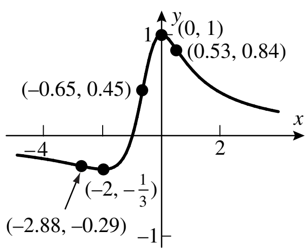
$y' = \dfrac{-1}{(x-2)^2}$



22. $y = \dfrac{x^2-1}{x^3-1} = \dfrac{x+1}{x^2+x+1};$

$y' = -\dfrac{x(x+2)}{(x^2+x+1)^2}$

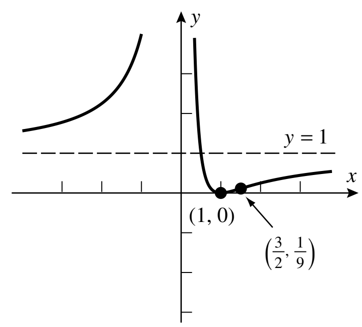
$y'' = 2\dfrac{x^3+3x^2-1}{(x^2+x+1)^3}$



23. $y = \dfrac{(x-1)^2}{x^2};$

$y' = \dfrac{2(x-1)}{x^3};$

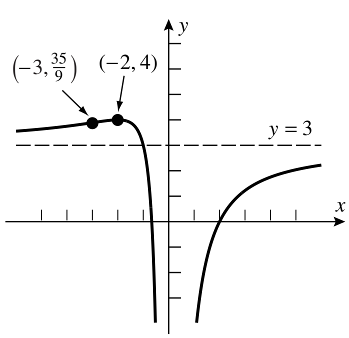
$y'' = \dfrac{2(3-2x)}{x^4}$



24. $y = 3 - \dfrac{4}{x} - \dfrac{4}{x^2};$

$y' = \dfrac{4(x+2)}{x^3};$

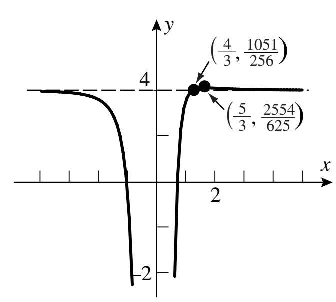
$y'' = -\dfrac{8(x+3)}{x^4}$



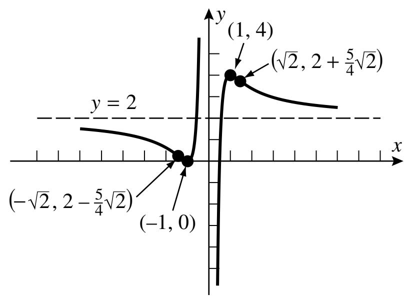
25. $y = 4 + \dfrac{x-1}{x^4};$

$y' = -\dfrac{3x-4}{x^5};$

$y'' = 4\dfrac{3x-5}{x^6}$

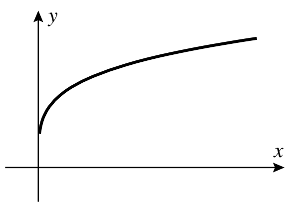


26. $y = 2 + \frac{3}{x} - \frac{1}{x^3}$;
$y' = \frac{3(1 - x^2)}{x^4}$;
$y'' = \frac{6(x^2 - 2)}{x^5}$

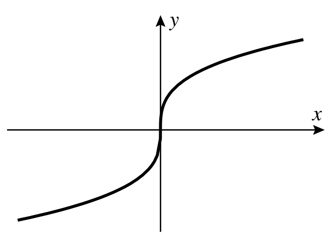


27. **(a)** VI **(b)** I **(c)** III **(d)** V **(e)** IV **(f)** II

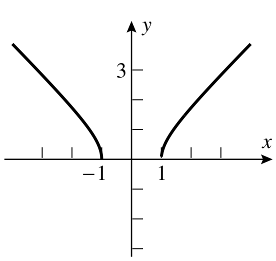
28. **(a)** When n is even the function is defined only for $x \geq 0$; as n increases the graph approaches the line $y = 1$ for $x > 0$.



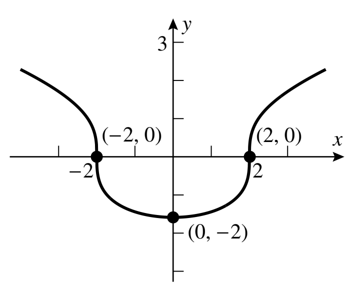
(b) When n is odd the graph is symmetric with respect to the origin; as n increases the graph approaches the line $y = 1$ for $x > 0$ and the line $y = -1$ for $x < 0$.



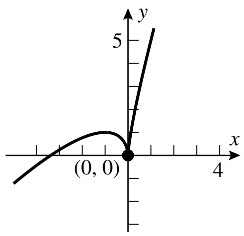
29. $y = \sqrt{x^2 - 1}$;
$y' = \frac{x}{\sqrt{x^2 - 1}}$;
$y'' = -\frac{1}{(x^2 - 1)^{3/2}}$



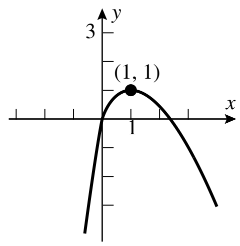
30. $y = \sqrt[3]{x^2 - 4}$;
$y' = \frac{2x}{3(x^2 - 4)^{2/3}}$;
$y'' = -\frac{2(3x^2 + 4)}{9(x^2 - 4)^{5/3}}$



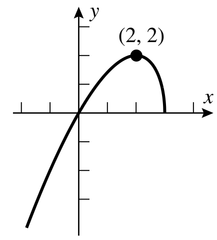
31. $y = 2x + 3x^{2/3}$;
$y' = 2 + 2x^{-1/3}$;
$y'' = -\dfrac{2}{3}x^{-4/3}$



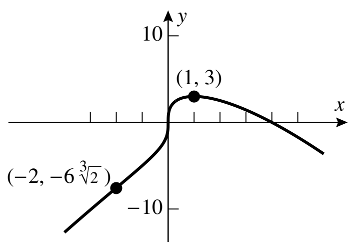
32. $y = 4x - 3x^{4/3}$;
$y' = 4 - 4x^{1/3}$;
$y'' = -\dfrac{4}{3}x^{-2/3}$



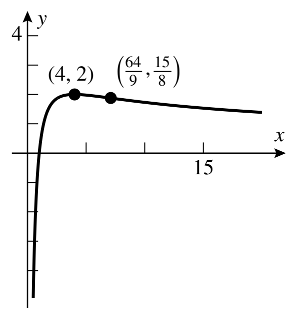
33. $y = x(3 - x)^{1/2}$;
$y' = \dfrac{3(2 - x)}{2\sqrt{3 - x}}$;
$y'' = \dfrac{3(x - 4)}{4(3 - x)^{3/2}}$



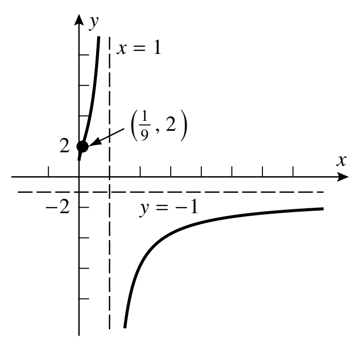
34. $y = x^{1/3}(4 - x)$;
$y' = \dfrac{4(1 - x)}{3x^{2/3}}$;
$y'' = -\dfrac{4(x + 2)}{9x^{5/3}}$



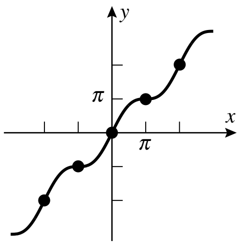
35. $y = \dfrac{8(\sqrt{x} - 1)}{x}$;
$y' = \dfrac{4(2 - \sqrt{x})}{x^2}$;
$y'' = \dfrac{2(3\sqrt{x} - 8)}{x^3}$



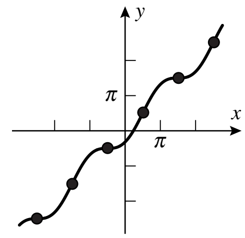
36. $y = \dfrac{1 + \sqrt{x}}{1 - \sqrt{x}}$;
$y' = \dfrac{1}{2\sqrt{x}(1-\sqrt{x})}$;
$y'' = \dfrac{3\sqrt{x} - 1}{2x^{3/2}(1 - \sqrt{x})^3}$



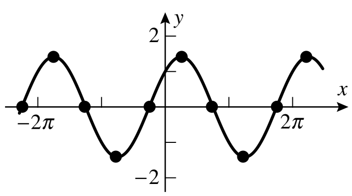
37. $y = x + \sin x$;
$y' = 1 + \cos x$, $y' = 0$ when $x = \pi + 2n\pi$;
$y'' = -\sin x$; $y'' = 0$ when $x = n\pi$
$n = 0, \pm 1, \pm 2, \ldots$



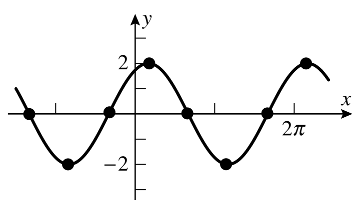
38. $y = x - \cos x$;
$y' = 1 + \sin x$;
$y' = 0$ when $x = -\pi/2 + 2n\pi$;
$y'' = \cos x$;
$y'' = 0$ when $x = \pi/2 + n\pi$
$n = 0, \pm 1, \pm 2, \ldots$



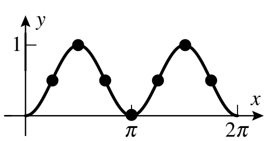
39. $y = \sin x + \cos x$;
$y' = \cos x - \sin x$;
$y' = 0$ when $x = \pi/4 + n\pi$;
$y'' = -\sin x - \cos x$;
$y'' = 0$ when $x = 3\pi/4 + n\pi$



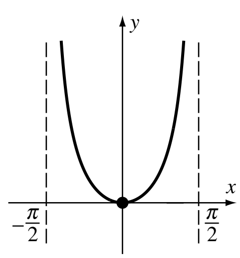
40. $y = \sqrt{3}\cos x + \sin x$;
$y' = -\sqrt{3}\sin x + \cos x$;
$y' = 0$ when $x = \pi/6 + n\pi$;
$y'' = -\sqrt{3}\cos x - \sin x$;
$y'' = 0$ when $x = 2\pi/3 + n\pi$



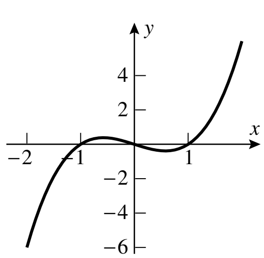
41. $y = \sin^2 x,\ 0 \le x \le 2\pi$;
$y' = 2\sin x \cos x = \sin 2x$;
$y'' = 2\cos 2x$



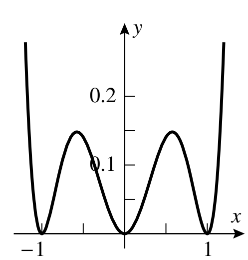
42. $y = x\tan x,\ -\pi/2 < x < \pi/2$;
$y' = x\sec^2 x + \tan x$;
$y' = 0$ when $x = 0$;
$y'' = 2\sec^2 x(x\tan x + 1)$, which is always positive for $-\pi/2 < x < \pi/2$



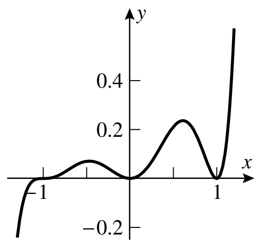
43. **(a)** $\lim_{x\to-\infty} y = -\infty$, $\lim_{x\to+\infty} y = +\infty$;
curve crosses x-axis at $x = 0, 1, -1$



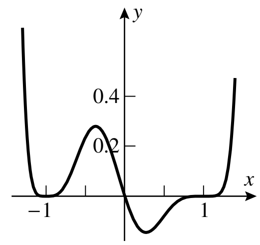
(b) $\lim_{x\to\pm\infty} y = +\infty$;
curve never crosses x-axis



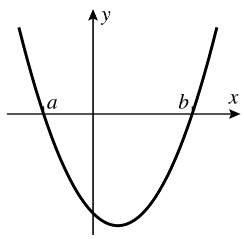
(c) $\lim_{x\to-\infty} y = -\infty$, $\lim_{x\to+\infty} y = +\infty$;
curve crosses x-axis at $x = -1$

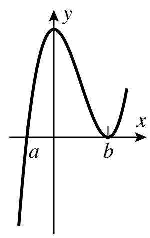


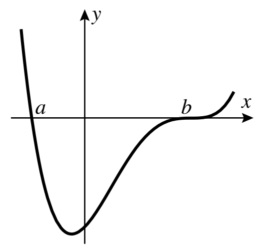
(d) $\lim_{x\to\pm\infty} y = +\infty$;
curve crosses x-axis at $x = 0, 1$



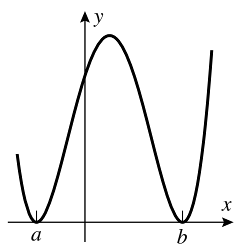
44. **(a)**

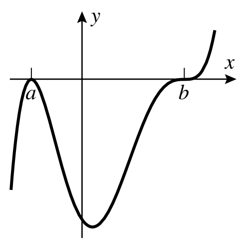




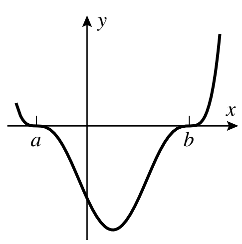


(b)

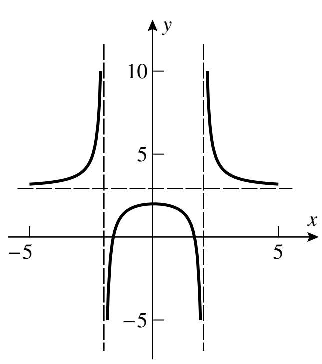




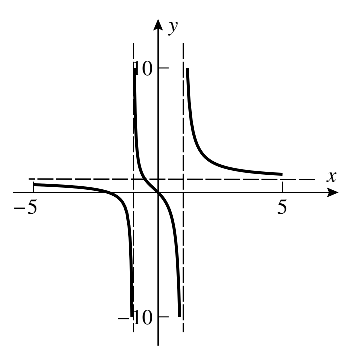
(c)



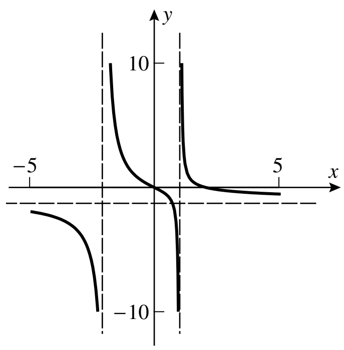
45. **(a)** horizontal asymptote $y = 3$ as $x \to \pm\infty$, vertical asymptotes at $x = \pm 2$



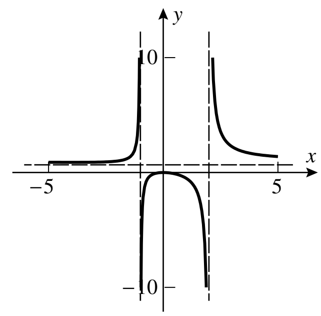
(b) horizontal asymptote $y = 1$ as $x \to \pm\infty$, vertical asymptotes at $x = \pm 1$



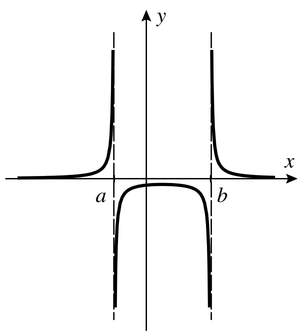
(c) horizontal asymptote $y = -1$ as $x \to \pm\infty$, vertical asymptotes at $x = -2, 1$



(d) horizontal asymptote $y = 1$ as $x \to \pm\infty$, vertical asymptote at $x = -1, 2$



46.



47. Symmetric about the line $x = \dfrac{a+b}{2}$ means $f\left(\dfrac{a+b}{2} + x\right) = f\left(\dfrac{a+b}{2} - x\right)$ for any x.

Note that $\dfrac{a+b}{2} + x - a = x - \dfrac{a-b}{2}$ and $\dfrac{a+b}{2} + x - b = x + \dfrac{a-b}{2}$, and the same equations are true with x replaced by $-x$. Hence

$$\left(\frac{a+b}{2} + x - a\right)\left(\frac{a+b}{2} + x - b\right) = \left(x - \frac{a-b}{2}\right)\left(x + \frac{a-b}{2}\right) = x^2 - \left(\frac{a-b}{2}\right)^2$$

The right hand side remains the same if we replace x with $-x$, and so the same is true of the left hand side, and the same is therefore true of the reciprocal of the left hand side. But the reciprocal of the left hand side is equal to $f\left(\dfrac{a+b}{2} + x\right)$. Since this quantity remains unchanged if we replace x with $-x$, the condition of symmetry about the line $x = \dfrac{a+b}{2}$ has been met.

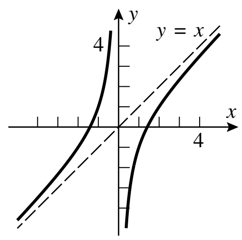
48. $\displaystyle\lim_{x\to\pm\infty} \left[\frac{P(x)}{Q(x)} - (ax+b)\right] = \lim_{x\to\pm\infty} \frac{R(x)}{Q(x)} = 0$ because the degree of $R(x)$ is less than the degree of $Q(x)$.

49. $y = \dfrac{x^2 - 2}{x} = x - \dfrac{2}{x}$ so

$y = x$ is an oblique asymptote;

$y' = \dfrac{x^2 + 2}{x^2}$,

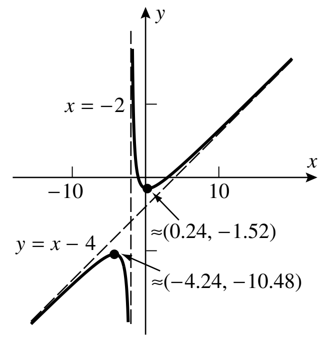
$y'' = -\dfrac{4}{x^3}$



50. $y = \dfrac{x^2 - 2x - 3}{x + 2} = x - 4 + \dfrac{5}{x + 2}$ so

$y = x - 4$ is an oblique asymptote;

$y' = \dfrac{x^2 + 4x - 1}{(x + 2)^2}$, $y'' = \dfrac{10}{(x + 2)^3}$

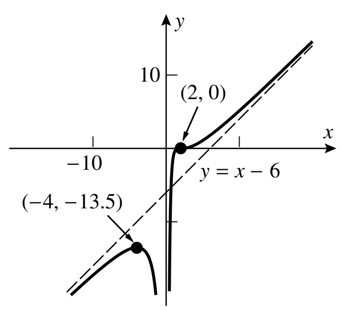


51. $y = \dfrac{(x - 2)^3}{x^2} = x - 6 + \dfrac{12x - 8}{x^2}$ so

$y = x - 6$ is an oblique asymptote;

$y' = \dfrac{(x - 2)^2(x + 4)}{x^3}$,

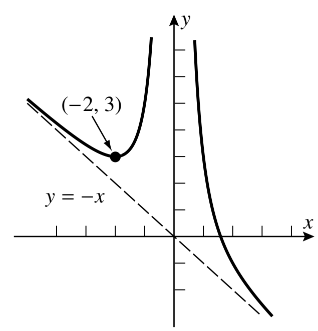
$y'' = \dfrac{24(x - 2)}{x^4}$



52. $y = \dfrac{4 - x^3}{x^2}$,

$y' = -\dfrac{x^3 + 8}{x^3}$,

$y'' = \dfrac{24}{x^4}$

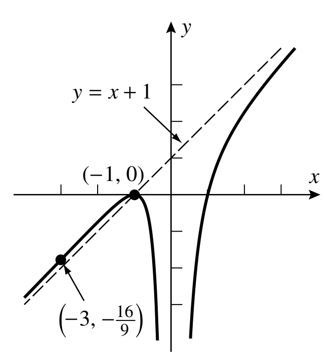


53. $y = x + 1 - \dfrac{1}{x} - \dfrac{1}{x^2} = \dfrac{(x - 1)(x + 1)^2}{x^2}$,

$y = x + 1$ is an oblique asymptote;

$y' = \dfrac{(x + 1)(x^2 - x + 2)}{x^3}$,

$y'' = -\dfrac{2(x + 3)}{x^4}$



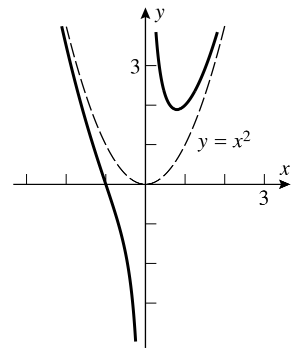
54. The oblique asymptote is $y = 2x$ so $(2x^3 - 3x + 4)/x^2 = 2x$, $-3x + 4 = 0$, $x = 4/3$.

55. $\lim_{x\to\pm\infty}[f(x) - x^2] = \lim_{x\to\pm\infty}(1/x) = 0$

$y = x^2 + \dfrac{1}{x} = \dfrac{x^3+1}{x}$, $y' = 2x - \dfrac{1}{x^2} = \dfrac{2x^3-1}{x^2}$,

$y'' = 2 + \dfrac{2}{x^3} = \dfrac{2(x^3+1)}{x^3}$, $y' = 0$ when $x = 1/\sqrt[3]{2} \approx 0.8$,

$y = 3\sqrt[3]{2}/2 \approx 1.9$; $y'' = 0$ when $x = -1, y = 0$

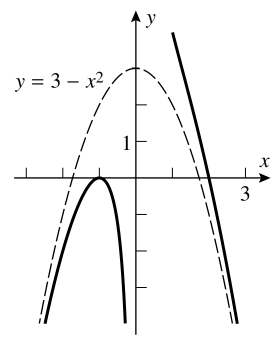


56. $\lim_{x\to\pm\infty}[f(x) - (3 - x^2)] = \lim_{x\to\pm\infty}(2/x) = 0$

$y = 3 - x^2 + \dfrac{2}{x} = \dfrac{2+3x-x^3}{x}$, $y' = -2x - \dfrac{2}{x^2} = -\dfrac{2(x^3+1)}{x^2}$,

$y'' = -2 + \dfrac{4}{x^3} = -\dfrac{2(x^3-2)}{x^3}$, $y' = 0$ when $x = -1, y = 0$;

$y'' = 0$ when $x = \sqrt[3]{2} \approx 1.3$, $y = 3$



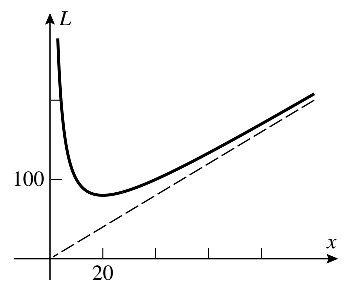
57. Let y be the length of the other side of the rectangle, then $L = 2x+2y$ and $xy = 400$ so $y = 400/x$ and hence $L = 2x + 800/x$. $L = 2x$ is an oblique asymptote (see Exercise 48)

$L = 2x + \dfrac{800}{x} = \dfrac{2(x^2+400)}{x}$,

$L' = 2 - \dfrac{800}{x^2} = \dfrac{2(x^2-400)}{x^2}$,

$L'' = \dfrac{1600}{x^3}$,

$L' = 0$ when $x = 20, L = 80$



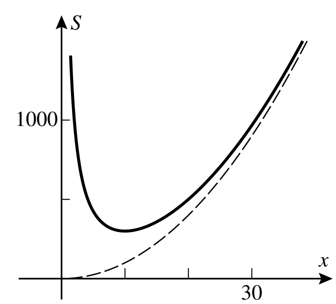
58. Let y be the height of the box, then $S = x^2 + 4xy$ and $x^2y = 500$ so $y = 500/x^2$ and hence $S = x^2 + 2000/x$. The graph approaches the curve $S = x^2$ asymptotically (see Exercise 63)

$S = x^2 + \dfrac{2000}{x} = \dfrac{x^3+2000}{x}$,

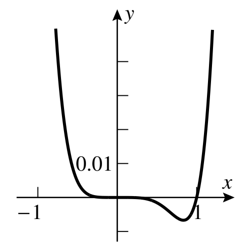
$S' = 2x - \dfrac{2000}{x^2} = \dfrac{2(x^3-1000)}{x^2}$,

$S'' = 2 + \dfrac{4000}{x^3} = \dfrac{2(x^3+2000)}{x^3}$,

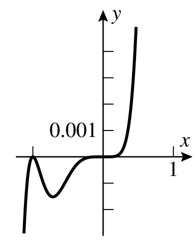
$S'' = 0$ when $x = 10, S = 300$



59. $y' = 0.1x^4(6x-5)$;
critical numbers: $x = 0$, $x = 5/6$;
relative minimum at $x = 5/6$,
$y \approx -6.7 \times 10^{-3}$



60. $y' = 0.1x^4(x+1)(7x+5)$;
critical numbers: $x = 0$, $x = -1$, $x = -5/7$,
relative maximum at $x = -1$, $y = 0$;
relative minimum at $x = -5/7$, $y \approx -1.5 \times 10^{-3}$



EXERCISE SET 4.4

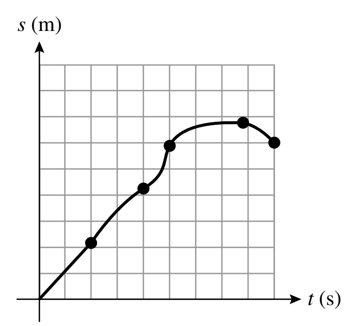
1. **(a)** positive, negative, slowing down
(b) positive, positive, speeding up
(c) negative, positive, slowing down

2. **(a)** positive, slowing down
(b) negative, slowing down
(c) positive, speeding up

3. **(a)** left because $v = ds/dt < 0$ at t_0
(b) negative because $a = d^2s/dt^2$ and the curve is concave down at $t_0(d^2s/dt^2 < 0)$
(c) speeding up because v and a have the same sign
(d) $v < 0$ and $a > 0$ at t_1 so the particle is slowing down because v and a have opposite signs.

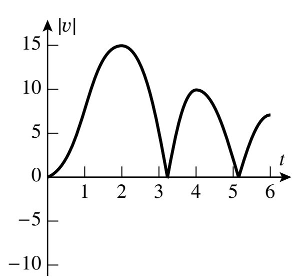
4. **(a)** III **(b)** I **(c)** II

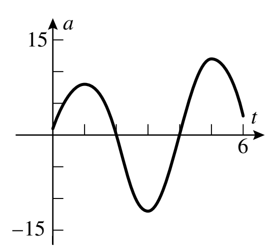
5.



6. **(a)** when $s \geq 0$, so $0 < t < 2$ and $4 < t \leq 8$ **(b)** when the slope is zero, at $t = 3$
(c) when s is decreasing, so $0 \leq t < 3$

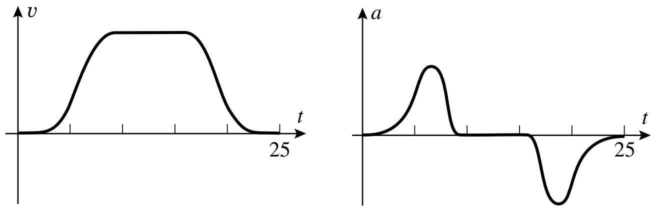
7.





8. **(a)** $v \approx (30 - 10)/(15 - 10) = 20/5 = 4$ m/s

(b)



(1) (2)

9. **(a)** At 60 mi/h the slope of the estimated tangent line is about 4.6 mi/h/s. Use 1 mi $= 5,280$ ft and 1 h $= 3600$ s to get $a = dv/dt \approx 4.6(5{,}280)/(3600) \approx 6.7$ ft/s^2.

(b) The slope of the tangent to the curve is maximum at $t = 0$ s.

10. **(a)**

t	1	2	3	4	5
s	0.71	1.00	0.71	0.00	−0.71
v	0.56	0.00	−0.56	−0.79	−0.56
a	−0.44	−0.62	−0.44	0.00	0.44

(b) to the right at $t = 1$, stopped at $t = 2$, otherwise to the left

(c) speeding up at $t = 3$; slowing down at $t = 1, 5$; neither at $t = 2, 4$

11. **(a)** $v(t) = 3t^2 - 12t$, $a(t) = 6t - 12$

(b) $s(1) = -5$ ft, $v(1) = -9$ ft/s, speed $= 9$ ft/s, $a(1) = -6$ ft/s^2

(c) $v = 0$ at $t = 0, 4$

(d) for $t \geq 0$, $v(t)$ changes sign at $t = 4$, and $a(t)$ changes sign at $t = 2$; so the particle is speeding up for $0 < t < 2$ and $4 < t$ and is slowing down for $2 < t < 4$

(e) total distance $= |s(4) - s(0)| + |s(5) - s(4)| = |-32 - 0| + |-25 - (-32)| = 39$ ft

12. **(a)** $v(t) = 4t^3 - 4$, $a(t) = 12t^2$

(b) $s(1) = -1$ ft, $v(1) = 0$ ft/s, speed $= 0$ ft/s, $a(1) = 12$ ft/s^2

(c) $v = 0$ at $t = 1$

(d) speeding up for $t > 1$, slowing down for $0 < t < 1$

(e) total distance $= |s(1) - s(0)| + |s(5) - s(1)| = |-1 - 2| + |607 - (-1)| = 611$ ft

13. **(a)** $v(t) = -(3\pi/2)\sin(\pi t/2)$, $a(t) = -(3\pi^2/4)\cos(\pi t/2)$

(b) $s(1) = 0$ ft, $v(1) = -3\pi/2$ ft/s, speed $= 3\pi/2$ ft/s, $a(1) = 0$ ft/s^2

(c) $v = 0$ at $t = 0, 2, 4$

(d) v changes sign at $t = 0, 2, 4$ and a changes sign at $t = 1, 3, 5$, so the particle is speeding up for $0 < t < 1$, $2 < t < 3$ and $4 < t < 5$, and it is slowing down for $1 < t < 2$ and $3 < t < 4$

(e)
$$\begin{aligned}\text{total distance } &= |s(2) - s(0)| + |s(4) - s(2)| + |s(5) - s(4)| \\ &= |-3 - 3| + |3 - (-3)| + |0 - 3| = 15 \text{ ft}\end{aligned}$$

14. **(a)** $v(t) = \dfrac{4 - t^2}{(t^2 + 4)^2}$, $a(t) = \dfrac{2t(t^2 - 12)}{(t^2 + 4)^3}$

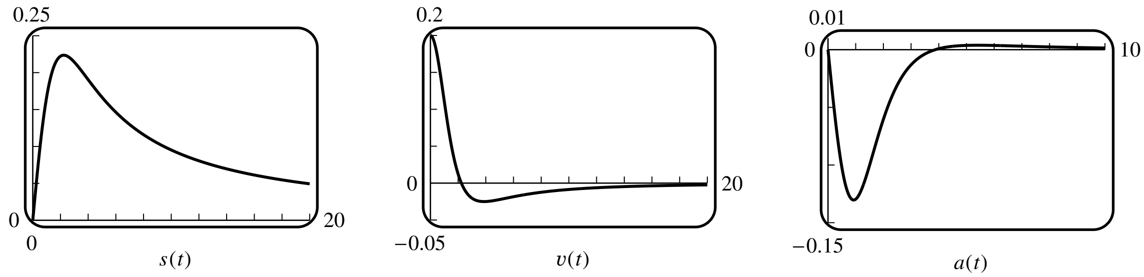
(b) $s(1) = 1/5$ ft, $v(1) = 3/25$ ft/s, speed $= 3/25$ ft/s, $a(1) = -22/125$ ft/s^2

(c) $v = 0$ at $t = 2$

(d) a changes sign at $t = 2\sqrt{3}$, so the particle is speeding up for $2 < t < 2\sqrt{3}$ and it is slowing down for $0 < t < 2$ and for $2\sqrt{3} < t$

(e) total distance $= |s(2) - s(0)| + |s(5) - s(2)| = \left|\dfrac{1}{4} - 0\right| + \left|\dfrac{5}{29} - \dfrac{1}{4}\right| = \dfrac{19}{58}$ ft

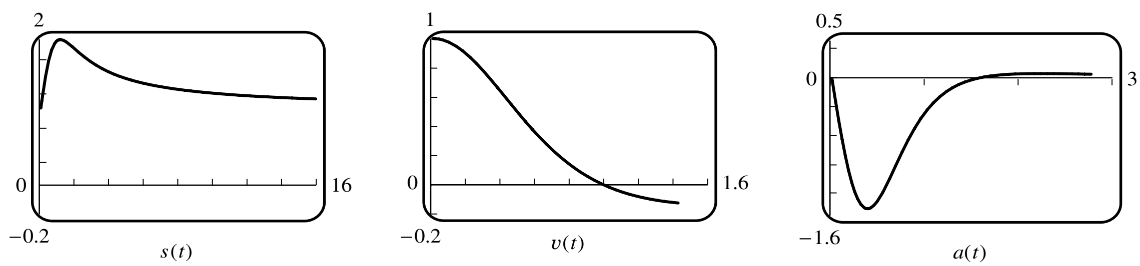
15. $v(t) = \dfrac{5 - t^2}{(t^2 + 5)^2}$, $a(t) = \dfrac{2t(t^2 - 15)}{(t^2 + 5)^3}$



(a) $v = 0$ at $t = \sqrt{5}$ **(b)** $s = \sqrt{5}/10$ at $t = \sqrt{5}$

(c) a changes sign at $t = \sqrt{15}$, so the particle is speeding up for $\sqrt{5} < t < \sqrt{15}$ and slowing down for $0 < t < \sqrt{5}$ and $\sqrt{15} < t$

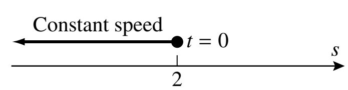
16. $v(t) = \dfrac{-t^2 + 1}{(t^2 + 1)^2}$, $a(t) = 2\dfrac{t(t^3 - 3)}{(t^2 + 1)^3}$



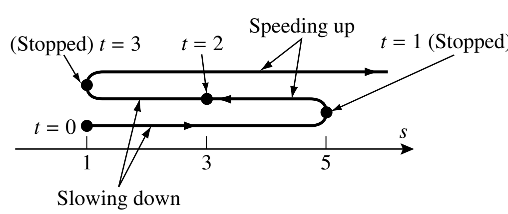
(a) $v = 0$ at $t = 1$ **(b)** $s = 3/2$ at $t = 1$

(c) a changes sign at $t = 3^{1/3}$, so the particle is slowing down for $t < 3^{1/3}$ and speeding up for $3^{1/3} < t$.

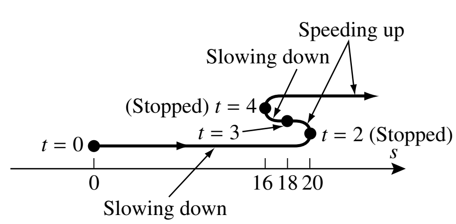
17. $s = -3t + 2$
$v = -3$
$a = 0$



18. $s = t^3 - 6t^2 + 9t + 1$
$v = 3(t - 1)(t - 3)$
$a = 6(t - 2)$



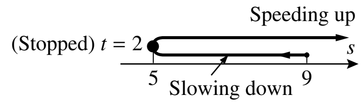
19. $s = t^3 - 9t^2 + 24t$
$v = 3(t - 2)(t - 4)$
$a = 6(t - 3)$



20. $s = t + \dfrac{9}{t+1}$

$v = \dfrac{(t+4)(t-2)}{(t+1)^2}$

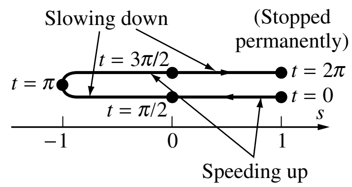
$a = \dfrac{18}{(t+1)^3}$



21. $s = \begin{cases} \cos t, & 0 \le t \le 2\pi \\ 1, & t > 2\pi \end{cases}$

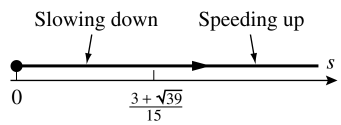
$v = \begin{cases} -\sin t, & 0 \le t \le 2\pi \\ 0, & t > 2\pi \end{cases}$

$a = \begin{cases} -\cos t, & 0 \le t < 2\pi \\ 0, & t > 2\pi \end{cases}$



22. $v(t) = \dfrac{5t^2 - 6t + 2}{\sqrt{t}}$ is always positive,

$a(t) = \dfrac{15t^2 - 6t - 2}{2t^{3/2}}$ has a positive root at $t = \dfrac{3+\sqrt{39}}{15}$



23. **(a)** $v = 10t - 22$, speed $= |v| = |10t - 22|$. $d|v|/dt$ does not exist at $t = 2.2$ which is the only critical point. If $t = 1, 2.2, 3$ then $|v| = 12, 0, 8$. The maximum speed is 12 ft/s.

(b) the distance from the origin is $|s| = |5t^2 - 22t| = |t(5t - 22)|$, but $t(5t - 22) < 0$ for $1 \le t \le 3$ so $|s| = -(5t^2 - 22t) = 22t - 5t^2$, $d|s|/dt = 22 - 10t$, thus the only critical point is $t = 2.2$. $d^2|s|/dt^2 < 0$ so the particle is farthest from the origin when $t = 2.2$. Its position is $s = 5(2.2)^2 - 22(2.2) = -24.2$.

24. $v = -\dfrac{200t}{(t^2+12)^2}$, speed $= |v| = \dfrac{200t}{(t^2+12)^2}$ for $t \ge 0$. $\dfrac{d|v|}{dt} = \dfrac{600(4-t^2)}{(t^2+12)^3} = 0$ when $t = 2$, which is the only critical point in $(0, +\infty)$. By the first derivative test there is a relative maximum, and hence an absolute maximum, at $t = 2$. The maximum speed is 25/16 ft/s to the left.

25. $s(t) = s_0 - \frac{1}{2}gt^2 = s_0 - 4.9t^2$ m, $v = -9.8t$ m/s, $a = -9.8$ m/s^2

(a) $|s(1.5) - s(0)| = 11.025$ m

(b) $v(1.5) = -14.7$ m/s

(c) $|v(t)| = 12$ when $t = 12/9.8 = 1.2245$ s

(d) $s(t) - s_0 = -100$ when $4.9t^2 = 100$, $t = 4.5175$ s

26. **(a)** $s(t) = s_0 - \frac{1}{2}gt^2 = 800 - 16t^2$ ft, $s(t) = 0$ when $t = \sqrt{\dfrac{800}{16}} = 5\sqrt{2}$

(b) $v(t) = -32t$ and $v(5\sqrt{2}) = -160\sqrt{2} \approx 226.27$ ft/s $= 154.28$ mi/h

27. $s(t) = s_0 + v_0 t - \frac{1}{2}gt^2 = 60t - 4.9t^2$ m and $v(t) = v_0 - gt = 60 - 9.8t$ m/s

(a) $v(t) = 0$ when $t = 60/9.8 \approx 6.12$ s

(b) $s(60/9.8) \approx 183.67$ m

(c) another 6.12 s; solve for t in $s(t) = 0$ to get this result, or use the symmetry of the parabola $s = 60t - 4.9t^2$ about the line $t = 6.12$ in the t-s plane

(d) also 60 m/s, as seen from the symmetry of the parabola (or compute $v(6.12)$)

28. **(a)** they are the same

(b) $s(t) = v_0t - \frac{1}{2}gt^2$ and $v(t) = v_0 - gt$; $s(t) = 0$ when $t = 0, 2v_0/g$;
$v(0) = v_0$ and $v(2v_0/g) = v_0 - g(2v_0/g) = -v_0$ so the speed is the same at launch ($t = 0$) and at return ($t = 2v_0/g$).

29. If g = 32 ft/s^2, $s_0 = 7$ and v_0 is unknown, then $s(t) = 7 + v_0t - 16t^2$ and $v(t) = v_0 - 32t$; $s = s_{\max}$ when $v = 0$, or $t = v_0/32$; and $s_{\max} = 208$ yields
$208 = s(v_0/32) = 7 + v_0(v_0/32) - 16(v_0/32)^2 = 7 + v_0^2/64$, so $v_0 = 8\sqrt{201} \approx 113.42$ ft/s.

30. **(a)** Use (6) and then (5) to get $v^2 = v_0^2 - 2v_0gt + g^2t^2 = v_0^2 - 2g(v_0t - \frac{1}{2}gt^2) = v_0^2 - 2g(s - s_0)$.

(b) Add v_0 to both sides of (6): $2v_0 - gt = v_0 + v$, $v_0 - \frac{1}{2}gt = \frac{1}{2}(v_0 + v)$;
from (5) $s = s_0 + t(v_0 - \frac{1}{2}gt) = s_0 + \frac{1}{2}(v_0 + v)t$

(c) Add v to both sides of (6): $2v + gt = v_0 + v$, $v + \frac{1}{2}gt = \frac{1}{2}(v_0 + v)$; from Part (b),
$s = s_0 + \frac{1}{2}(v_0 + v)t = s_0 + vt + \frac{1}{2}gt^2$

31. $v_0 = 0$ and $g = 9.8$, so $v^2 = -19.6(s - s_0)$; since $v = 24$ when $s = 0$ it follows that $19.6s_0 = 24^2$ or $s_0 = 29.39$ m.

32. $s = 1000 + vt + \frac{1}{2}(32)t^2 = 1000 + vt + 16t^2$; $s = 0$ when $t = 5$, so $v = -(1000 + 16 \cdot 5^2)/5 = -280$ ft/s.

33. **(a)** $s = s_{\max}$ when $v = 0$, so $0 = v_0^2 - 2g(s_{\max} - s_0)$, $s_{\max} = v_0^2/2g + s_0$.

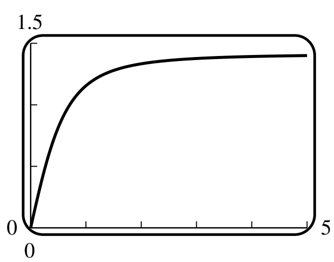
(b) $s_0 = 7$, $s_{\max} = 208$, $g = 32$ and v_0 is unknown, so from Part (a) $v_0^2 = 2g(208 - 7) = 64 \cdot 201$, $v_0 = 8\sqrt{201} \approx 113.42$ ft/s.

34. $s = t^3 - 6t^2 + 1$, $v = 3t^2 - 12t$, $a = 6t - 12$.

(a) $a = 0$ when $t = 2$; $s = -15$, $v = -12$.

(b) $v = 0$ when $3t^2 - 12t = 3t(t - 4) = 0$, $t = 0$ or $t = 4$. If $t = 0$, then $s = 1$ and $a = -12$; if $t = 4$, then $s = -31$ and $a = 12$.

35. **(a)**



(b) $v = \dfrac{2t}{\sqrt{2t^2 + 1}}$, $\displaystyle\lim_{t \to +\infty} v = \frac{2}{\sqrt{2}} = \sqrt{2}$

36. **(a)** $a = \dfrac{dv}{dt} = \dfrac{dv}{ds}\dfrac{ds}{dt} = v\dfrac{dv}{ds}$ because $v = \dfrac{ds}{dt}$

(b) $v = \dfrac{3}{2\sqrt{3t + 7}} = \dfrac{3}{2s}$; $\dfrac{dv}{ds} = -\dfrac{3}{2s^2}$; $a = -\dfrac{9}{4s^3} = -9/500$

37. **(a)** $s_1 = s_2$ if they collide, so $\frac{1}{2}t^2 - t + 3 = -\frac{1}{4}t^2 + t + 1$, $\frac{3}{4}t^2 - 2t + 2 = 0$ which has no real solution.

(b) Find the minimum value of $D = |s_1 - s_2| = \left|\frac{3}{4}t^2 - 2t + 2\right|$. From Part (a), $\frac{3}{4}t^2 - 2t + 2$ is never zero, and for $t = 0$ it is positive, hence it is always positive, so $D = \frac{3}{4}t^2 - 2t + 2$.

$\dfrac{dD}{dt} = \frac{3}{2}t - 2 = 0$ when $t = \frac{4}{3}$. $\dfrac{d^2D}{dt^2} > 0$ so D is minimum when $t = \frac{4}{3}$, $D = \frac{2}{3}$.

(c) $v_1 = t - 1$, $v_2 = -\dfrac{1}{2}t + 1$. $v_1 < 0$ if $0 \le t < 1$, $v_1 > 0$ if $t > 1$; $v_2 < 0$ if $t > 2$, $v_2 > 0$ if $0 \le t < 2$. They are moving in opposite directions during the intervals $0 \le t < 1$ and $t > 2$.

38. **(a)** $s_A - s_B = 20 - 0 = 20$ ft

(b) $s_A = s_B$, $15t^2 + 10t + 20 = 5t^2 + 40t$, $10t^2 - 30t + 20 = 0$, $(t-2)(t-1) = 0$, $t = 1$ or $t = 2$ s.

(c) $v_A = v_B$, $30t + 10 = 10t + 40$, $20t = 30$, $t = 3/2$ s. When $t = 3/2$, $s_A = 275/4$ and $s_B = 285/4$ so car B is ahead of car A.

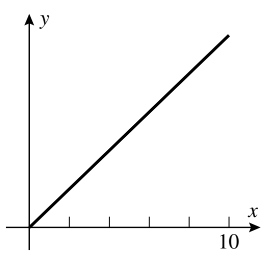
39. $r(t) = \sqrt{v^2(t)}$, $r'(t) = 2v(t)v'(t)/[2\sqrt{v(t)}] = v(t)a(t)/|v(t)|$ so $r'(t) > 0$ (speed is increasing) if v and a have the same sign, and $r'(t) < 0$ (speed is decreasing) if v and a have opposite signs.
If $v(t) > 0$ then $r(t) = v(t)$ and $r'(t) = a(t)$, so if $a(t) > 0$ then the particle is speeding up and a and v have the same sign; if $a(t) < 0$, then the particle is slowing down, and a and v have opposite signs.
If $v(t) < 0$ then $r(t) = -v(t)$, $r'(t) = -a(t)$, and if $a(t) > 0$ then the particle is speeding up and a and v have opposite signs; if $a(t) < 0$ then the particle is slowing down and a and v have the same sign.

EXERCISE SET 4.5

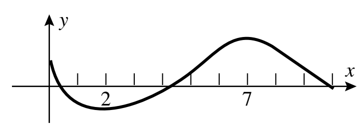
1. relative maxima at $x = 2, 6$; absolute maximum at $x = 6$; relative and absolute minimum at $x = 4$

2. relative maximum at $x = 3$; absolute maximum at $x = 7$; relative minima at $x = 1, 5$; absolute minima at $x = 1, 5$

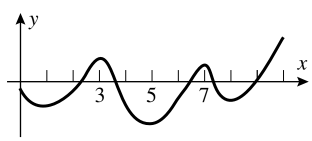
3. **(a)**



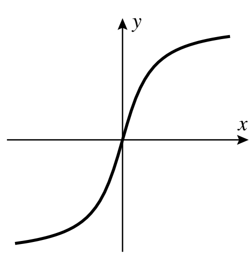
(b)



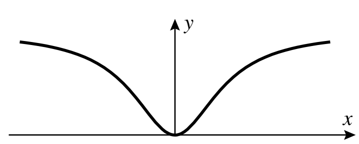
(c)



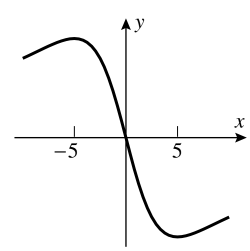
4. **(a)**



(b)



(c)



5. $f'(x) = 8x - 4$, $f'(x) = 0$ when $x = 1/2$; $f(0) = 1$, $f(1/2) = 0$, $f(1) = 1$ so the maximum value is 1 at $x = 0, 1$ and the minimum value is 0 at $x = 1/2$.

6. $f'(x) = 8 - 2x$, $f'(x) = 0$ when $x = 4$; $f(0) = 0$, $f(4) = 16$, $f(6) = 12$ so the maximum value is 16 at $x = 4$ and the minimum value is 0 at $x = 0$.

7. $f'(x) = 3(x-1)^2$, $f'(x) = 0$ when $x = 1$; $f(0) = -1$, $f(1) = 0$, $f(4) = 27$ so the maximum value is 27 at $x = 4$ and the minimum value is -1 at $x = 0$.

8. $f'(x) = 6x^2 - 6x - 12 = 6(x+1)(x-2)$, $f'(x) = 0$ when $x = -1, 2$; $f(-2) = -4$, $f(-1) = 7$, $f(2) = -20$, $f(3) = -9$ so the maximum value is 7 at $x = -1$ and the minimum value is -20 at $x = 2$.

9. $f'(x) = 3/(4x^2+1)^{3/2}$, no critical points; $f(-1) = -3/\sqrt{5}$, $f(1) = 3/\sqrt{5}$ so the maximum value is $3/\sqrt{5}$ at $x = 1$ and the minimum value is $-3/\sqrt{5}$ at $x = -1$.

10. $f'(x) = \dfrac{2(2x+1)}{3(x^2+x)^{1/3}}$, $f'(x) = 0$ when $x = -1/2$ and $f'(x)$ does not exist when $x = -1, 0$; $f(-2) = 2^{2/3}$, $f(-1) = 0$, $f(-1/2) = 4^{-2/3}$, $f(0) = 0$, $f(3) = 12^{2/3}$ so the maximum value is $12^{2/3}$ at $x = 3$ and the minimum value is 0 at $x = -1, 0$.

11. $f'(x) = 1 - \sec^2 x$, $f'(x) = 0$ for x in $(-\pi/4, \pi/4)$ when $x = 0$; $f(-\pi/4) = 1 - \pi/4$, $f(0) = 0$, $f(\pi/4) = \pi/4 - 1$ so the maximum value is $1 - \pi/4$ at $x = -\pi/4$ and the minimum value is $\pi/4 - 1$ at $x = \pi/4$.

12. $f'(x) = \cos x + \sin x$, $f'(x) = 0$ for x in $(0, \pi)$ when $x = 3\pi/4$; $f(0) = -1$, $f(3\pi/4) = \sqrt{2}$, $f(\pi) = 1$ so the maximum value is $\sqrt{2}$ at $x = 3\pi/4$ and the minimum value is -1 at $x = 0$.

13. $f(x) = 1 + |9 - x^2| = \begin{cases} 10 - x^2, & |x| \le 3 \\ -8 + x^2, & |x| > 3 \end{cases}$, $f'(x) = \begin{cases} -2x, & |x| < 3 \\ 2x, & |x| > 3 \end{cases}$ thus $f'(x) = 0$ when $x = 0$, $f'(x)$ does not exist for x in $(-5, 1)$ when $x = -3$ because $\lim_{x\to -3^-} f'(x) \ne \lim_{x\to -3^+} f'(x)$ (see Theorem preceding Exercise 75, Section 3.3); $f(-5) = 17$, $f(-3) = 1$, $f(0) = 10$, $f(1) = 9$ so the maximum value is 17 at $x = -5$ and the minimum value is 1 at $x = -3$.

14. $f(x) = |6 - 4x| = \begin{cases} 6 - 4x, & x \le 3/2 \\ -6 + 4x, & x > 3/2 \end{cases}$, $f'(x) = \begin{cases} -4, & x < 3/2 \\ 4, & x > 3/2 \end{cases}$, $f'(x)$ does not exist when $x = 3/2$ thus $3/2$ is the only critical point in $(-3, 3)$; $f(-3) = 18$, $f(3/2) = 0$, $f(3) = 6$ so the maximum value is 18 at $x = -3$ and the minimum value is 0 at $x = 3/2$.

15. $f'(x) = 2x - 3$; critical point $x = 3/2$. Minimum value $f(3/2) = -13/4$, no maximum.

16. $f'(x) = -4(x+1)$; critical point $x = -1$. Maximum value $f(-1) = 5$, no minimum.

17. $f'(x) = 12x^2(1-x)$; critical points $x = 0, 1$. Maximum value $f(1) = 1$, no minimum because $\lim_{x\to +\infty} f(x) = -\infty$.

18. $f'(x) = 4(x^3+1)$; critical point $x = -1$. Minimum value $f(-1) = -3$, no maximum.

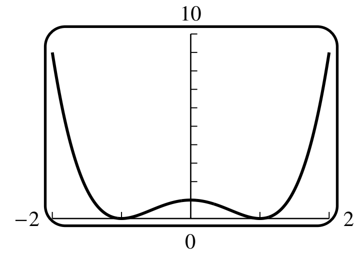
19. No maximum or minimum because $\lim_{x\to +\infty} f(x) = +\infty$ and $\lim_{x\to -\infty} f(x) = -\infty$.

20. No maximum or minimum because $\lim_{x\to +\infty} f(x) = +\infty$ and $\lim_{x\to -\infty} f(x) = -\infty$.

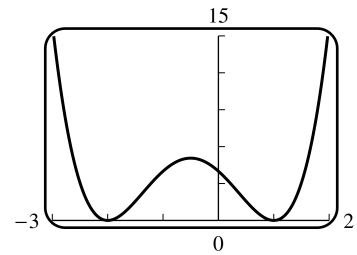
21. $f'(x) = x(x+2)/(x+1)^2$; critical point $x = -2$ in $(-5, -1)$. Maximum value $f(-2) = -4$, no minimum.

22. $f'(x) = -6/(x-3)^2$; no critical points in $[-5, 5]$ ($x = 3$ is not in the domain of f). No maximum or minimum because $\lim_{x\to 3^+} f(x) = +\infty$ and $\lim_{x\to 3^-} f(x) = -\infty$.

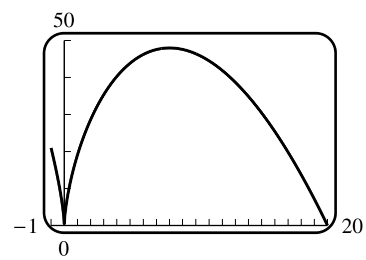
23. $(x^2-1)^2$ can never be less than zero because it is the square of x^2-1; the minimum value is 0 for $x=\pm 1$, no maximum because $\lim_{x\to+\infty} f(x)=+\infty$.



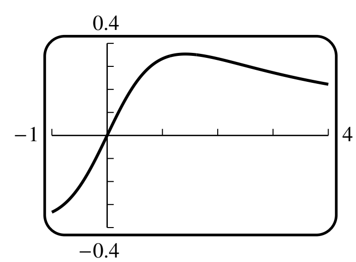
24. $(x-1)^2(x+2)^2$ can never be less than zero because it is the product of two squares; the minimum value is 0 for $x=1$ or -2, no maximum because $\lim_{x\to+\infty} f(x)=+\infty$.



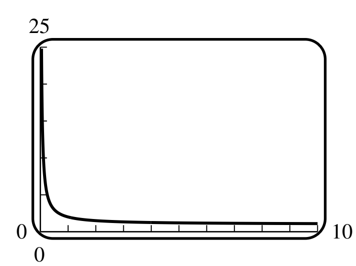
25. $f'(x)=\dfrac{5(8-x)}{3x^{1/3}}$, $f'(x)=0$ when $x=8$ and $f'(x)$ does not exist when $x=0$; $f(-1)=21$, $f(0)=0$, $f(8)=48$, $f(20)=0$ so the maximum value is 48 at $x=8$ and the minimum value is 0 at $x=0,20$.



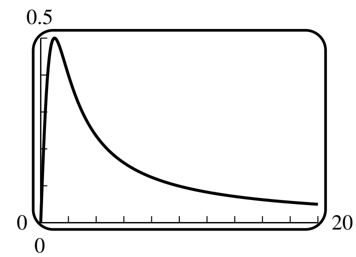
26. $f'(x)=(2-x^2)/(x^2+2)^2$, $f'(x)=0$ for x in the interval $(-1,4)$ when $x=\sqrt{2}$; $f(-1)=-1/3$, $f(\sqrt{2})=\sqrt{2}/4$, $f(4)=2/9$ so the maximum value is $\sqrt{2}/4$ at $x=\sqrt{2}$ and the minimum value is $-1/3$ at $x=-1$.



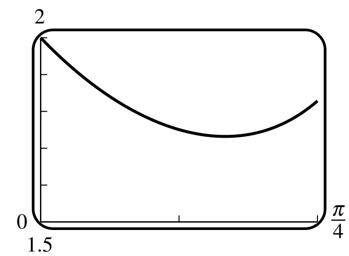
27. $f'(x)=-1/x^2$; no maximum or minimum because there are no critical points in $(0,+\infty)$.



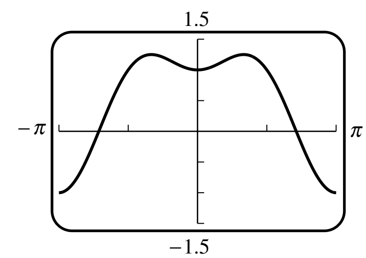
28. $f'(x) = (1-x^2)/(x^2+1)^2$; critical point $x = 1$. Maximum value $f(1) = 1/2$, minimum value 0 because $f(x)$ is never less than zero on $[0, +\infty)$ and $f(0) = 0$.



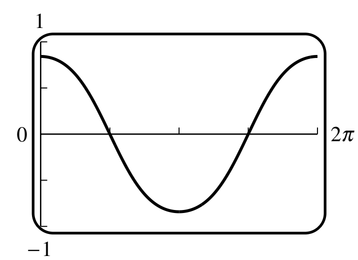
29. $f'(x) = 2\sec x \tan x - \sec^2 x = (2\sin x - 1)/\cos^2 x$, $f'(x) = 0$ for x in $(0, \pi/4)$ when $x = \pi/6$; $f(0) = 2$, $f(\pi/6) = \sqrt{3}$, $f(\pi/4) = 2\sqrt{2}-1$ so the maximum value is 2 at $x = 0$ and the minimum value is $\sqrt{3}$ at $x = \pi/6$.



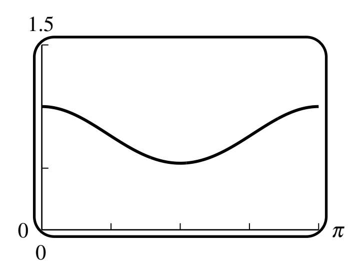
30. $f'(x) = 2\sin x \cos x - \sin x = \sin x(2\cos x - 1)$, $f'(x) = 0$ for x in $(-\pi, \pi)$ when $x = 0, \pm\pi/3$; $f(-\pi) = -1$, $f(-\pi/3) = 5/4$, $f(0) = 1$, $f(\pi/3) = 5/4$, $f(\pi) = -1$ so the maximum value is $5/4$ at $x = \pm\pi/3$ and the minimum value is -1 at $x = \pm\pi$.



31. $f'(x) = -[\cos(\cos x)]\sin x$; $f'(x) = 0$ if $\sin x = 0$ or if $\cos(\cos x) = 0$. If $\sin x = 0$, then $x = \pi$ is the critical point in $(0, 2\pi)$; $\cos(\cos x) = 0$ has no solutions because $-1 \le \cos x \le 1$. Thus $f(0) = \sin(1)$, $f(\pi) = \sin(-1) = -\sin(1)$, and $f(2\pi) = \sin(1)$ so the maximum value is $\sin(1) \approx 0.84147$ and the minimum value is $-\sin(1) \approx -0.84147$.



32. $f'(x) = -[\sin(\sin x)]\cos x$; $f'(x) = 0$ if $\cos x = 0$ or if $\sin(\sin x) = 0$. If $\cos x = 0$, then $x = \pi/2$ is the critical point in $(0, \pi)$; $\sin(\sin x) = 0$ if $\sin x = 0$, which gives no critical points in $(0, \pi)$. Thus $f(0) = 1$, $f(\pi/2) = \cos(1)$, and $f(\pi) = 1$ so the maximum value is 1 and the minimum value is $\cos(1) \approx 0.54030$.



33. $f'(x) = \begin{cases} 4, & x < 1 \\ 2x-5, & x > 1 \end{cases}$ so $f'(x) = 0$ when $x = 5/2$, and $f'(x)$ does not exist when $x = 1$ because $\lim_{x\to 1^-} f'(x) \neq \lim_{x\to 1^+} f'(x)$ (see Theorem preceding Exercise 75, Section 3.3); $f(1/2) = 0$, $f(1) = 2$, $f(5/2) = -1/4$, $f(7/2) = 3/4$ so the maximum value is 2 and the minimum value is $-1/4$.

34. $f'(x) = 2x + p$ which exists throughout the interval $(0, 2)$ for all values of p so $f'(1) = 0$ because $f(1)$ is an extreme value, thus $2 + p = 0$, $p = -2$. $f(1) = 3$ so $1^2 + (-2)(1) + q = 3$, $q = 4$ thus $f(x) = x^2 - 2x + 4$ and $f(0) = 4$, $f(2) = 4$ so $f(1)$ is the minimum value.

35. $\sin 2x$ has a period of π, and $\sin 4x$ a period of $\pi/2$ so $f(x)$ is periodic with period π. Consider the interval $[0, \pi]$. $f'(x) = 4\cos 2x + 4\cos 4x$, $f'(x) = 0$ when $\cos 2x + \cos 4x = 0$, but $\cos 4x = 2\cos^2 2x - 1$ (trig identity) so

$$\begin{aligned} 2\cos^2 2x + \cos 2x - 1 &= 0 \\ (2\cos 2x - 1)(\cos 2x + 1) &= 0 \\ \cos 2x = 1/2 \quad \text{or} \quad \cos 2x &= -1. \end{aligned}$$

From $\cos 2x = 1/2$, $2x = \pi/3$ or $5\pi/3$ so $x = \pi/6$ or $5\pi/6$. From $\cos 2x = -1$, $2x = \pi$ so $x = \pi/2$. $f(0) = 0$, $f(\pi/6) = 3\sqrt{3}/2$, $f(\pi/2) = 0$, $f(5\pi/6) = -3\sqrt{3}/2$, $f(\pi) = 0$. The maximum value is $3\sqrt{3}/2$ at $x = \pi/6 + n\pi$ and the minimum value is $-3\sqrt{3}/2$ at $x = 5\pi/6 + n\pi$, $n = 0, \pm 1, \pm 2, \ldots$.

36. $\cos\dfrac{x}{3}$ has a period of 6π, and $\cos\dfrac{x}{2}$ a period of 4π, so $f(x)$ has a period of 12π. Consider the interval $[0, 12\pi]$. $f'(x) = -\sin\dfrac{x}{3} - \sin\dfrac{x}{2}$, $f'(x) = 0$ when $\sin\dfrac{x}{3} + \sin\dfrac{x}{2} = 0$ thus, by use of the trig identity $\sin a + \sin b = 2\sin\dfrac{a+b}{2}\cos\dfrac{a-b}{2}$, $2\sin\left(\dfrac{5x}{12}\right)\cos\left(-\dfrac{x}{12}\right) = 0$ so $\sin\dfrac{5x}{12} = 0$ or $\cos\dfrac{x}{12} = 0$. Solve $\sin\dfrac{5x}{12} = 0$ to get $x = 12\pi/5,\ 24\pi/5,\ 36\pi/5,\ 48\pi/5$ and then solve $\cos\dfrac{x}{12} = 0$ to get $x = 6\pi$. The corresponding values of $f(x)$ are $-4.0450, 1.5450, 1.5450, -4.0450, 1, 5, 5$ so the maximum value is 5 and the minimum value is -4.0450 (approximately).

37. Let $f(x) = x - \sin x$, then $f'(x) = 1 - \cos x$ and so $f'(x) = 0$ when $\cos x = 1$ which has no solution for $0 < x < 2\pi$ thus the minimum value of f must occur at 0 or 2π. $f(0) = 0$, $f(2\pi) = 2\pi$ so 0 is the minimum value on $[0, 2\pi]$ thus $x - \sin x \ge 0$, $\sin x \le x$ for all x in $[0, 2\pi]$.

38. Let $h(x) = \cos x - 1 + x^2/2$. Then $h(0) = 0$, and it is sufficient to show that $h'(x) \ge 0$ for $0 < x < 2\pi$. But $h'(x) = -\sin x + x \ge 0$ by Exercise 37.

39. Let $m =$ slope at x, then $m = f'(x) = 3x^2 - 6x + 5$, $dm/dx = 6x - 6$; critical point for m is $x = 1$, minimum value of m is $f'(1) = 2$

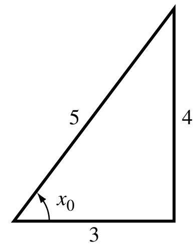
40. **(a)** $f'(x) = -\dfrac{64\cos x}{\sin^2 x} + \dfrac{27\sin x}{\cos^2 x} = \dfrac{-64\cos^3 x + 27\sin^3 x}{\sin^2 x\cos^2 x}$, $f'(x) = 0$ when

$27\sin^3 x = 64\cos^3 x$, $\tan^3 x = 64/27$, $\tan x = 4/3$ so the critical point is $x = x_0$ where $\tan x_0 = 4/3$ and $0 < x_0 < \pi/2$. To test x_0 first rewrite $f'(x)$ as

$$f'(x) = \frac{27\cos^3 x(\tan^3 x - 64/27)}{\sin^2 x\cos^2 x} = \frac{27\cos x(\tan^3 x - 64/27)}{\sin^2 x};$$

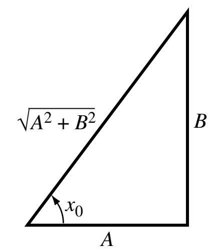
if $x < x_0$ then $\tan x < 4/3$ and $f'(x) < 0$, if $x > x_0$ then $\tan x > 4/3$ and $f'(x) > 0$ so $f(x_0)$ is the minimum value. f has no maximum because $\lim\limits_{x\to 0^+} f(x) = +\infty$.

(b) If $\tan x_0 = 4/3$ then (see figure)
$\sin x_0 = 4/5$ and $\cos x_0 = 3/5$
so $f(x_0) = 64/\sin x_0 + 27/\cos x_0$
$= 64/(4/5) + 27/(3/5)$
$= 80 + 45 = 125$



41. $f'(x) = \dfrac{2x(x^3 - 24x^2 + 192x - 640)}{(x-8)^3}$; real root of $x^3 - 24x^2 + 192x - 640$ at $x = 4(2 + \sqrt[3]{2})$. Since $\lim_{x\to 8^+} f(x) = \lim_{x\to+\infty} f(x) = +\infty$ and there is only one relative extremum, it must be a minimum.

42. Assume A and B nonzero, since if A or $B = 0$ the solution is obvious. Otherwise
$f'(x) = -A\sin x + B\cos x = 0$ when $x = x_0 = \tan^{-1}(B/A)$.
From the figure we get
$f(x_0) = A^2/\sqrt{A^2+B^2} + B^2/\sqrt{A^2+B^2} = \sqrt{A^2+B^2}$.
This must be a maximum since $f''(x_0) = -f(x_0) < 0$ there.



43. The slope of the line is -1, and the slope of the tangent to $y = -x^2$ is $-2x$ so $-2x = -1$, $x = 1/2$. The line lies above the curve so the vertical distance is given by $F(x) = 2 - x + x^2$; $F(-1) = 4$, $F(1/2) = 7/4$, $F(3/2) = 11/4$. The point $(1/2, -1/4)$ is closest, the point $(-1, -1)$ farthest.

44. The slope of the line is $4/3$; and the slope of the tangent to $y = x^3$ is $3x^2$ so $3x^2 = 4/3$, $x^2 = 4/9$, $x = \pm 2/3$. The line lies below the curve so the vertical distance is given by $F(x) = x^3 - 4x/3 + 1$; $F(-1) = 4/3$, $F(-2/3) = 43/27$, $F(2/3) = 11/27$, $F(1) = 2/3$. The closest point is $(2/3, 8/27)$, the farthest is $(-2/3, -8/27)$.

45. The absolute extrema of $y(t)$ can occur at the endpoints $t = 0, 12$ or when $dy/dt = 2\sin t = 0$, i.e. $t = 0, 12, k\pi$, $k = 1, 2, 3$; the absolute maximum is $y = 4$ at $t = \pi, 3\pi$; the absolute minimum is $y = 0$ at $t = 0, 2\pi$.

46. **(a)** The absolute extrema of $y(t)$ can occur at the endpoints $t = 0, 2\pi$ or when $dy/dt = 2\cos 2t - 4\sin t\cos t = 2\cos 2t - 2\sin 2t = 0$, $t = 0, 2\pi, \pi/8, 5\pi/8, 9\pi/8, 13\pi/8$; the absolute maximum is $y = 3.4142$ at $t = \pi/8, 9\pi/8$; the absolute minimum is $y = 0.5859$ at $t = 5\pi/8, 13\pi/8$.

(b) The absolute extrema of $x(t)$ occur at the endpoints $t = 0, 2\pi$ or when

$\dfrac{dx}{dt} = -\dfrac{2\sin t + 1}{(2+\sin t)^2} = 0$, $t = 7\pi/6, 11\pi/6$. The absolute maximum is $x = 0.5774$ at $t = 11\pi/6$

and the absolute minimum is $x = -0.5774$ at $t = 7\pi/6$.

47. $f'(x) = 2ax + b$; critical point is $x = -\dfrac{b}{2a}$

$f''(x) = 2a > 0$ so $f\left(-\dfrac{b}{2a}\right)$ is the minimum value of f, but

$f\left(-\dfrac{b}{2a}\right) = a\left(-\dfrac{b}{2a}\right)^2 + b\left(-\dfrac{b}{2a}\right) + c = \dfrac{-b^2+4ac}{4a}$ thus $f(x) \geq 0$ if and only if

$f\left(-\dfrac{b}{2a}\right) \geq 0$, $\dfrac{-b^2+4ac}{4a} \geq 0$, $-b^2+4ac \geq 0$, $b^2 - 4ac \leq 0$

48. Use the proof given in the text, replacing "maximum" by "minimum" and "largest" by "smallest" and reversing the order of all inequality symbols.

EXERCISE SET 4.6

1. Let x = one number, y = the other number, and $P = xy$ where $x + y = 10$. Thus $y = 10 - x$ so $P = x(10-x) = 10x - x^2$ for x in $[0, 10]$. $dP/dx = 10 - 2x$, $dP/dx = 0$ when $x = 5$. If $x = 0, 5, 10$ then $P = 0, 25, 0$ so P is maximum when $x = 5$ and, from $y = 10 - x$, when $y = 5$.

2. Let x and y be nonnegative numbers and z the sum of their squares, then $z = x^2 + y^2$. But $x + y = 1$, $y = 1 - x$ so $z = x^2 + (1-x)^2 = 2x^2 - 2x + 1$ for $0 \leq x \leq 1$. $dz/dx = 4x - 2$, $dz/dx = 0$ when $x = 1/2$. If $x = 0, 1/2, 1$ then $z = 1, 1/2, 1$ so

(a) z is as large as possible when one number is 0 and the other is 1.

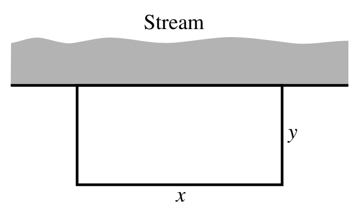
(b) z is as small as possible when both numbers are 1/2.

3. If $y = x + 1/x$ for $1/2 \leq x \leq 3/2$ then $dy/dx = 1 - 1/x^2 = (x^2-1)/x^2$, $dy/dx = 0$ when $x = 1$. If $x = 1/2, 1, 3/2$ then $y = 5/2, 2, 13/6$ so

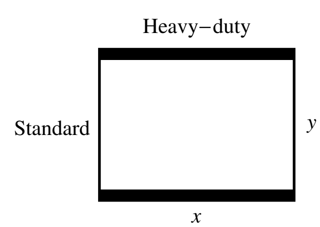
(a) y is as small as possible when $x = 1$.

(b) y is as large as possible when $x = 1/2$.

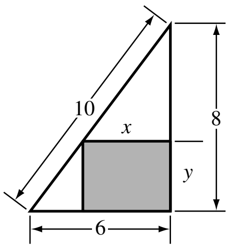
4. $A = xy$ where $x + 2y = 1000$ so $y = 500 - x/2$ and $A = 500x - x^2/2$ for x in $[0, 1000]$; $dA/dx = 500 - x$, $dA/dx = 0$ when $x = 500$. If $x = 0$ or 1000 then $A = 0$, if $x = 500$ then $A = 125{,}000$ so the area is maximum when $x = 500$ ft and $y = 500 - 500/2 = 250$ ft.



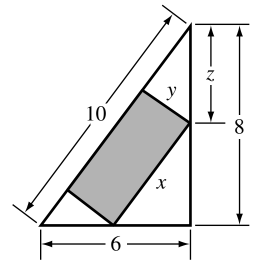
5. Let x and y be the dimensions shown in the figure and A the area, then $A = xy$ subject to the cost condition $3(2x) + 2(2y) = 6000$, or $y = 1500 - 3x/2$. Thus $A = x(1500 - 3x/2) = 1500x - 3x^2/2$ for x in $[0, 1000]$. $dA/dx = 1500 - 3x$, $dA/dx = 0$ when $x = 500$. If $x = 0$ or 1000 then $A = 0$, if $x = 500$ then $A = 375{,}000$ so the area is greatest when $x = 500$ ft and (from $y = 1500 - 3x/2$) when $y = 750$ ft.



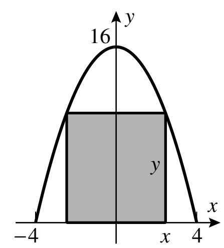
6. Let x and y be the dimensions shown in the figure and A the area of the rectangle, then $A = xy$ and, by similar triangles, $x/6 = (8-y)/8$, $y = 8 - 4x/3$ so $A = x(8 - 4x/3) = 8x - 4x^2/3$ for x in $[0, 6]$. $dA/dx = 8 - 8x/3$, $dA/dx = 0$ when $x = 3$. If $x = 0, 3, 6$ then $A = 0, 12, 0$ so the area is greatest when $x = 3$ in and (from $y = 8 - 4x/3$) $y = 4$ in.



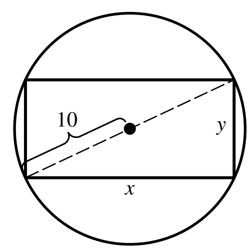
7. Let x, y, and z be as shown in the figure and A the area of the rectangle, then $A = xy$ and, by similar triangles, $z/10 = y/6$, $z = 5y/3$; also $x/10 = (8-z)/8 = (8 - 5y/3)/8$ thus
$y = 24/5 - 12x/25$ so $A = x(24/5 - 12x/25) = 24x/5 - 12x^2/25$ for x in $[0, 10]$. $dA/dx = 24/5 - 24x/25$, $dA/dx = 0$ when $x = 5$. If $x = 0, 5, 10$ then $A = 0, 12, 0$ so the area is greatest when $x = 5$ in. and $y = 12/5$ in.



8. $A = (2x)y = 2xy$ where $y = 16 - x^2$ so $A = 32x - 2x^3$ for $0 \le x \le 4$; $dA/dx = 32 - 6x^2$, $dA/dx = 0$ when $x = 4/\sqrt{3}$. If $x = 0, 4/\sqrt{3}, 4$ then $A = 0, 256/(3\sqrt{3}), 0$ so the area is largest when $x = 4/\sqrt{3}$ and $y = 32/3$. The dimensions of the rectangle with largest area are $8/\sqrt{3}$ by $32/3$.



9. $A = xy$ where $x^2 + y^2 = 20^2 = 400$ so $y = \sqrt{400 - x^2}$ and
$A = x\sqrt{400 - x^2}$ for $0 \le x \le 20$;
$dA/dx = 2(200 - x^2)/\sqrt{400 - x^2}$, $dA/dx = 0$ when
$x = \sqrt{200} = 10\sqrt{2}$. If $x = 0, 10\sqrt{2}, 20$ then $A = 0, 200, 0$ so the area is maximum when $x = 10\sqrt{2}$ and $y = \sqrt{400 - 200} = 10\sqrt{2}$.



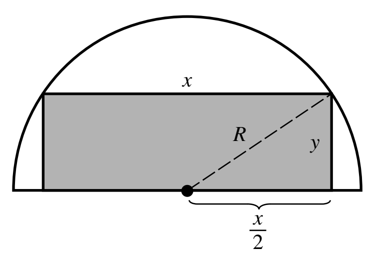
10. Let x and y be the dimensions shown in the figure, then the area of the rectangle is $A = xy$.

But $\left(\dfrac{x}{2}\right)^2 + y^2 = R^2$, thus

$y = \sqrt{R^2 - x^2/4} = \dfrac{1}{2}\sqrt{4R^2 - x^2}$ so

$A = \dfrac{1}{2}x\sqrt{4R^2 - x^2}$ for $0 \le x \le 2R$.

$dA/dx = (2R^2 - x^2)/\sqrt{4R^2 - x^2}$, $dA/dx = 0$ when $x = \sqrt{2}R$. If $x = 0, \sqrt{2}R, 2R$ then $A = 0, R^2, 0$ so the greatest area occurs when $x = \sqrt{2}R$ and $y = \sqrt{2}R/2$.

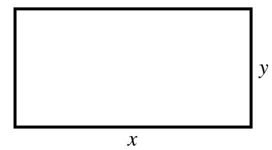


11. Let $x =$ length of each side that uses the \$1 per foot fencing,
$y =$ length of each side that uses the \$2 per foot fencing.
The cost is $C = (1)(2x) + (2)(2y) = 2x + 4y$, but $A = xy = 3200$ thus $y = 3200/x$ so

$$C = 2x + 12800/x \text{ for } x > 0,$$
$$dC/dx = 2 - 12800/x^2,\ dC/dx = 0 \text{ when } x = 80,\ d^2C/dx^2 > 0 \text{ so}$$

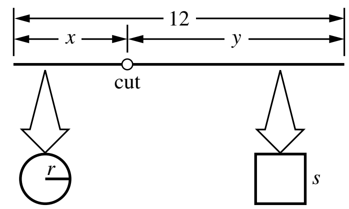
C is least when $x = 80$, $y = 40$.

12. $A = xy$ where $2x + 2y = p$ so $y = p/2 - x$ and $A = px/2 - x^2$ for x in $[0, p/2]$; $dA/dx = p/2 - 2x$, $dA/dx = 0$ when $x = p/4$. If $x = 0$ or $p/2$ then $A = 0$, if $x = p/4$ then $A = p^2/16$ so the area is maximum when $x = p/4$ and $y = p/2 - p/4 = p/4$, which is a square.



13. Let x and y be the dimensions of a rectangle; the perimeter is $p = 2x + 2y$. But $A = xy$ thus $y = A/x$ so $p = 2x + 2A/x$ for $x > 0$, $dp/dx = 2 - 2A/x^2 = 2(x^2 - A)/x^2$, $dp/dx = 0$ when $x = \sqrt{A}$, $d^2p/dx^2 = 4A/x^3 > 0$ if $x > 0$ so p is a minimum when $x = \sqrt{A}$ and $y = \sqrt{A}$ and thus the rectangle is a square.

14. With x, y, r, and s as shown in the figure, the sum of the enclosed areas is $A = \pi r^2 + s^2$ where $r = \dfrac{x}{2\pi}$ and $s = \dfrac{y}{4}$ because x is the circumference of the circle and y is the perimeter of the square, thus $A = \dfrac{x^2}{4\pi} + \dfrac{y^2}{16}$. But $x + y = 12$, so $y = 12 - x$ and



$A = \dfrac{x^2}{4\pi} + \dfrac{(12-x)^2}{16} = \dfrac{\pi+4}{16\pi}x^2 - \dfrac{3}{2}x + 9$ for $0 \le x \le 12$.

$\dfrac{dA}{dx} = \dfrac{\pi+4}{8\pi}x - \dfrac{3}{2}$, $\dfrac{dA}{dx} = 0$ when $x = \dfrac{12\pi}{\pi+4}$. If $x = 0, \dfrac{12\pi}{\pi+4}, 12$ then $A = 9, \dfrac{36}{\pi+4}, \dfrac{36}{\pi}$ so the sum of the enclosed areas is

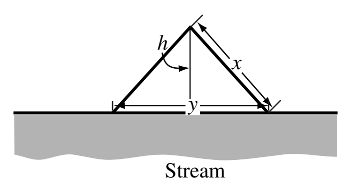
(a) a maximum when $x = 12$ in. (when all of the wire is used for the circle)

(b) a minimum when $x = 12\pi/(\pi + 4)$ in.

15. The area of the field is $A = 2bh$, where $b = y/2$ is the base of the half-triangle and h is the height. The length of fencing is $2x = 300$. Thus $x = 150$ and

$A = yh = y\sqrt{x^2 - y^2/4} = y\sqrt{(150)^2 - y^2/4}$

$dA/dy = \dfrac{y(-y/2)}{2\sqrt{(150)^2 - y^2/4)}} + \sqrt{(150)^2 - y^2/4} = 0$



when $-y^2/4 + (150)^2 - y^2/4 = 0$, or $y^2 = 2(150)^2$. Thus $y = 150\sqrt{2}$ yd, $x = 150$ yd; $b = 75\sqrt{2}$ yd, $h = 75\sqrt{2}$ yd, and $A = yh = (150)^2 = 22{,}500$ yd^2.

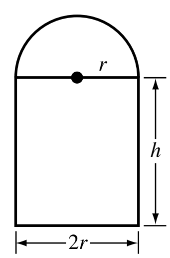
16. The area of the window is $A = 2rh + \pi r^2/2$, the perimeter is $p = 2r + 2h + \pi r$ thus

$h = \dfrac{1}{2}[p - (2+\pi)r]$ so

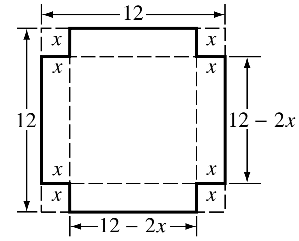
$A = r[p - (2+\pi)r] + \pi r^2/2$

$\quad = pr - (2 + \pi/2)r^2$ for $0 \le r \le p/(2+\pi)$,

$dA/dr = p - (4+\pi)r$, $dA/dr = 0$ when $r = p/(4+\pi)$ and $d^2A/dr^2 < 0$, so A is maximum when $r = p/(4+\pi)$.



17. $V = x(12-2x)^2$ for $0 \le x \le 6$; $dV/dx = 12(x-2)(x-6)$, $dV/dx = 0$ when $x = 2$ for $0 < x < 6$. If $x = 0, 2, 6$ then $V = 0, 128, 0$ so the volume is largest when $x = 2$ in.



18. The dimensions of the box will be $(k-2x)$ by $(k-2x)$ by x so $V = (k-2x)^2 x = 4x^3 - 4kx^2 + k^2 x$ for x in $[0, k/2]$. $dV/dx = 12x^2 - 8kx + k^2 = (6x-k)(2x-k)$, $dV/dx = 0$ for x in $(0, k/2)$ when $x = k/6$. If $x = 0, k/6, k/2$ then $V = 0, 2k^3/27, 0$ so V is maximum when $x = k/6$. The squares should have dimensions $k/6$ by $k/6$.

19. Let x be the length of each side of a square, then $V = x(3-2x)(8-2x) = 4x^3 - 22x^2 + 24x$ for $0 \le x \le 3/2$; $dV/dx = 12x^2 - 44x + 24 = 4(3x-2)(x-3)$, $dV/dx = 0$ when $x = 2/3$ for $0 < x < 3/2$. If $x = 0, 2/3, 3/2$ then $V = 0, 200/27, 0$ so the maximum volume is $200/27$ ft^3.

20. Let x = length of each edge of base, y = height. The cost is $C =$ (cost of top and bottom) + (cost of sides) $= (2)(2x^2) + (3)(4xy) = 4x^2 + 12xy$, but $V = x^2 y = 2250$ thus $y = 2250/x^2$ so $C = 4x^2 + 27000/x$ for $x > 0$, $dC/dx = 8x - 27000/x^2$, $dC/dx = 0$ when $x = \sqrt[3]{3375} = 15$, $d^2C/dx^2 > 0$ so C is least when $x = 15$, $y = 10$.

21. Let x = length of each edge of base, y = height, $k = \$/\text{cm}^2$ for the sides. The cost is $C = (2k)(2x^2) + (k)(4xy) = 4k(x^2 + xy)$, but $V = x^2 y = 2000$ thus $y = 2000/x^2$ so $C = 4k(x^2 + 2000/x)$ for $x > 0$; $dC/dx = 4k(2x - 2000/x^2)$, $dC/dx = 0$ when $x = \sqrt[3]{1000} = 10$, $d^2C/dx^2 > 0$ so C is least when $x = 10$, $y = 20$.

22. Let x and y be the dimensions shown in the figure and V the volume, then $V = x^2 y$. The amount of material is to be 1000 ft^2, thus (area of base) + (area of sides) = 1000, $x^2 + 4xy = 1000$, $y = \dfrac{1000 - x^2}{4x}$ so

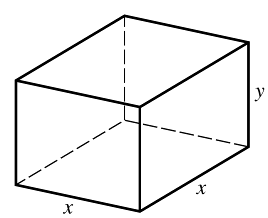
$$V = x^2 \frac{1000 - x^2}{4x} = \frac{1}{4}(1000x - x^3) \text{ for } 0 < x \le 10\sqrt{10}.$$

$$\frac{dV}{dx} = \frac{1}{4}(1000 - 3x^2), \ \frac{dV}{dx} = 0$$

when $x = \sqrt{1000/3} = 10\sqrt{10/3}$.

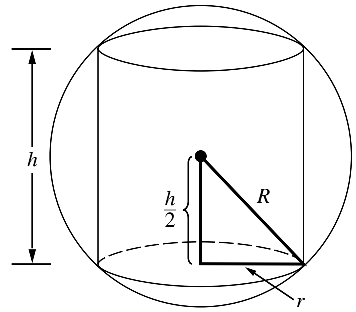
If $x = 0, 10\sqrt{10/3}, 10\sqrt{10}$ then $V = 0, \dfrac{5000}{3}\sqrt{10/3}, 0$;

the volume is greatest for $x = 10\sqrt{10/3}$ ft and $y = 5\sqrt{10/3}$ ft.



23. Let x = height and width, y = length. The surface area is $S = 2x^2 + 3xy$ where $x^2 y = V$, so $y = V/x^2$ and $S = 2x^2 + 3V/x$ for $x > 0$; $dS/dx = 4x - 3V/x^2$, $dS/dx = 0$ when $x = \sqrt[3]{3V/4}$, $d^2S/dx^2 > 0$ so S is minimum when $x = \sqrt[3]{\dfrac{3V}{4}}$, $y = \dfrac{4}{3}\sqrt[3]{\dfrac{3V}{4}}$.

24. Let r and h be the dimensions shown in the figure, then the volume of the inscribed cylinder is $V = \pi r^2 h$. But $r^2 + \left(\dfrac{h}{2}\right)^2 = R^2$ thus $r^2 = R^2 - \dfrac{h^2}{4}$ so $V = \pi\left(R^2 - \dfrac{h^2}{4}\right)h = \pi\left(R^2 h - \dfrac{h^3}{4}\right)$ for $0 \le h \le 2R$. $\dfrac{dV}{dh} = \pi\left(R^2 - \dfrac{3}{4}h^2\right)$, $\dfrac{dV}{dh} = 0$ when $h = 2R/\sqrt{3}$. If $h = 0, 2R/\sqrt{3}, 2R$ then $V = 0, \dfrac{4\pi}{3\sqrt{3}}R^3, 0$ so the volume is largest when $h = 2R/\sqrt{3}$ and $r = \sqrt{2/3}R$.



25. Let r and h be the dimensions shown in the figure, then the surface area is $S = 2\pi rh + 2\pi r^2$.
But $r^2 + \left(\dfrac{h}{2}\right)^2 = R^2$ thus $h = 2\sqrt{R^2 - r^2}$ so

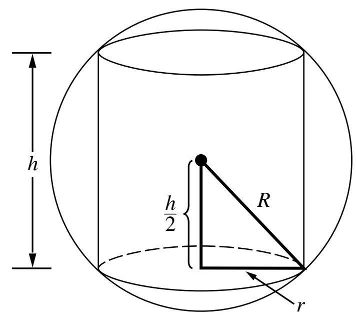
$S = 4\pi r\sqrt{R^2 - r^2} + 2\pi r^2$ for $0 \le r \le R$,

$\dfrac{dS}{dr} = \dfrac{4\pi(R^2 - 2r^2)}{\sqrt{R^2 - r^2}} + 4\pi r$; $\dfrac{dS}{dr} = 0$ when

$$\frac{R^2 - 2r^2}{\sqrt{R^2 - r^2}} = -r \qquad (1)$$

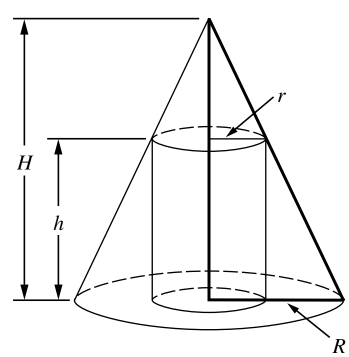
$$R^2 - 2r^2 = -r\sqrt{R^2 - r^2}$$

$R^4 - 4R^2 r^2 + 4r^4 = r^2(R^2 - r^2)$
$5r^4 - 5R^2 r^2 + R^4 = 0$



and using the quadratic formula $r^2 = \dfrac{5R^2 \pm \sqrt{25R^4 - 20R^4}}{10} = \dfrac{5 \pm \sqrt{5}}{10}R^2$, $r = \sqrt{\dfrac{5 \pm \sqrt{5}}{10}}R$, of which only $r = \sqrt{\dfrac{5 + \sqrt{5}}{10}}R$ satisfies (1). If $r = 0, \sqrt{\dfrac{5 + \sqrt{5}}{10}}R, 0$ then $S = 0, (5 + \sqrt{5})\pi R^2, 2\pi R^2$ so the surface area is greatest when $r = \sqrt{\dfrac{5 + \sqrt{5}}{10}}R$ and, from $h = 2\sqrt{R^2 - r^2}$, $h = 2\sqrt{\dfrac{5 - \sqrt{5}}{10}}R$.

26. Let R and H be the radius and height of the cone, and r and h the radius and height of the cylinder (see figure), then the volume of the cylinder is $V = \pi r^2 h$. By similar triangles (see figure) $\dfrac{H - h}{H} = \dfrac{r}{R}$ thus $h = \dfrac{H}{R}(R - r)$ so $V = \pi\dfrac{H}{R}(R - r)r^2 = \pi\dfrac{H}{R}(Rr^2 - r^3)$ for $0 \le r \le R$. $\dfrac{dV}{dr} = \pi\dfrac{H}{R}(2Rr - 3r^2) = \pi\dfrac{H}{R}r(2R - 3r)$, $\dfrac{dV}{dr} = 0$ for $0 < r < R$ when $r = 2R/3$. If $r = 0, 2R/3, R$ then $V = 0, 4\pi R^2 H/27, 0$ so the maximum volume is $\dfrac{4\pi R^2 H}{27} = \dfrac{4}{9}\dfrac{1}{3}\pi R^2 H = \dfrac{4}{9}\cdot$ (volume of cone).

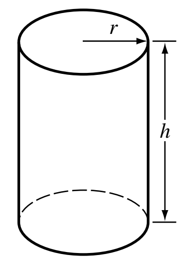


27. From (13), $S = 2\pi r^2 + 2\pi rh$. But $V = \pi r^2 h$ thus $h = V/(\pi r^2)$ and so $S = 2\pi r^2 + 2V/r$ for $r > 0$. $dS/dr = 4\pi r - 2V/r^2, dS/dr = 0$ if $r = \sqrt[3]{V/(2\pi)}$. Since $d^2S/dr^2 = 4\pi + 4V/r^3 > 0$, the minimum surface area is achieved when $r = \sqrt[3]{V/2\pi}$ and so $h = V/(\pi r^2) = [V/(\pi r^3)]r = 2r$.

28. $V = \pi r^2 h$ where $S = 2\pi r^2 + 2\pi rh$ so $h = \dfrac{S - 2\pi r^2}{2\pi r}$, $V = \dfrac{1}{2}(Sr - 2\pi r^3)$ for $r > 0$.
$\dfrac{dV}{dr} = \dfrac{1}{2}(S - 6\pi r^2) = 0$ if $r = \sqrt{S/(6\pi)}$, $\dfrac{d^2V}{dr^2} = -6\pi r < 0$ so V is maximum when $r = \sqrt{S/(6\pi)}$ and $h = \dfrac{S - 2\pi r^2}{2\pi r} = \dfrac{S - 2\pi r^2}{2\pi r^2}r = \dfrac{S - S/3}{S/3}r = 2r$, thus the height is equal to the diameter of the base.

29. The surface area is $S = \pi r^2 + 2\pi rh$ where $V = \pi r^2 h = 500$ so $h = 500/(\pi r^2)$ and $S = \pi r^2 + 1000/r$ for $r > 0$;
$dS/dr = 2\pi r - 1000/r^2 = (2\pi r^3 - 1000)/r^2$, $dS/dr = 0$ when $r = \sqrt[3]{500/\pi}$, $d^2S/dr^2 > 0$ for $r > 0$ so S is minimum when $r = \sqrt[3]{500/\pi}$ cm, and

$$h = \frac{500}{\pi r^2} = \frac{500}{\pi}\left(\frac{\pi}{500}\right)^{2/3} = \sqrt[3]{500/\pi} \text{ cm.}$$



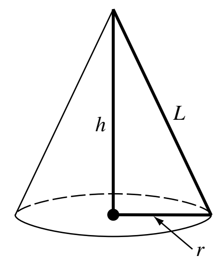
30. The total area of material used is
$A = A_{\text{top}} + A_{\text{bottom}} + A_{\text{side}} = (2r)^2 + (2r)^2 + 2\pi rh = 8r^2 + 2\pi rh$.
The volume is $V = \pi r^2 h$ thus $h = V/(\pi r^2)$ so $A = 8r^2 + 2V/r$ for $r > 0$,
$dA/dr = 16r - 2V/r^2 = 2(8r^3 - V)/r^2$, $dA/dr = 0$ when $r = \sqrt[3]{V}/2$. This is the only critical point, $d^2A/dr^2 > 0$ there so the least material is used when $r = \sqrt[3]{V}/2$, $\dfrac{r}{h} = \dfrac{r}{V/(\pi r^2)} = \dfrac{\pi}{V}r^3$ and, for $r = \sqrt[3]{V}/2$, $\dfrac{r}{h} = \dfrac{\pi}{V}\dfrac{V}{8} = \dfrac{\pi}{8}$.

31. Let x be the length of each side of the squares and y the height of the frame, then the volume is $V = x^2 y$. The total length of the wire is L thus $8x + 4y = L$, $y = (L - 8x)/4$ so $V = x^2(L - 8x)/4 = (Lx^2 - 8x^3)/4$ for $0 \le x \le L/8$. $dV/dx = (2Lx - 24x^2)/4$, $dV/dx = 0$ for $0 < x < L/8$ when $x = L/12$. If $x = 0, L/12, L/8$ then $V = 0, L^3/1728, 0$ so the volume is greatest when $x = L/12$ and $y = L/12$.

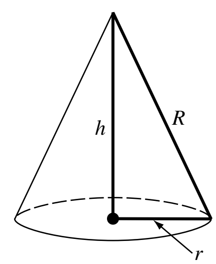
32. **(a)** Let x = diameter of the sphere, y = length of an edge of the cube. The combined volume is $V = \dfrac{1}{6}\pi x^3 + y^3$ and the surface area is $S = \pi x^2 + 6y^2$ = constant. Thus $y = \dfrac{(S - \pi x^2)^{1/2}}{6^{1/2}}$ and $V = \dfrac{\pi}{6}x^3 + \dfrac{(S - \pi x^2)^{3/2}}{6^{3/2}}$ for $0 \le x \le \sqrt{\dfrac{S}{\pi}}$;
$\dfrac{dV}{dx} = \dfrac{\pi}{2}x^2 - \dfrac{3\pi}{6^{3/2}}x(S - \pi x^2)^{1/2} = \dfrac{\pi}{2\sqrt{6}}x(\sqrt{6}x - \sqrt{S - \pi x^2})$. $\dfrac{dV}{dx} = 0$ when $x = 0$, or when $\sqrt{6}x = \sqrt{S - \pi x^2}$, $6x^2 = S - \pi x^2$, $x^2 = \dfrac{S}{6 + \pi}$, $x = \sqrt{\dfrac{S}{6 + \pi}}$. If $x = 0, \sqrt{\dfrac{S}{6 + \pi}}, \sqrt{\dfrac{S}{\pi}}$, then $V = \dfrac{S^{3/2}}{6^{3/2}}, \dfrac{S^{3/2}}{6\sqrt{6 + \pi}}, \dfrac{S^{3/2}}{6\sqrt{\pi}}$ so that V is smallest when $x = \sqrt{\dfrac{S}{6 + \pi}}$, and hence when $y = \sqrt{\dfrac{S}{6 + \pi}}$, thus $x = y$.

(b) From Part (a), the sum of the volumes is greatest when there is no cube.

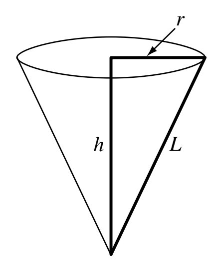
33. Let h and r be the dimensions shown in the figure, then the volume is $V = \frac{1}{3}\pi r^2 h$. But $r^2 + h^2 = L^2$ thus $r^2 = L^2 - h^2$ so $V = \frac{1}{3}\pi(L^2 - h^2)h = \frac{1}{3}\pi(L^2 h - h^3)$ for $0 \le h \le L$. $\frac{dV}{dh} = \frac{1}{3}\pi(L^2 - 3h^2)$. $\frac{dV}{dh} = 0$ when $h = L/\sqrt{3}$. If $h = 0, L/\sqrt{3}, 0$ then $V = 0, \frac{2\pi}{9\sqrt{3}}L^3, 0$ so the volume is as large as possible when $h = L/\sqrt{3}$ and $r = \sqrt{2/3}L$.



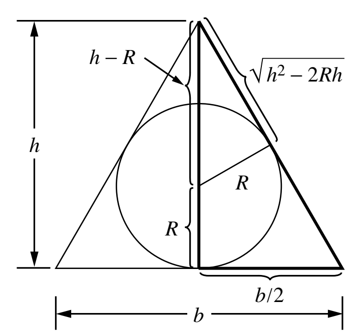
34. Let r and h be the radius and height of the cone (see figure). The slant height of any such cone will be R, the radius of the circular sheet. Refer to the solution of Exercise 33 to find that the largest volume is $\frac{2\pi}{9\sqrt{3}}R^3$.



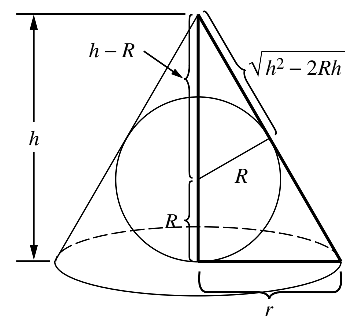
35. The area of the paper is $A = \pi r L = \pi r\sqrt{r^2 + h^2}$, but $V = \frac{1}{3}\pi r^2 h = 10$ thus $h = 30/(\pi r^2)$
so $A = \pi r\sqrt{r^2 + 900/(\pi^2 r^4)}$.
To simplify the computations let $S = A^2$,
$S = \pi^2 r^2\left(r^2 + \frac{900}{\pi^2 r^4}\right) = \pi^2 r^4 + \frac{900}{r^2}$ for $r > 0$,
$\frac{dS}{dr} = 4\pi^2 r^3 - \frac{1800}{r^3} = \frac{4(\pi^2 r^6 - 450)}{r^3}$, $dS/dr = 0$ when $r = \sqrt[6]{450/\pi^2}$, $d^2S/dr^2 > 0$, so S and hence A is least when $r = \sqrt[6]{450/\pi^2}$ cm, $h = \frac{30}{\pi}\sqrt[3]{\pi^2/450}$ cm.



36. The area of the triangle is $A = \frac{1}{2}hb$. By similar triangles (see figure) $\frac{b/2}{h} = \frac{R}{\sqrt{h^2 - 2Rh}}$, $b = \frac{2Rh}{\sqrt{h^2 - 2Rh}}$ so $A = \frac{Rh^2}{\sqrt{h^2 - 2Rh}}$ for $h > 2R$, $\frac{dA}{dh} = \frac{Rh^2(h - 3R)}{(h^2 - 2Rh)^{3/2}}$, $\frac{dA}{dh} = 0$ for $h > 2R$ when $h = 3R$, by the first derivative test A is minimum when $h = 3R$. If $h = 3R$ then $b = 2\sqrt{3}R$ (the triangle is equilateral).



37. The volume of the cone is $V = \dfrac{1}{3}\pi r^2 h$. By similar triangles (see figure) $\dfrac{r}{h} = \dfrac{R}{\sqrt{h^2 - 2Rh}}$, $r = \dfrac{Rh}{\sqrt{h^2 - 2Rh}}$ so $V = \dfrac{1}{3}\pi R^2 \dfrac{h^3}{h^2 - 2Rh} = \dfrac{1}{3}\pi R^2 \dfrac{h^2}{h - 2R}$ for $h > 2R$, $\dfrac{dV}{dh} = \dfrac{1}{3}\pi R^2 \dfrac{h(h - 4R)}{(h - 2R)^2}$, $\dfrac{dV}{dh} = 0$ for $h > 2R$ when $h = 4R$, by the first derivative test V is minimum when $h = 4R$. If $h = 4R$ then $r = \sqrt{2}R$.



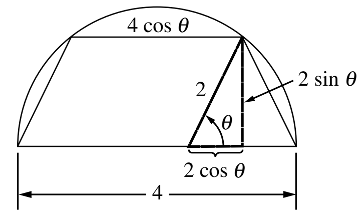
38. The area is (see figure)

$$A = \frac{1}{2}(2\sin\theta)(4 + 4\cos\theta)$$
$$= 4(\sin\theta + \sin\theta\cos\theta)$$

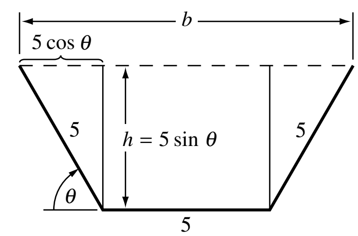
for $0 \le \theta \le \pi/2$;

$$dA/d\theta = 4(\cos\theta - \sin^2\theta + \cos^2\theta)$$
$$= 4(\cos\theta - [1 - \cos^2\theta] + \cos^2\theta)$$
$$= 4(2\cos^2\theta + \cos\theta - 1)$$
$$= 4(2\cos\theta - 1)(\cos\theta + 1)$$

$dA/d\theta = 0$ when $\theta = \pi/3$ for $0 < \theta < \pi/2$. If $\theta = 0, \pi/3, \pi/2$ then $A = 0, 3\sqrt{3}, 4$ so the maximum area is $3\sqrt{3}$.



39. Let b and h be the dimensions shown in the figure, then the cross-sectional area is $A = \dfrac{1}{2}h(5 + b)$. But $h = 5\sin\theta$ and $b = 5 + 2(5\cos\theta) = 5 + 10\cos\theta$ so $A = \dfrac{5}{2}\sin\theta(10 + 10\cos\theta) = 25\sin\theta(1 + \cos\theta)$ for $0 \le \theta \le \pi/2$.

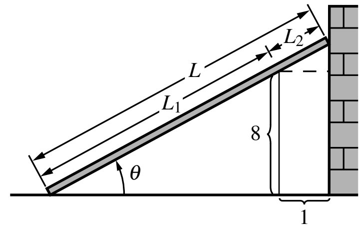


$$dA/d\theta = -25\sin^2\theta + 25\cos\theta(1 + \cos\theta)$$
$$= 25(-\sin^2\theta + \cos\theta + \cos^2\theta)$$
$$= 25(-1 + \cos^2\theta + \cos\theta + \cos^2\theta)$$
$$= 25(2\cos^2\theta + \cos\theta - 1) = 25(2\cos\theta - 1)(\cos\theta + 1).$$

$dA/d\theta = 0$ for $0 < \theta < \pi/2$ when $\cos\theta = 1/2$, $\theta = \pi/3$. If $\theta = 0, \pi/3, \pi/2$ then $A = 0, 75\sqrt{3}/4, 25$ so the cross-sectional area is greatest when $\theta = \pi/3$.

40. $I = k\dfrac{\cos\phi}{\ell^2}$, k the constant of proportionality. If h is the height of the lamp above the table then $\cos\phi = h/\ell$ and $\ell = \sqrt{h^2 + r^2}$ so $I = k\dfrac{h}{\ell^3} = k\dfrac{h}{(h^2 + r^2)^{3/2}}$ for $h > 0$, $\dfrac{dI}{dh} = k\dfrac{r^2 - 2h^2}{(h^2 + r^2)^{5/2}}$, $\dfrac{dI}{dh} = 0$ when $h = r/\sqrt{2}$, by the first derivative test I is maximum when $h = r/\sqrt{2}$.

41. Let L, L_1, and L_2 be as shown in the figure, then $L = L_1 + L_2 = 8\csc\theta + \sec\theta$,



$$\frac{dL}{d\theta} = -8\csc\theta\cot\theta + \sec\theta\tan\theta,\ 0 < \theta < \pi/2$$

$$= -\frac{8\cos\theta}{\sin^2\theta} + \frac{\sin\theta}{\cos^2\theta} = \frac{-8\cos^3\theta + \sin^3\theta}{\sin^2\theta\cos^2\theta};$$

$\dfrac{dL}{d\theta} = 0$ if $\sin^3\theta = 8\cos^3\theta$, $\tan^3\theta = 8$, $\tan\theta = 2$ which gives the absolute minimum for L because $\lim\limits_{\theta\to 0^+} L = \lim\limits_{\theta\to\pi/2^-} L = +\infty$.

If $\tan\theta = 2$, then $\csc\theta = \sqrt{5}/2$ and $\sec\theta = \sqrt{5}$ so $L = 8(\sqrt{5}/2) + \sqrt{5} = 5\sqrt{5}$ ft.

42. Let $x =$ number of steers per acre
$w =$ average market weight per steer
$T =$ total market weight per acre
then $T = xw$ where $w = 2000 - 50(x - 20) = 3000 - 50x$
so $T = x(3000 - 50x) = 3000x - 50x^2$ for $0 \le x \le 60$,
$dT/dx = 3000 - 100x$ and $dT/dx = 0$ when $x = 30$. If $x = 0, 30, 60$ then $T = 0, 45000, 0$ so the total market weight per acre is largest when 30 steers per acre are allowed.

43. **(a)** The daily profit is
$P =$ (revenue) $-$ (production cost) $= 100x - (100,000 + 50x + 0.0025x^2)$
$= -100,000 + 50x - 0.0025x^2$
for $0 \le x \le 7000$, so $dP/dx = 50 - 0.005x$ and $dP/dx = 0$ when $x = 10,000$. Because 10,000 is not in the interval $[0, 7000]$, the maximum profit must occur at an endpoint. When $x = 0$, $P = -100,000$; when $x = 7000$, $P = 127,500$ so 7000 units should be manufactured and sold daily.

(b) Yes, because $dP/dx > 0$ when $x = 7000$ so profit is increasing at this production level.

(c) $dP/dx = 15$ when $x = 7000$, so $P(7001) - P(7000) \approx 15$, and the marginal profit is \$15.

44. **(a)** $R(x) = px$ but $p = 1000 - x$ so $R(x) = (1000 - x)x$

(b) $P(x) = R(x) - C(x) = (1000 - x)x - (3000 + 20x) = -3000 + 980x - x^2$

(c) $P'(x) = 980 - 2x$, $P'(x) = 0$ for $0 < x < 500$ when $x = 490$; test the points $0, 490, 500$ to find that the profit is a maximum when $x = 490$.

(d) $P(490) = 237{,}100$

(e) $p = 1000 - x = 1000 - 490 = 510$.

45. The profit is

$$P = \text{(profit on nondefective)} - \text{(loss on defective)} = 100(x - y) - 20y = 100x - 120y$$

but $y = 0.01x + 0.00003x^2$ so $P = 100x - 120(0.01x + 0.00003x^2) = 98.8x - 0.0036x^2$ for $x > 0$, $dP/dx = 98.8 - 0.0072x$, $dP/dx = 0$ when $x = 98.8/0.0072 \approx 13,722$, $d^2P/dx^2 < 0$ so the profit is maximum at a production level of about 13,722 pounds.

46. The total cost C is
$C = c\cdot$ (hours to travel 3000 mi at a speed of v mi/h)
$= c\cdot\dfrac{3000}{v} = (a+bv^n)\dfrac{3000}{v} = 3000(av^{-1}+bv^{n-1})$ for $v>0$,
$dC/dv = 3000[-av^{-2}+b(n-1)v^{n-2}] = 3000[-a+b(n-1)v^n]/v^2$,
$dC/dv = 0$ when $v = \left[\dfrac{a}{b(n-1)}\right]^{1/n}$. This is the only critical point and dC/dv changes sign from $-$ to $+$ at this point so the total cost is least when $v = \left[\dfrac{a}{b(n-1)}\right]^{1/n}$ mi/h.

47. The distance between the particles is $D = \sqrt{(1-t-t)^2+(t-2t)^2} = \sqrt{5t^2-4t+1}$ for $t\ge 0$. For convenience, we minimize D^2 instead, so $D^2 = 5t^2-4t+1$, $dD^2/dt = 10t-4$, which is 0 when $t = 2/5$. $d^2D^2/dt^2 > 0$ so D^2 and hence D is minimum when $t = 2/5$. The minimum distance is $D = 1/\sqrt{5}$.

48. The distance between the particles is $D = \sqrt{(2t-t)^2+(2-t^2)^2} = \sqrt{t^4-3t^2+4}$ for $t\ge 0$. For convenience we minimize D^2 instead so $D^2 = t^4-3t^2+4$, $dD^2/dt = 4t^3-6t = 4t(t^2-3/2)$, which is 0 for $t>0$ when $t=\sqrt{3/2}$. $d^2D^2/dt^2 = 12t^2-6>0$ when $t=\sqrt{3/2}$ so D^2 and hence D is minimum there. The minimum distance is $D = \sqrt{7}/2$.

49. Let $P(x,y)$ be a point on the curve $x^2+y^2=1$. The distance between $P(x,y)$ and $P_0(2,0)$ is $D = \sqrt{(x-2)^2+y^2}$, but $y^2 = 1-x^2$ so $D = \sqrt{(x-2)^2+1-x^2} = \sqrt{5-4x}$ for $-1\le x\le 1$, $\dfrac{dD}{dx} = -\dfrac{2}{\sqrt{5-4x}}$ which has no critical points for $-1<x<1$. If $x=-1,1$ then $D=3,1$ so the closest point occurs when $x=1$ and $y=0$.

50. Let $P(x,y)$ be a point on $y=\sqrt{x}$, then the distance D between P and $(2,0)$ is
$D = \sqrt{(x-2)^2+y^2} = \sqrt{(x-2)^2+x} = \sqrt{x^2-3x+4}$, for $0\le x\le 3$. For convenience we find the extrema for D^2 instead, so $D^2 = x^2-3x+4$, $dD^2/dx = 2x-3 = 0$ when $x = 3/2$. If $x = 0, 3/2, 3$ then $D^2 = 4, 7/4, 4$ so $D = 2, \sqrt{7}/2, 2$. The points $(0,0)$ and $(3,\sqrt{3})$ are at the greatest distance, and $(3/2, \sqrt{3/2})$ the shortest distance from $(2,0)$.

51. Let (x,y) be a point on the curve, then the square of the distance between (x,y) and $(0,2)$ is $S = x^2+(y-2)^2$ where $x^2-y^2=1$, $x^2 = y^2+1$ so
$S = (y^2+1)+(y-2)^2 = 2y^2-4y+5$ for any y, $dS/dy = 4y-4$, $dS/dy = 0$ when $y=1$,
$d^2S/dy^2>0$ so S is least when $y=1$ and $x=\pm\sqrt{2}$.

52. The square of the distance between a point (x,y) on the curve and the point $(0,9)$ is
$S = x^2+(y-9)^2$ where $x = 2y^2$ so $S = 4y^4+(y-9)^2$ for any y,
$dS/dy = 16y^3+2(y-9) = 2(8y^3+y-9)$, $dS/dy = 0$ when $y=1$ (which is the only real solution),
$d^2S/dy^2>0$ so S is least when $y=1$, $x=2$.

53. If $P(x_0,y_0)$ is on the curve $y=1/x^2$, then $y_0 = 1/x_0^2$. At P the slope of the tangent line is $-2/x_0^3$ so its equation is $y - \dfrac{1}{x_0^2} = -\dfrac{2}{x_0^3}(x-x_0)$, or $y = -\dfrac{2}{x_0^3}x + \dfrac{3}{x_0^2}$. The tangent line crosses the y-axis at $\dfrac{3}{x_0^2}$, and the x-axis at $\dfrac{3}{2}x_0$. The length of the segment then is $L = \sqrt{\dfrac{9}{x_0^4}+\dfrac{9}{4}x_0^2}$ for $x_0>0$. For convenience, we minimize L^2 instead, so $L^2 = \dfrac{9}{x_0^4}+\dfrac{9}{4}x_0^2$, $\dfrac{dL^2}{dx_0} = -\dfrac{36}{x_0^5}+\dfrac{9}{2}x_0 = \dfrac{9(x_0^6-8)}{2x_0^5}$, which is 0 when $x_0^6 = 8$, $x_0 = \sqrt{2}$. $\dfrac{d^2L^2}{dx_0^2}>0$ so L^2 and hence L is minimum when $x_0=\sqrt{2}$, $y_0 = 1/2$.

54. If $P(x_0, y_0)$ is on the curve $y = 1 - x^2$, then $y_0 = 1 - x_0^2$. At P the slope of the tangent line is $-2x_0$ so its equation is $y - (1 - x_0^2) = -2x_0(x - x_0)$, or $y = -2x_0 x + x_0^2 + 1$. The y-intercept is $x_0^2 + 1$ and the x-intercept is $\frac{1}{2}(x_0 + 1/x_0)$ so the area A of the triangle is $A = \frac{1}{4}(x_0^2 + 1)(x_0 + 1/x_0) = \frac{1}{4}(x_0^3 + 2x_0 + 1/x_0)$ for $0 \le x_0 \le 1$. $dA/dx_0 = \frac{1}{4}(3x_0^2 + 2 - 1/x_0^2) = \frac{1}{4}(3x_0^4 + 2x_0^2 - 1)/x_0^2$ which is 0 when $x_0^2 = -1$ (reject), or when $x_0^2 = 1/3$ so $x_0 = 1/\sqrt{3}$. $d^2A/dx_0^2 = \frac{1}{4}(6x_0 + 2/x_0^3) > 0$ at $x_0 = 1/\sqrt{3}$ so a relative minimum and hence the absolute minimum occurs there.

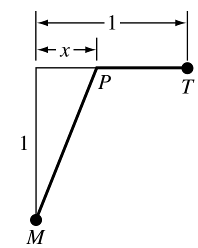
55. At each point (x, y) on the curve the slope of the tangent line is $m = \frac{dy}{dx} = -\frac{2x}{(1+x^2)^2}$ for any x, $\frac{dm}{dx} = \frac{2(3x^2 - 1)}{(1+x^2)^3}$, $\frac{dm}{dx} = 0$ when $x = \pm 1/\sqrt{3}$, by the first derivative test the only relative maximum occurs at $x = -1/\sqrt{3}$, which is the absolute maximum because $\lim_{x \to \pm\infty} m = 0$. The tangent line has greatest slope at the point $(-1/\sqrt{3}, 3/4)$.

56. Let x be how far P is upstream from where the man starts (see figure), then the total time to reach T is

$t =$ (time from M to P) + (time from P to T)

$$= \frac{\sqrt{x^2+1}}{r_R} + \frac{1-x}{r_W} \text{ for } 0 \le x \le 1,$$

where r_R and r_W are the rates at which he can row and walk, respectively.



(a) $t = \frac{\sqrt{x^2+1}}{3} + \frac{1-x}{5}$, $\frac{dt}{dx} = \frac{x}{3\sqrt{x^2+1}} - \frac{1}{5}$ so $\frac{dt}{dx} = 0$ when $5x = 3\sqrt{x^2+1}$, $25x^2 = 9(x^2+1), x^2 = 9/16$, $x = 3/4$. If $x = 0, 3/4, 1$ then $t = 8/15, 7/15, \sqrt{2}/3$ so the time is a minimum when $x = 3/4$ mile.

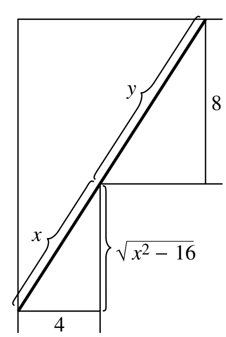
(b) $t = \frac{\sqrt{x^2+1}}{4} + \frac{1-x}{5}$, $\frac{dt}{dx} = \frac{x}{4\sqrt{x^2+1}} - \frac{1}{5}$ so $\frac{dt}{dx} = 0$ when $x = 4/3$ which is not in the interval $[0, 1]$. Check the endpoints to find that the time is a minimum when $x = 1$ (he should row directly to the town).

57. With x and y as shown in the figure, the maximum length of pipe will be the smallest value of $L = x + y$. By similar triangles

$\frac{y}{8} = \frac{x}{\sqrt{x^2-16}}$, $y = \frac{8x}{\sqrt{x^2-16}}$ so

$L = x + \frac{8x}{\sqrt{x^2-16}}$ for $x > 4$, $\frac{dL}{dx} = 1 - \frac{128}{(x^2-16)^{3/2}}$,

$\frac{dL}{dx} = 0$ when



$$\begin{aligned}(x^2-16)^{3/2} &= 128\\ x^2 - 16 &= 128^{2/3} = 16(2^{2/3})\\ x^2 &= 16(1 + 2^{2/3})\\ x &= 4(1 + 2^{2/3})^{1/2},\end{aligned}$$

$d^2L/dx^2 = 384x/(x^2-16)^{5/2} > 0$ if $x > 4$ so L is smallest when $x = 4(1 + 2^{2/3})^{1/2}$. For this value of x, $L = 4(1 + 2^{2/3})^{3/2}$ ft.

58.
$$s = (x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2,$$
$$ds/d\bar{x} = -2(x_1 - \bar{x}) - 2(x_2 - \bar{x}) - \cdots - 2(x_n - \bar{x}),$$
$ds/d\bar{x} = 0$ when

$$(x_1 - \bar{x}) + (x_2 - \bar{x}) + \cdots + (x_n - \bar{x}) = 0$$
$$(x_1 + x_2 + \cdots x_n) - (\bar{x} + \bar{x} + \cdots + \bar{x}) = 0$$
$$(x_1 + x_2 + \cdots + x_n) - n\bar{x} = 0$$
$$\bar{x} = \frac{1}{n}(x_1 + x_2 + \cdots + x_n),$$

$d^2s/d\bar{x}^2 = 2 + 2 + \cdots + 2 = 2n > 0$, so s is minimum when $\bar{x} = \dfrac{1}{n}(x_1 + x_2 + \cdots + x_n)$.

59. Let x = distance from the weaker light source, I = the intensity at that point, and k the constant of proportionality. Then

$I = \dfrac{kS}{x^2} + \dfrac{8kS}{(90-x)^2}$ if $0 < x < 90$;

$$\frac{dI}{dx} = -\frac{2kS}{x^3} + \frac{16kS}{(90-x)^3} = \frac{2kS[8x^3 - (90-x)^3]}{x^3(90-x)^3} = 18\frac{kS(x-30)(x^2+2700)}{x^3(x-90)^3},$$

which is 0 when $x = 30$; $\dfrac{dI}{dx} < 0$ if $x < 30$, and $\dfrac{dI}{dx} > 0$ if $x > 30$, so the intensity is minimum at a distance of 30 cm from the weaker source.

60. If $f(x_0)$ is a maximum then $f(x) \le f(x_0)$ for all x in some open interval containing x_0 thus $\sqrt{f(x)} \le \sqrt{f(x_0)}$ because $\sqrt{x}$ is an increasing function, so $\sqrt{f(x_0)}$ is a maximum of $\sqrt{f(x)}$ at x_0. The proof is similar for a minimum value, simply replace $\le$ by $\ge$.

61. Let v = speed of light in the medium. The total time required for the light to travel from A to P to B is

$t =$ (total distance from A to P to B)$/v = \dfrac{1}{v}(\sqrt{(c-x)^2 + a^2} + \sqrt{x^2 + b^2})$,

$$\frac{dt}{dx} = \frac{1}{v}\left[-\frac{c-x}{\sqrt{(c-x)^2+a^2}} + \frac{x}{\sqrt{x^2+b^2}}\right]$$

and $\dfrac{dt}{dx} = 0$ when $\dfrac{x}{\sqrt{x^2+b^2}} = \dfrac{c-x}{\sqrt{(c-x)^2+a^2}}$. But $x/\sqrt{x^2+b^2} = \sin\theta_2$ and

$(c-x)/\sqrt{(c-x)^2+a^2} = \sin\theta_1$ thus $dt/dx = 0$ when $\sin\theta_2 = \sin\theta_1$ so $\theta_2 = \theta_1$.

62. The total time required for the light to travel from A to P to B is

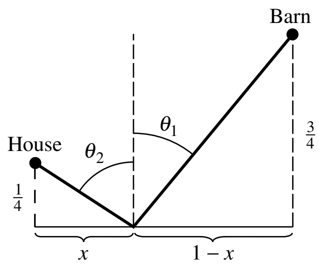
$t =$ (time from A to P) + (time from P to B) $= \dfrac{\sqrt{x^2+a^2}}{v_1} + \dfrac{\sqrt{(c-x)^2+b^2}}{v_2}$,

$\dfrac{dt}{dx} = \dfrac{x}{v_1\sqrt{x^2+a^2}} - \dfrac{c-x}{v_2\sqrt{(c-x)^2+b^2}}$ but $x/\sqrt{x^2+a^2} = \sin\theta_1$ and

$(c-x)/\sqrt{(c-x)^2+b^2} = \sin\theta_2$ thus $\dfrac{dt}{dx} = \dfrac{\sin\theta_1}{v_1} - \dfrac{\sin\theta_2}{v_2}$ so $\dfrac{dt}{dx} = 0$ when $\dfrac{\sin\theta_1}{v_1} = \dfrac{\sin\theta_2}{v_2}$.

63. **(a)** The rate at which the farmer walks is analogous to the speed of light in Fermat's principle.

(b) the best path occurs when $\theta_1 = \theta_2$ (see figure).



(c) by similar triangles,

$$x/(1/4) = (1-x)/(3/4)$$
$$3x = 1 - x$$
$$4x = 1$$
$$x = 1/4 \text{ mi.}$$

EXERCISE SET 4.7

1. $f(x) = x^2 - 2,\ f'(x) = 2x,\ x_{n+1} = x_n - \dfrac{x_n^2 - 2}{2x_n}$

$x_1 = 1,\ x_2 = 1.5,\ x_3 = 1.416666667, \ldots,\ x_5 = x_6 = 1.414213562$

2. $f(x) = x^2 - 7,\ f'(x) = 2x,\ x_{n+1} = x_n - \dfrac{x_n^2 - 7}{2x_n}$

$x_1 = 3,\ x_2 = 2.666666667,\ x_3 = 2.645833333, \ldots,\ x_5 = x_6 = 2.645751311$

3. $f(x) = x^3 - 6,\ f'(x) = 3x^2,\ x_{n+1} = x_n - \dfrac{x_n^3 - 6}{3x_n^2}$

$x_1 = 2,\ x_2 = 1.833333333,\ x_3 = 1.817263545, \ldots,\ x_5 = x_6 = 1.817120593$

4. $x^n - a = 0$

5. $f(x) = x^3 - x + 3,\ f'(x) = 3x^2 - 1,\ x_{n+1} = x_n - \dfrac{x_n^3 - x_n + 3}{3x_n^2 - 1}$

$x_1 = -2,\ x_2 = -1.727272727,\ x_3 = -1.673691174, \ldots, x_5 = x_6 = -1.671699882$

6. $f(x) = x^3 + x - 1,\ f'(x) = 3x^2 + 1,\ x_{n+1} = x_n - \dfrac{x_n^3 + x_n - 1}{3x_n^2 + 1}$

$x_1 = 1,\ x_2 = 0.75,\ x_3 = 0.686046512, \ldots,\ x_5 = x_6 = 0.682327804$

7. $f(x) = x^5 + x^4 - 5,\ f'(x) = 5x^4 + 4x^3,\ x_{n+1} = x_n - \dfrac{x_n^5 + x_n^4 - 5}{5x_n^4 + 4x_n^3}$

$x_1 = 1,\ x_2 = 1.333333333,\ x_3 = 1.239420573, \ldots, x_6 = x_7 = 1.224439550$

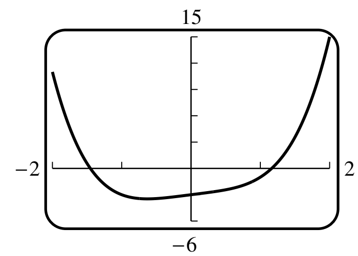
8. $f(x) = x^5 - x + 1,\ f'(x) = 5x^4 - 1,\ x_{n+1} = x_n - \dfrac{x_n^5 - x_n + 1}{5x_n^4 - 1}$

$x_1 = -1,\ x_2 = -1.25,\ x_3 = -1.178459394, \ldots, x_6 = x_7 = -1.167303978$

9. $f(x) = x^4 + x - 3,\ f'(x) = 4x^3 + 1,$

$$x_{n+1} = x_n - \frac{x_n^4 + x_n - 3}{4x_n^3 + 1}$$

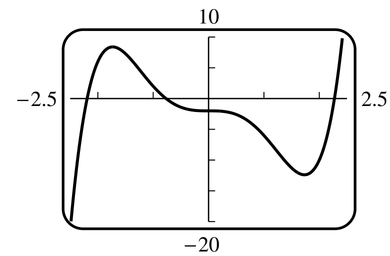
$x_1 = -2,\ x_2 = -1.645161290,$
$x_3 = -1.485723955, \ldots, x_6 = x_7 = -1.452626879$



10. $f(x) = x^5 - 5x^3 - 2,\ f'(x) = 5x^4 - 15x^2,$

$$x_{n+1} = x_n - \frac{x_n^5 - 5x_n^3 - 2}{5x_n^4 - 15x_n^2}$$

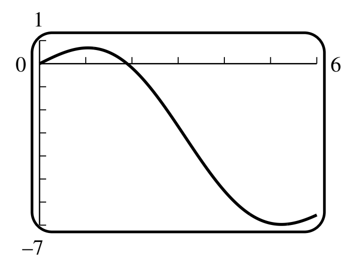
$x_1 = 2,\ x_2 = 2.5,$
$x_3 = 2.327384615, \ldots, x_7 = x_8 = 2.273791732$



11. $f(x) = 2\sin x - x,\ f'(x) = 2\cos x - 1,$

$$x_{n+1} = x_n - \frac{2\sin x_n - x_n}{2\cos x_n - 1}$$

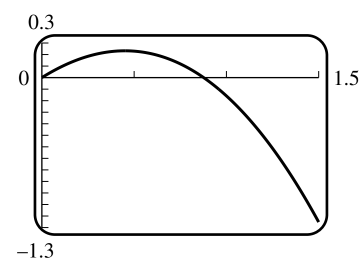
$x_1 = 2,\ x_2 = 1.900995594,$
$x_3 = 1.895511645,\ x_4 = x_5 = 1.895494267$



12. $f(x) = \sin x - x^2,$
$f'(x) = \cos x - 2x,$

$$x_{n+1} = x_n - \frac{\sin x_n - x_n^2}{\cos x_n - 2x_n}$$

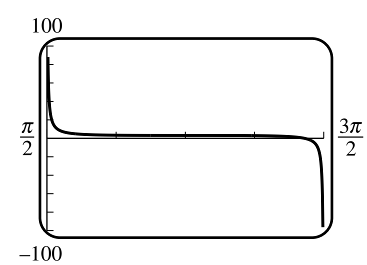
$x_1 = 1,\ x_2 = 0.891395995,$
$x_3 = 0.876984845, \ldots, x_5 = x_6 = 0.876726215$



13. $f(x) = x - \tan x,$

$$f'(x) = 1 - \sec^2 x = -\tan^2 x,\ x_{n+1} = x_n + \frac{x_n - \tan x_n}{\tan^2 x_n}$$

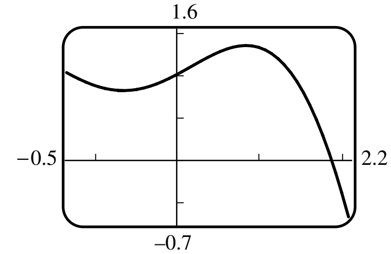
$x_1 = 4.5,\ x_2 = 4.493613903,\ x_3 = 4.493409655,$
$x_4 = x_5 = 4.493409458$



14. $f(x) = 1 + x^2 \cos x,\ f'(x) = 2x \cos x - x^2 \sin x$

$$x_{n+1} = x_n - \frac{1 + x_n^2 \cos x_n}{2x_n \cos x_n - x_n^2 \sin x_n}$$

$x_1 = 2,\ x_2 = 1.8746,\ x_3 = 1.8631,$
$x_4 = 1.863045316, x_5 = x_6 = 1.863045308$

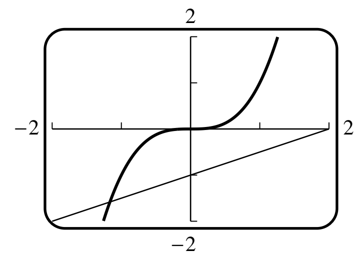


15. At the point of intersection, $x^3 = 0.5x - 1$, $x^3 - 0.5x + 1 = 0$. Let $f(x) = x^3 - 0.5x + 1$. By graphing $y = x^3$ and $y = 0.5x - 1$ it is evident that there is only one point of intersection and it occurs in the interval $[-2, -1]$; note that $f(-2) < 0$ and $f(-1) > 0$.
$f'(x) = 3x^2 - 0.5$ so

$$x_{n+1} = x_n - \frac{x_n^3 - 0.5x + 1}{3x_n^2 - 0.5};$$

$x_1 = -1,\ x_2 = -1.2,$

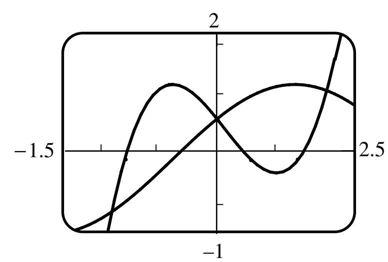
$x_3 = -1.166492147, \ldots,$
$x_5 = x_6 = -1.165373043$



16. The graphs of $y = \sin x$ and $y = x^3 - 2x^2 + 1$ intersect at points near $x = -0.8$ and $x = 0.6$ and $x = 2$. Let $f(x) = \sin x - x^3 + 2x^2 - 1$, then $f'(x) = \cos x - 3x^2 + 4x$, so

$$x_{n+1} = x_n - \frac{\cos x - 3x^2 + 4x}{\sin x - x^3 + 2x^2 + 1}.$$

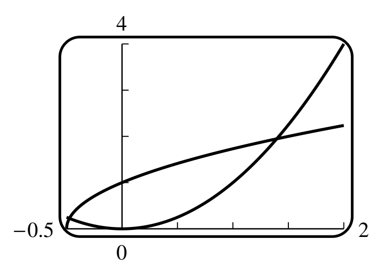
If $x_1 = -0.8$, then $x_2 = -0.783124811$,
$x_3 = -0.782808234$,
$x_4 = x_5 = -0.782808123$; ; if $x_1 = 0.6$, then
$x_2 = 0.568003853, x_3 = x_4 = 0.568025739$; if $x_1 = 2$, then
$x_2 = 1.979461151,\ x_3 = 1.979019264, x_4 = x_5 = 1.979019061$



17. The graphs of $y = x^2$ and $y = \sqrt{2x + 1}$ intersect at points near $x = -0.5$ and $x = 1$; $x^2 = \sqrt{2x + 1}$, $x^4 - 2x - 1 = 0$. Let $f(x) = x^4 - 2x - 1$, then $f'(x) = 4x^3 - 2$ so

$$x_{n+1} = x_n - \frac{x_n^4 - 2x_n - 1}{4x_n^3 - 2}.$$

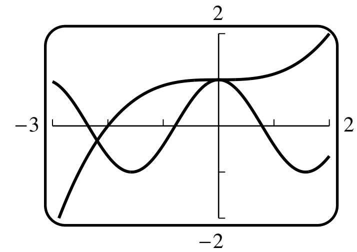
If $x_1 = -0.5$, then $x_2 = -0.475$,
$x_3 = -0.474626695$,
$x_4 = x_5 = -0.474626618$; if
$x_1 = 1$, then $x_2 = 2$,
$x_3 = 1.633333333, \ldots, x_8 = x_9 = 1.395336994.$



18. The graphs of $y = x^3/8 + 1$ and $y = \cos 2x$ intersect at $x = 0$ and at a point near $x = -2$; $x^3/8 + 1 = \cos 2x$, $x^3 - 8\cos 2x + 8 = 0$. Let $f(x) = x^3 - 8\cos 2x + 8$, then $f'(x) = 3x^2 + 16\sin 2x$ so

$$x_{n+1} = x_n - \frac{x_n^3 - 8\cos 2x_n + 8}{3x_n^2 + 16\sin 2x_n}.$$

$x_1 = -2$, $x_2 = -2.216897577$, $x_3 = -2.193821581, \ldots, x_5 = x_6 = -2.193618950$.



19. **(a)** $f(x) = x^2 - a$, $f'(x) = 2x$, $x_{n+1} = x_n - \dfrac{x_n^2 - a}{2x_n} = \dfrac{1}{2}\left(x_n + \dfrac{a}{x_n}\right)$

(b) $a = 10$; $x_1 = 3$, $x_2 = 3.166666667$, $x_3 = 3.162280702$, $x_4 = x_5 = 3.162277660$

20. **(a)** $f(x) = \dfrac{1}{x} - a$, $f'(x) = -\dfrac{1}{x^2}$, $x_{n+1} = x_n(2 - ax_n)$

(b) $a = 17$; $x_1 = 0.05$, $x_2 = 0.0575$, $x_3 = 0.058793750$, $x_5 = x_6 = 0.058823529$

21. $f'(x) = x^3 + 2x + 5$; solve $f'(x) = 0$ to find the critical points. Graph $y = x^3$ and $y = -2x - 5$ to see that they intersect at a point near $x = -1$; $f''(x) = 3x^2 + 2$ so $x_{n+1} = x_n - \dfrac{x_n^3 + 2x_n + 5}{3x_n^2 + 2}$.
$x_1 = -1$, $x_2 = -1.4$, $x_3 = -1.330964467, \ldots, x_5 = x_6 = -1.328268856$ so the minimum value of $f(x)$ occurs at $x \approx -1.328268856$ because $f''(x) > 0$; its value is approximately -4.098859132.

22. From a rough sketch of $y = x\sin x$ we see that the maximum occurs at a point near $x = 2$, which will be a point where $f'(x) = x\cos x + \sin x = 0$. $f''(x) = 2\cos x - x\sin x$ so

$$x_{n+1} = x_n - \frac{x_n\cos x_n + \sin x_n}{2\cos x_n - x_n\sin x_n} = x_n - \frac{x_n + \tan x_n}{2 - x_n\tan x_n}.$$

$x_1 = 2$, $x_2 = 2.029048281$, $x_3 = 2.028757866$, $x_4 = x_5 = 2.028757838$; the maximum value is approximately 1.819705741.

23. Let $f(x)$ be the square of the distance between $(1, 0)$ and any point (x, x^2) on the parabola, then $f(x) = (x-1)^2 + (x^2 - 0)^2 = x^4 + x^2 - 2x + 1$ and $f'(x) = 4x^3 + 2x - 2$. Solve $f'(x) = 0$ to find the critical points; $f''(x) = 12x^2 + 2$ so $x_{n+1} = x_n - \dfrac{4x_n^3 + 2x_n - 2}{12x_n^2 + 2} = x_n - \dfrac{2x_n^3 + x_n - 1}{6x_n^2 + 1}$.
$x_1 = 1$, $x_2 = 0.714285714$, $x_3 = 0.605168701, \ldots, x_6 = x_7 = 0.589754512$; the coordinates are approximately $(0.589754512, 0.347810385)$.

24. The area is $A = xy = x\cos x$ so $dA/dx = \cos x - x\sin x$. Find x so that $dA/dx = 0$;

$$d^2A/dx^2 = -2\sin x - x\cos x \text{ so } x_{n+1} = x_n + \frac{\cos x_n - x_n\sin x_n}{2\sin x_n + x_n\cos x_n} = x_n + \frac{1 - x_n\tan x_n}{2\tan x_n + x_n}.$$

$x_1 = 1$, $x_2 = 0.864536397$, $x_3 = 0.860339078$, $x_4 = x_5 = 0.860333589$; $y \approx 0.652184624$.

25. **(a)** Let s be the arc length, and L the length of the chord, then $s = 1.5L$. But $s = r\theta$ and $L = 2r\sin(\theta/2)$ so $r\theta = 3r\sin(\theta/2)$, $\theta - 3\sin(\theta/2) = 0$.

(b) Let $f(\theta) = \theta - 3\sin(\theta/2)$, then $f'(\theta) = 1 - 1.5\cos(\theta/2)$ so $\theta_{n+1} = \theta_n - \dfrac{\theta_n - 3\sin(\theta_n/2)}{1 - 1.5\cos(\theta_n/2)}$.

$\theta_1 = 3$, $\theta_2 = 2.991592920$, $\theta_3 = 2.991563137$, $\theta_4 = \theta_5 = 2.991563136$ rad so $\theta \approx 171°$.

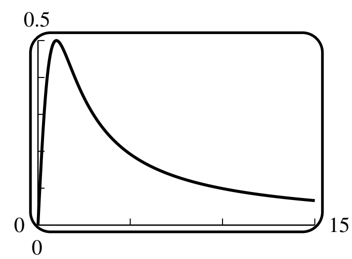
26. $r^2(\theta - \sin\theta)/2 = \pi r^2/4$ so $\theta - \sin\theta - \pi/2 = 0$. Let $f(\theta) = \theta - \sin\theta - \pi/2$, then $f'(\theta) = 1 - \cos\theta$ so $\theta_{n+1} = \dfrac{\theta_n - \sin\theta_n - \pi/2}{1 - \cos\theta_n}$.
$\theta_1 = 2$, $\theta_2 = 2.339014106$, $\theta_3 = 2.310063197, \ldots, \theta_5 = \theta_6 = 2.309881460$ rad; $\theta \approx 132°$.

27. If $x = 1$, then $y^4 + y = 1$, $y^4 + y - 1 = 0$. Graph $z = y^4$ and $z = 1 - y$ to see that they intersect near $y = -1$ and $y = 1$. Let $f(y) = y^4 + y - 1$, then $f'(y) = 4y^3 + 1$ so $y_{n+1} = y_n - \dfrac{y_n^4 + y_n - 1}{4y_n^3 + 1}$.
If $y_1 = -1$, then $y_2 = -1.333333333$, $y_3 = -1.235807860, \ldots, y_6 = y_7 = -1.220744085$;
if $y_1 = 1$, then $y_2 = 0.8$, $y_3 = 0.731233596, \ldots, y_6 = y_7 = 0.724491959$.

28. If $x = 1$, then $2y - \cos y = 0$. Graph $z = 2y$ and $z = \cos y$ to see that they intersect near $y = 0.5$. Let $f(y) = 2y - \cos y$, then $f'(y) = 2 + \sin y$ so $y_{n+1} = y_n - \dfrac{2y_n - \cos y_n}{2 + \sin y_n}$.
$y_1 = 0.5$, $y_2 = 0.450626693$, $y_3 = 0.450183648$, $y_4 = y_5 = 0.450183611$.

29. $S(25) = 250{,}000 = \dfrac{5000}{i}\left[(1+i)^{25} - 1\right]$; set $f(i) = 50i - (1+i)^{25} + 1$, $f'(i) = 50 - 25(1+i)^{24}$; solve $f(i) = 0$. Set $i_0 = .06$ and $i_{k+1} = i_k - \left[50i - (1+i)^{25} + 1\right] / \left[50 - 25(1+i)^{24}\right]$. Then $i_1 = 0.05430$, $i_2 = 0.05338$, $i_3 = 0.05336$, $\ldots$, $i = 0.053362$.

30. **(a)** $x_1 = 2$, $x_2 = 5.3333$,
$x_3 = 11.055$, $x_4 = 22.293$,
$x_5 = 44.676$



(b) $x_1 = 0.5$, $x_2 = -0.3333$, $x_3 = 0.0833$, $x_4 = -0.0012$, $x_5 = 0.0000$ (and $x_n = 0$ for $n \geq 6$)

31. **(a)**

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
0.5000	−0.7500	0.2917	−1.5685	−0.4654	0.8415	−0.1734	2.7970	1.2197	0.1999

(b) The sequence x_n must diverge, since if it did converge then $f(x) = x^2 + 1 = 0$ would have a solution. It seems the x_n are oscillating back and forth in a quasi-cyclical fashion.

EXERCISE SET 4.8

1. $f(0) = f(4) = 0$; $f'(3) = 0$; $[0, 4]$, $c = 3$

2. $f(-3) = f(3) = 0$; $f'(0) = 0$

3. $f(2) = f(4) = 0$, $f'(x) = 2x - 6$, $2c - 6 = 0$, $c = 3$

4. $f(0) = f(2) = 0$, $f'(x) = 3x^2 - 6x + 2$, $3c^2 - 6c + 2 = 0$; $c = \dfrac{6 \pm \sqrt{36 - 24}}{6} = 1 \pm \sqrt{3}/3$

5. $f(\pi/2) = f(3\pi/2) = 0$, $f'(x) = -\sin x$, $-\sin c = 0$, $c = \pi$

6. $f(-1) = f(1) = 0$, $f'(x) = \dfrac{x^2 - 4x + 1}{(x-2)^2}$, $\dfrac{c^2 - 4c + 1}{(c-2)^2} = 0$, $c^2 - 4c + 1 = 0$

$c = \dfrac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$, of which only $c = 2 - \sqrt{3}$ is in $(-1, 1)$

7. $f(0) = f(4) = 0$, $f'(x) = \dfrac{1}{2} - \dfrac{1}{2\sqrt{x}}$, $\dfrac{1}{2} - \dfrac{1}{2\sqrt{c}} = 0$, $c = 1$

8. $f(1) = f(3) = 0$, $f'(x) = -\dfrac{2}{x^3} + \dfrac{4}{3x^2}$, $-\dfrac{2}{c^3} + \dfrac{4}{3c^2} = 0$, $-6 + 4c = 0$, $c = 3/2$

9. $\dfrac{f(8) - f(0)}{8 - 0} = \dfrac{6}{8} = \dfrac{3}{4} = f'(1.54)$; $c = 1.54$

10. $\dfrac{f(4) - f(0)}{4 - 0} = 1.19 = f'(0.77)$

11. $f(-4) = 12$, $f(6) = 42$, $f'(x) = 2x + 1$, $2c + 1 = \dfrac{42 - 12}{6 - (-4)} = 3$, $c = 1$

12. $f(-1) = -6$, $f(2) = 6$, $f'(x) = 3x^2 + 1$, $3c^2 + 1 = \dfrac{6 - (-6)}{2 - (-1)} = 4$, $c^2 = 1$, $c = \pm 1$ of which only $c = 1$ is in $(-1, 2)$

13. $f(0) = 1$, $f(3) = 2$, $f'(x) = \dfrac{1}{2\sqrt{x+1}}$, $\dfrac{1}{2\sqrt{c+1}} = \dfrac{2-1}{3-0} = \dfrac{1}{3}$, $\sqrt{c+1} = 3/2$, $c + 1 = 9/4$, $c = 5/4$

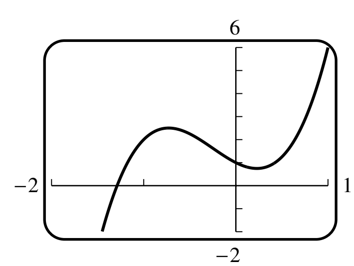
14. $f(3) = 10/3$, $f(4) = 17/4$, $f'(x) = 1 - 1/x^2$, $1 - 1/c^2 = \dfrac{17/4 - 10/3}{4 - 3} = 11/12$, $c^2 = 12$, $c = \pm 2\sqrt{3}$ of which only $c = 2\sqrt{3}$ is in $(3, 4)$

15. $f(-5) = 0$, $f(3) = 4$, $f'(x) = -\dfrac{x}{\sqrt{25 - x^2}}$, $-\dfrac{c}{\sqrt{25 - c^2}} = \dfrac{4 - 0}{3 - (-5)} = \dfrac{1}{2}$, $-2c = \sqrt{25 - c^2}$,
$4c^2 = 25 - c^2$, $c^2 = 5$, $c = -\sqrt{5}$
(we reject $c = \sqrt{5}$ because it does not satisfy the equation $-2c = \sqrt{25 - c^2}$)

16. $f(2) = 1$, $f(5) = 1/4$, $f'(x) = -1/(x-1)^2$, $-\dfrac{1}{(c-1)^2} = \dfrac{1/4 - 1}{5 - 2} = -\dfrac{1}{4}$, $(c-1)^2 = 4$, $c - 1 = \pm 2$, $c = -1$ (reject), or $c = 3$

17. **(a)** $f(-2) = f(1) = 0$
The interval is $[-2, 1]$

(b) $c = -1.29$



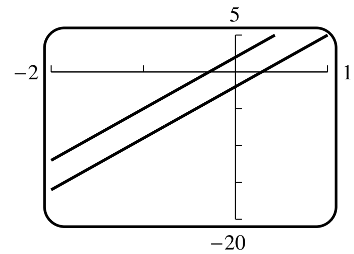
(c) $x_0 = -1$, $x_1 = -1.5$, $x_2 = -1.32$, $x_3 = -1.290$, $x_4 = -1.2885843$

18. **(a)** $m = \dfrac{f(-2) - f(1)}{-2 - 1} = \dfrac{-16 - 5}{-3} = 7$ so $y - 5 = 7(x - 1)$, $y = 7x - 2$

(b) $f'(x) = 3x^2 + 4 = 7$ has solutions $x = \pm 1$; discard $x = 1$, so $c = -1$

(c) $y - (-5) = 7(x - (-1))$ or $y = 7x + 2$

(d)



19. **(a)** $f'(x) = \sec^2 x$, $\sec^2 c = 0$ has no solution **(b)** $\tan x$ is not continuous on $[0, \pi]$

20. **(a)** $f(-1) = 1$, $f(8) = 4$, $f'(x) = \dfrac{2}{3}x^{-1/3}$

$\dfrac{2}{3}c^{-1/3} = \dfrac{4 - 1}{8 - (-1)} = \dfrac{1}{3}$, $c^{1/3} = 2$, $c = 8$ which is not in $(-1, 8)$.

(b) $x^{2/3}$ is not differentiable at $x = 0$, which is in $(-1, 8)$.

21. **(a)** Two x-intercepts of f determine two solutions a and b of $f(x) = 0$; by Rolle's Theorem there exists a point c between a and b such that $f'(c) = 0$, i.e. c is an x-intercept for f'.

(b) $f(x) = \sin x = 0$ at $x = n\pi$, and $f'(x) = \cos x = 0$ at $x = n\pi + \pi/2$, which lies between $n\pi$ and $(n + 1)\pi$, $(n = 0, \pm 1, \pm 2, \ldots)$

22. $\dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$ is the average rate of change of y with respect to x on the interval $[x_0, x_1]$. By the Mean-Value Theorem there is a value c in (x_0, x_1) such that the instantaneous rate of change $f'(c) = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$.

23. Let $s(t)$ be the position function of the automobile for $0 \le t \le 5$, then by the Mean-Value Theorem there is at least one point c in $(0, 5)$ where
$s'(c) = v(c) = [s(5) - s(0)]/(5 - 0) = 4/5 = 0.8$ mi/min $= 48$ mi/h.

24. Let $T(t)$ denote the temperature at time with $t = 0$ denoting 11 AM, then $T(0) = 76$ and $T(12) = 52$.

(a) By the Mean-Value Theorem there is a value c between 0 and 12 such that
$T'(c) = [T(12) - T(0)]/(12 - 0) = (52 - 76)/(12) = -2°$ F/h.

(b) Assume that $T(t_1) = 88°$F where $0 < t_1 < 12$, then there is at least one point c in $(t_1, 12)$ where $T'(c) = [T(12) - T(t_1)]/(12 - t_1) = (52 - 88)/(12 - t_1) = -36/(12 - t_1)$. But $12 - t_1 < 12$ so $T'(c) < -36/12 = -3°$F/h.

25. Let $f(t)$ and $g(t)$ denote the distances from the first and second runners to the starting point, and let $h(t) = f(t) - g(t)$. Since they start (at $t = 0$) and finish (at $t = t_1$) at the same time, $h(0) = h(t_1) = 0$, so by Rolle's Theorem there is a time t_2 for which $h'(t_2) = 0$, i.e. $f'(t_2) = g'(t_2)$; so they have the same velocity at time t_2.

26. $f(x) = x^6 - 2x^2 + x$ satisfies $f(0) = f(1) = 0$, so by Rolle's Theorem $f'(c) = 0$ for some c in $(0, 1)$.

27. **(a)** By the Constant Difference Theorem $f(x) - g(x) = k$ for some k; since $f(x_0) = g(x_0)$, $k = 0$, so $f(x) = g(x)$ for all x.

(b) Set $f(x) = \sin^2 x + \cos^2 x$, $g(x) = 1$; then $f'(x) = 2\sin x\cos x - 2\cos x\sin x = 0 = g'(x)$. Since $f(0) = 1 = g(0)$, $f(x) = g(x)$ for all x.

28. **(a)** By the Constant Difference Theorem $f(x) - g(x) = k$ for some k; since $f(x_0) - g(x_0) = c$, $k = c$, so $f(x) - g(x) = c$ for all x.

(b) Set $f(x) = (x-1)^3$, $g(x) = (x^2+3)(x-3)$. Then
$f'(x) = 3(x-1)^2$, $g'(x) = (x^2+3) + 2x(x-3) = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x-1)^2$, so $f'(x) = g'(x)$ and hence $f(x) - g(x) = k$. Expand $f(x)$ and $g(x)$ to get
$h(x) = f(x) - g(x) = (x^3 - 3x^2 + 3x - 1) - (x^3 - 3x^2 + 3x - 9) = 8$.

(c) $h(x) = x^3 - 3x^2 + 3x - 1 - (x^3 - 3x^2 + 3x - 9) = 8$

29. **(a)** If x, y belong to I and $x < y$ then for some c in I, $\dfrac{f(y) - f(x)}{y - x} = f'(c)$,
so $|f(x) - f(y)| = |f'(c)||x - y| \le M|x - y|$; if $x > y$ exchange x and y; if $x = y$ the inequality also holds.

(b) $f(x) = \sin x$, $f'(x) = \cos x$, $|f'(x)| \le 1 = M$, so $|f(x) - f(y)| \le |x - y|$ or
$|\sin x - \sin y| \le |x - y|$.

30. **(a)** If x, y belong to I and $x < y$ then for some c in I, $\dfrac{f(y) - f(x)}{y - x} = f'(c)$,
so $|f(x) - f(y)| = |f'(c)||x - y| \ge M|x - y|$; if $x > y$ exchange x and y; if $x = y$ the inequality also holds.

(b) If x and y belong to $(-\pi/2, \pi/2)$ and $f(x) = \tan x$, then $|f'(x)| = \sec^2 x \ge 1$ and
$|\tan x - \tan y| \ge |x - y|$

(c) y lies in $(-\pi/2, \pi/2)$ if and only if $-y$ does; use Part (b) and replace y with $-y$

31. **(a)** Let $f(x) = \sqrt{x}$. By the Mean-Value Theorem there is a number c between x and y such that
$\dfrac{\sqrt{y} - \sqrt{x}}{y - x} = \dfrac{1}{2\sqrt{c}} < \dfrac{1}{2\sqrt{x}}$ for c in (x, y), thus $\sqrt{y} - \sqrt{x} < \dfrac{y - x}{2\sqrt{x}}$

(b) multiply through and rearrange to get $\sqrt{xy} < \dfrac{1}{2}(x + y)$.

32. Suppose that $f(x)$ has at least two distinct real solutions r_1 and r_2 in I. Then $f(r_1) = f(r_2) = 0$ so by Rolle's Theorem there is at least one number between r_1 and r_2 where $f'(x) = 0$, but this contradicts the assumption that $f'(x) \neq 0$, so $f(x) = 0$ must have fewer than two distinct solutions in I.

33. **(a)** If $f(x) = x^3 + 4x - 1$ then $f'(x) = 3x^2 + 4$ is never zero, so by Exercise 32 f has at most one real root; since f is a cubic polynomial it has at least one real root, so it has exactly one real root.

(b) Let $f(x) = ax^3 + bx^2 + cx + d$. If $f(x) = 0$ has at least two distinct real solutions r_1 and r_2, then $f(r_1) = f(r_2) = 0$ and by Rolle's Theorem there is at least one number between r_1 and r_2 where $f'(x) = 0$. But $f'(x) = 3ax^2 + 2bx + c = 0$ for
$x = (-2b \pm \sqrt{4b^2 - 12ac})/(6a) = (-b \pm \sqrt{b^2 - 3ac})/(3a)$, which are not real if $b^2 - 3ac < 0$ so $f(x) = 0$ must have fewer than two distinct real solutions.

34. $f'(x) = \dfrac{1}{2\sqrt{x}}$, $\dfrac{1}{2\sqrt{c}} = \dfrac{\sqrt{4} - \sqrt{3}}{4 - 3} = 2 - \sqrt{3}$. But $\dfrac{1}{4} < \dfrac{1}{2\sqrt{c}} < \dfrac{1}{2\sqrt{3}}$ for c in $(3, 4)$ so
$\dfrac{1}{4} < 2 - \sqrt{3} < \dfrac{1}{2\sqrt{3}}$, $0.25 < 2 - \sqrt{3} < 0.29$, $-1.75 < -\sqrt{3} < -1.71$, $1.71 < \sqrt{3} < 1.75$.

35. **(a)** $\dfrac{d}{dx}[f^2(x) + g^2(x)] = 2f(x)f'(x) + 2g(x)g'(x) = 2f(x)g(x) + 2g(x)[-f(x)] = 0$,
so $f^2(x) + g^2(x)$ is constant.

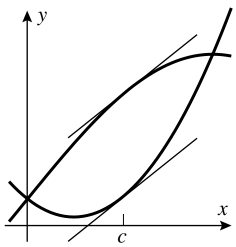
(b) $f(x) = \sin x$ and $g(x) = \cos x$

36. $\frac{d}{dx}[f^2(x) - g^2(x)] = 2f(x)f'(x) - 2g(x)g'(x) = 2f(x)g(x) - 2g(x)f(x) = 0$ so $f^2(x) - g^2(x)$ is constant.

37. If $f'(x) = g'(x)$, then $f(x) = g(x) + k$. Let $x = 1$,
$f(1) = g(1) + k = (1)^3 - 4(1) + 6 + k = 3 + k = 2$, so $k = -1$. $f(x) = x^3 - 4x + 5$.

38. Let $h = f - g$, then h is continuous on $[a, b]$, differentiable on (a, b), and $h(a) = f(a) - g(a) = 0$, $h(b) = f(b) - g(b) = 0$. By Rolle's Theorem there is some c in (a, b) where $h'(c) = 0$. But $h'(c) = f'(c) - g'(c)$ so $f'(c) - g'(c) = 0$, $f'(c) = g'(c)$.

39.



40. **(a)** Suppose $f'(x) = 0$ more than once in (a, b), say at c_1 and c_2. Then $f'(c_1) = f'(c_2) = 0$ and by using Rolle's Theorem on f', there is some c between c_1 and c_2 where $f''(c) = 0$, which contradicts the fact that $f''(x) > 0$ so $f'(x) = 0$ at most once in (a, b).

(b) If $f''(x) > 0$ for all x in (a, b), then f is concave up on (a, b) and has at most one relative extremum, which would be a relative minimum, on (a, b).

41. **(a)** similar to the proof of Part (a) with $f'(c) < 0$

(b) similar to the proof of Part (a) with $f'(c) = 0$

42. Let $x \neq x_0$ be sufficiently near x_0 so that there exists (by the Mean-Value Theorem) a number c (which depends on x) between x and x_0, such that

$$\frac{f(x) - f(x_0}{x - x_0} = f'(c).$$

Since c is between x and x_0 it follows that

$$\begin{aligned} f'(x_0) &= \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} && \text{(by definition of derivative)} \\ &= \lim_{x\to x_0} f'(c) && \text{(by the Mean-Value Theorem)} \\ &= \lim_{x\to x_0} f'(x) && \text{(since } \lim f'(x) \text{ exists and } c \text{ is between } x \text{ and } x_0\text{).} \end{aligned}$$

43. If f is differentiable at $x = 1$, then f is continuous there;
$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = f(1) = 3$, $a + b = 3$; $\lim_{x\to 1^+} f'(x) = a$ and $\lim_{x\to 1^-} f'(x) = 6$ so $a = 6$ and $b = 3 - 6 = -3$.

44. **(a)** $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} 2x = 0$ and $\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} 2x = 0$; $f'(0)$ does not exist because f is not continuous at $x = 0$.

(b) $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x) = 0$ and f is continuous at $x = 0$, so $f'(0) = 0$;
$\lim_{x\to 0^-} f''(x) = \lim_{x\to 0^-} (2) = 2$ and $\lim_{x\to 0^+} f''(x) = \lim_{x\to 0^+} 6x = 0$, so $f''(0)$ does not exist.

45. From Section 3.2 a function has a vertical tangent line at a point of its graph if the slopes of secant lines through the point approach $+\infty$ or $-\infty$. Suppose f is continuous at $x = x_0$ and $\lim_{x \to x_0^+} f(x) = +\infty$. Then a secant line through $(x_1, f(x_1))$ and $(x_0, f(x_0))$, assuming $x_1 > x_0$, will have slope $\dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$. By the Mean Value Theorem, this quotient is equal to $f'(c)$ for some c between x_0 and x_1. But as x_1 approaches x_0, c must also approach x_0, and it is given that $\lim_{c \to x_0^+} f'(c) = +\infty$, so the slope of the secant line approaches $+\infty$. The argument can be altered appropriately for $x_1 < x_0$, and/or for $f'(c)$ approaching $-\infty$.

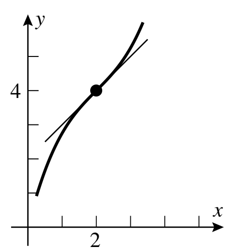
SUPPLEMENTARY EXERCISES FOR CHAPTER 4

4. **(a)** False; an example is $y = \dfrac{x^3}{3} - \dfrac{x^2}{2}$ on $[-2, 2]$; $x = 0$ is a relative maximum and $x = 1$ is a relative minimum, but y = 0 is not the largest value of y on the interval, nor is $y = -\dfrac{1}{6}$ the smallest.

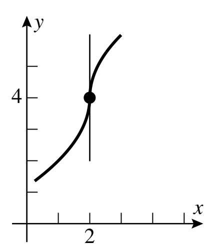
(b) true

(c) False; for example $y = x^3$ on $(-1, 1)$ which has a critical number but no relative extrema

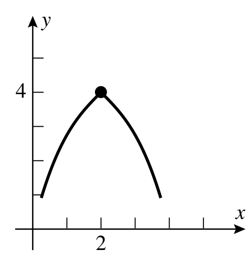
6. **(a)**



(b)



(c)



7. **(a)** $f'(x) = \dfrac{7(x-7)(x-1)}{3x^{2/3}}$; critical numbers at $x = 0, 1, 7$;
neither at $x = 0$, relative maximum at $x = 1$, relative minimum at $x = 7$ (First Derivative Test)

(b) $f'(x) = 2\cos x(1 + 2\sin x)$; critical numbers at $x = \pi/2, 3\pi/2, 7\pi/6, 11\pi/6$;
relative maximum at $x = \pi/2, 3\pi/2$, relative minimum at $x = 7\pi/6, 11\pi/6$

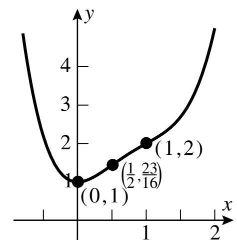
(c) $f'(x) = 3 - \dfrac{3\sqrt{x-1}}{2}$; critical numbers at $x = 5$; relative maximum at $x = 5$

8. **(a)** $f'(x) = \dfrac{x-9}{18x^{3/2}}$, $f''(x) = \dfrac{27-x}{36x^{5/2}}$; critical number at $x = 9$;
$f''(9) > 0$, relative minimum at $x = 9$

(b) $f'(x) = 2\dfrac{x^3-4}{x^2}$, $f''(x) = 2\dfrac{x^3+8}{x^3}$;
critical number at $x = 4^{1/3}$, $f''(4^{1/3}) > 0$, relative minimum at $x = 4^{1/3}$

(c) $f'(x) = \sin x(2\cos x + 1)$, $f''(x) = 2\cos^2 x - 2\sin^2 x + \cos x$; critical numbers at $x = 2\pi/3, \pi, 4\pi/3$; $f''(2\pi/3) < 0$, relative maximum at $x = 2\pi/3$; $f''(\pi) > 0$, relative minimum at $x = \pi$; $f''(4\pi/3) < 0$, relative maximum at $x = 4\pi/3$

9. $\lim_{x\to-\infty} f(x) = +\infty$, $\lim_{x\to+\infty} f(x) = +\infty$
$f'(x) = x(4x^2 - 9x + 6)$, $f''(x) = 6(2x - 1)(x - 1)$
relative minimum at $x = 0$,
points of inflection when $x = 1/2, 1$,
no asymptotes



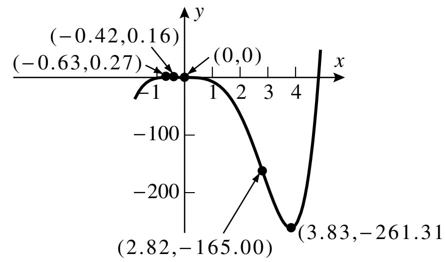
10. $\lim_{x\to-\infty} f(x) = -\infty$, $\lim_{x\to+\infty} f(x) = +\infty$
$f(x) = x^3(x-2)^2$, $f'(x) = x^2(5x-6)(x-2)$,
$f''(x) = 4x(5x^2 - 12x + 6)$

critical numbers at $x = 0, \dfrac{8 \pm 2\sqrt{31}}{5}$

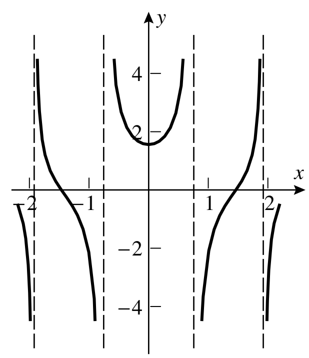
relative maximum at $x = \dfrac{8 - 2\sqrt{31}}{5} \approx -0.63$

relative minimum at $x = \dfrac{8 + 2\sqrt{31}}{5} \approx 3.83$

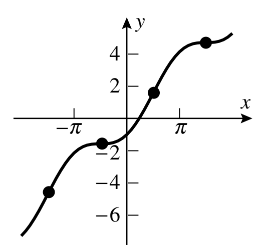
points of inflection at $x = 0, \dfrac{6 \pm \sqrt{66}}{5} \approx 0, -0.42, 2.82$
no asymptotes



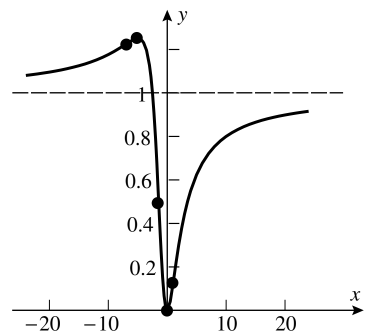
11. $\lim_{x\to\pm\infty} f(x)$ doesn't exist
$f'(x) = 2x\sec^2(x^2+1)$,
$f''(x) = 2\sec^2(x^2+1)\left[1 + 4x^2\tan(x^2+1)\right]$
critical number at $x = 0$; relative minimum at $x = 0$
point of inflection when $1 + 4x^2\tan(x^2+1) = 0$
vertical asymptotes at $x = \pm\sqrt{\pi(n+\frac{1}{2}) - 1}$, $n = 0, 1, 2, \ldots$



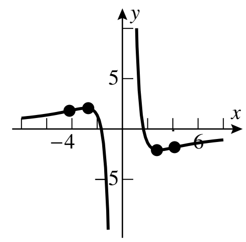
12. $\lim_{x\to-\infty} f(x) = -\infty$, $\lim_{x\to+\infty} f(x) = +\infty$
$f'(x) = 1 + \sin x$, $f''(x) = \cos x$
critical numbers at $x = 2n\pi + \pi/2$, $n = 0, \pm1, \pm2, \ldots$,
no extrema because $f' \geq 0$ and by Exercise 51 of Section 5.1,
f is increasing on $(-\infty, +\infty)$
inflections points at $x = n\pi + \pi/2$, $n = 0, \pm1, \pm2, \ldots$
no asymptotes



13. $f'(x) = 2\dfrac{x(x+5)}{(x^2+2x+5)^2}$, $f''(x) = -2\dfrac{2x^3+15x^2-25}{(x^2+2x+5)^3}$
critical numbers at $x = -5, 0$;
relative maximum at $x = -5$,
relative minimum at $x = 0$
points of inflection at $x = -7.26, -1.44, 1.20$
horizontal asymptote $y = 1$ as $x \to \pm\infty$



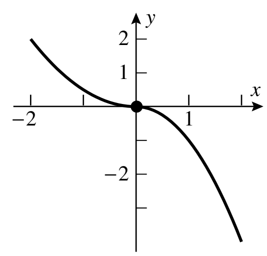
14. $f'(x) = 3\dfrac{3x^2-25}{x^4}$, $f''(x) = -6\dfrac{3x^2-50}{x^5}$
critical numbers at $x = \pm 5\sqrt{3}/3$;
relative maximum at $x = -5\sqrt{3}/3$,
relative minimum at $x = +5\sqrt{3}/3$
inflection points at $x = \pm 5\sqrt{2/3}$
horizontal asymptote of $y = 0$ as $x \to \pm\infty$,
vertical asymptote $x = 0$



15. $\lim_{x\to-\infty} f(x) = +\infty$, $\lim_{x\to+\infty} f(x) = -\infty$

$$f'(x) = \begin{cases} x \\ -2x \end{cases} \text{ if } \begin{cases} x \le 0 \\ x > 0 \end{cases}$$

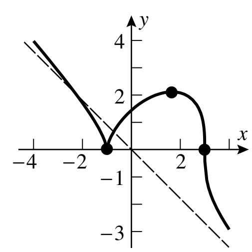
critical number at $x = 0$, no extrema
inflection point at $x = 0$ (f changes concavity)
no asymptotes



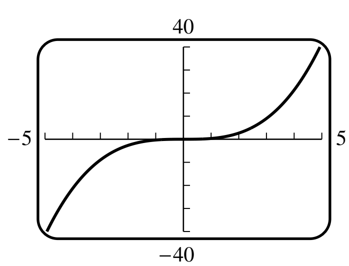
16. $f'(x) = \dfrac{5-3x}{3(1+x)^{1/3}(3-x)^{2/3}}$,

$f''(x) = \dfrac{-32}{9(1+x)^{4/3}(3-x)^{5/3}}$

critical number at $x = 5/3$;
relative maximum at $x = 5/3$
cusp at $x = -1$;
point of inflection at $x = 3$
oblique asymptote $y = -x$ as $x \to \pm\infty$



17. **(a)**



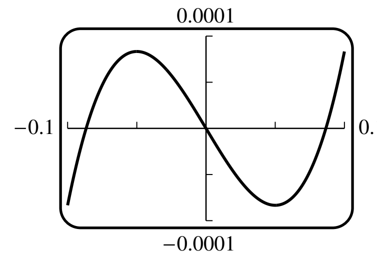
(b) $f'(x) = x^2 - \dfrac{1}{400}$, $f''(x) = 2x$

critical points at $x = \pm\dfrac{1}{20}$;

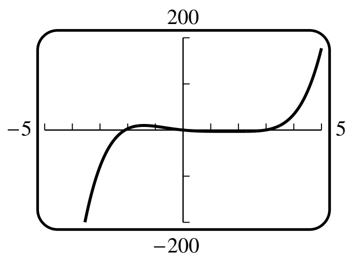
relative maximum at $x = -\dfrac{1}{20}$,

relative minimum at $x = \dfrac{1}{20}$

(c) The finer details can be seen when graphing over a much smaller x-window.

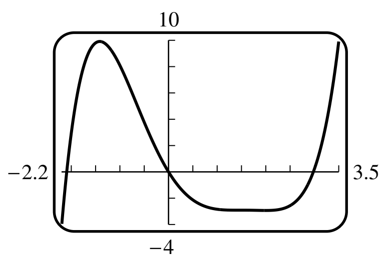


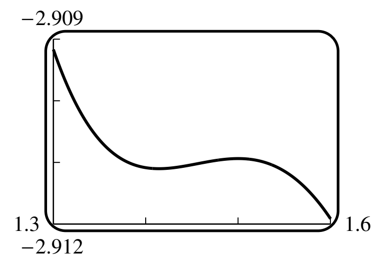
18. **(a)**



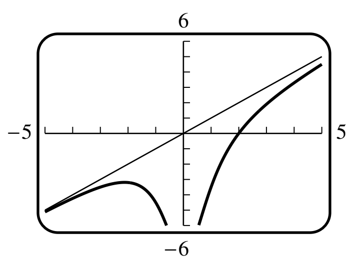
(b) critical points at $x = \pm\sqrt{2}, \frac{3}{2}, 2$;
relative maximum at $x = -\sqrt{2}$,
relative minimum at $x = \sqrt{2}$,
relative maximum at $x = \frac{3}{2}$,
relative minimum at $x = 2$

(c)





19. **(a)**



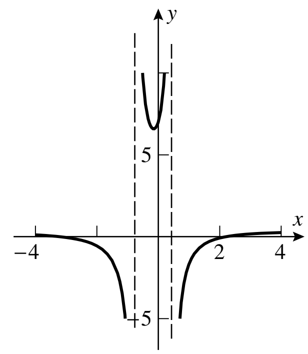
(b) Divide $y = x^2 + 1$ into $y = x^3 - 8$ to get the asymptote $ax + b = x$

20. **(a)** $p(x) = x^3 - x$ **(b)** $p(x) = x^4 - x^2$
(c) $p(x) = x^5 - x^4 - x^3 + x^2$ **(d)** $p(x) = x^5 - x^3$

21. $f'(x) = 4x^3 - 18x^2 + 24x - 8$, $f''(x) = 12(x-1)(x-2)$
$f''(1) = 0$, $f'(1) = 2$, $f(1) = 2$; $f''(2) = 0$, $f'(2) = 0$, $f(2) = 3$,
so the tangent lines at the inflection points are $y = 2x$ and $y = 3$.

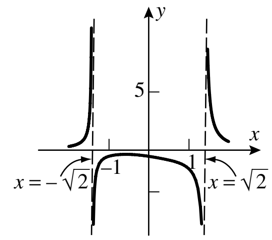
22. $\cos x - (\sin y)\dfrac{dy}{dx} = 2\dfrac{dy}{dx}$; $\dfrac{dy}{dx} = 0$ when $\cos x = 0$. Use the first derivative test: $\dfrac{dy}{dx} = \dfrac{\cos x}{2 + \sin y}$ and $2 + \sin y > 0$, so critical points when $\cos x = 0$, relative maxima when $x = 2n\pi + \pi/2$, relative minima when $x = 2n\pi - \pi/2$, $n = 0, \pm 1, \pm 2, \ldots$

23. $f(x) = \dfrac{(2x-1)(x^2+x-7)}{(2x-1)(3x^2+x-1)} = \dfrac{x^2+x-7}{3x^2+x-1}, \qquad x \neq 1/2$
horizontal asymptote: $y = 1/3$,
vertical asymptotes: $x = (-1 \pm \sqrt{13})/6$



24. **(a)** $f(x) = \dfrac{(x-2)(x^2+x+1)(x^2-2)}{(x-2)(x^2-2)^2(x^2+1)} = \dfrac{x^2+x+1}{(x^2-2)(x^2+1)}$

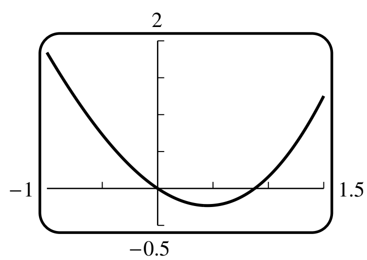
(b)



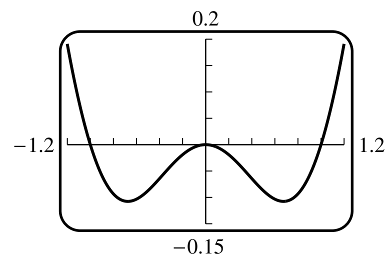
25. $f'(x) = 2ax + b$; $f'(x) > 0$ or $f'(x) < 0$ on $[0, +\infty)$ if $f'(x) = 0$ has no positive solution, so the polynomial is always increasing or always decreasing on $[0, +\infty)$ provided $-b/2a \leq 0$.

26. $f'(x) = 3ax^2 + 2bx + c$; $f'(x) > 0$ or $f'(x) < 0$ on $(-\infty, +\infty)$ if $f'(x) = 0$ has no real solutions so from the quadratic formula $(2b)^2 - 4(3a)c < 0$, $4b^2 - 12ac < 0$, $b^2 - 3ac < 0$. If $b^2 - 3ac = 0$, then $f'(x) = 0$ has only one real solution at, say, $x = c$ so f is always increasing or always decreasing on both $(-\infty, c]$ and $[c, +\infty)$, and hence on $(-\infty, +\infty)$ because f is continuous everywhere. Thus f is always increasing or decreasing if $b^2 - 3ac \leq 0$.

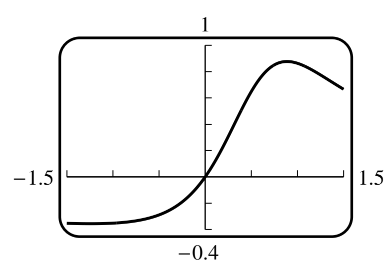
27. **(a)** relative minimum -0.232466 at $x = 0.450184$



(b) relative maximum 0 at $x = 0$; relative minimum -0.107587 at $x = \pm 0.674841$

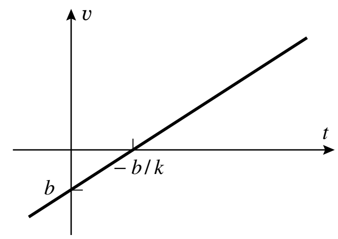


(c) relative maximum 0.876839; at $x = 0.886352$; relative minimum -0.355977 at $x = -1.244155$

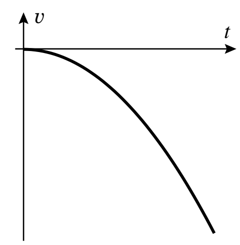


28. $f'(x) = 2 + 3x^2 - 4x^3$ has one real root at $x_0 \approx 1.136861168$, so $f(x)$ is increasing on $(-\infty, k))$ at least for $k = x_0$. But $f''(x_0) < 0$, so $f(x)$ has a relative maximum at $x = x_0$, and is thus decreasing to the right of $x = x_0$. So f is increasing on $(-\infty, x_0]$, where $x_0 \approx 1.136861$.

29. **(a)** If $a = k$, a constant, then $v = kt + b$ where b is constant; so the velocity changes sign at $t = -b/k$.

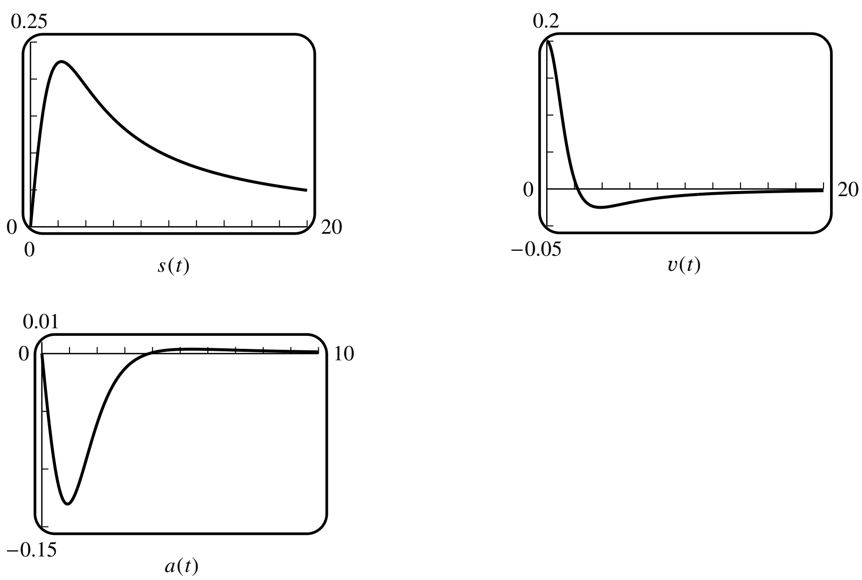


(b) Consider the equation $s = 5 - t^3/6$, $v = -t^2/2$, $a = -t$. Then for $t > 0$, a is decreasing and $av > 0$, so the particle is speeding up.



30. $s(t) = t/(t^2 + 5)$, $v(t) = (5 - t^2)/(t^2 + 5)^2$, $a(t) = 2t(t^2 - 15)/(t^2 + 5)^3$

(a)



(b) v changes sign at $t = \sqrt{5}$

(c) $s = \sqrt{5}/10$, $v = 0$, $a = -\sqrt{5}/50$

(d) a changes sign at $t = \sqrt{15}$, so the particle is speeding up for $\sqrt{5} < t < \sqrt{15}$, and it is slowing down for $0 < t < \sqrt{5}$ and $\sqrt{15} < t$

(e) $v(0) = 1/5$, $\lim_{t\to+\infty} v(t) = 0$, $v(t)$ has one t-intercept at $t = \sqrt{5}$ and $v(t)$ has one critical point at $t = \sqrt{15}$. Consequently the maximum velocity occurs when $t = 0$ and the minimum velocity occurs when $t = \sqrt{15}$.

31. **(a)** $s(t) = s_0 + v_0 t - \frac{1}{2}gt^2 = v_0 t - 4.9t^2$, $v(t) = v_0 - 9.8t$; $s_{\max}$ occurs when $v = 0$, i.e. $t = v_0/9.8$, and then $0.76 = s_{\max} = v_0(v_0/9.8) - 4.9(v_0/9.8)^2 = v_0^2/19.6$, so $v_0 = \sqrt{0.76 \cdot 19.6} = 3.86$ m/s and $s(t) = 3.86t - 4.9t^2$. Then $s(t) = 0$ when $t = 0, 0.7878$, $s(t) = 0.15$ when $t = 0.0410, 0.7468$, and $s(t) = 0.76 - 0.15 = 0.61$ when $t = 0.2188, 0.5689$, so the player spends $0.5689 - 0.2188 = 0.3501$ s in the top 15.0 cm of the jump and $0.0410 + (0.7878 - 0.7468) = 0.0820$ s in the bottom 15.0 cm.

(b) The height *vs* time plot is a parabola that opens down, and the slope is smallest near the top of the parabola, so a given change Δh in height corresponds to a large time change Δt near the top of the parabola and a narrower time change at points farther away from the top.

32. **(a)** $s(t) = s_0 + v_0 - 4.9t^2$; assume $s_0 = v_0 = 0$, so $s(t) = -4.9t^2$, $v(t) = -9.8t$

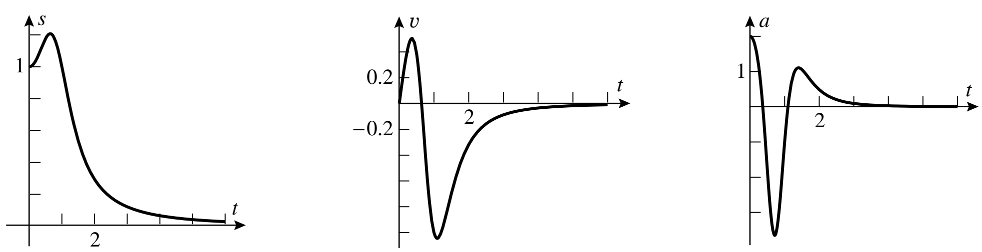
t	0	1	2	3	4
s	0	-4.9	-19.6	-44.1	-78.4
v	0	-9.8	-19.6	-29.4	-39.2

(b) The formula for v is linear (with no constant term).

(c) The formula for s is quadratic (with no linear or constant term).

33. **(a)** $v = -2\dfrac{t(t^4 + 2t^2 - 1)}{(t^4+1)^2}$, $a = 2\dfrac{3t^8 + 10t^6 - 12t^4 - 6t^2 + 1}{(t^4+1)^3}$

(b)



(c) It is farthest from the origin at approximately $t = 0.64$ (when $v = 0$) and $s = 1.2$

(d) Find t so that the velocity $v = ds/dt > 0$. The particle is moving in the positive direction for $0 \le t \le 0.64$ s.

(e) It is speeding up when $a, v > 0$ or $a, v < 0$, so for $0 \le t < 0.36$ and $0.64 < t < 1.1$, otherwise it is slowing down.

(f) Find the maximum value of $|v|$ to obtain: maximum speed = 1.05 m/s when $t = 1.10$ s.

34. No; speeding up means the velocity and acceleration have the same sign, i.e. $av > 0$; the velocity is increasing when the acceleration is positive, i.e. $a > 0$. These are not the same thing. An example is $s = t - t^2$ at $t = 1$, where $v = -1$ and $a = -2$, so $av > 0$ but $a < 0$.

37. **(a)** If f has an absolute extremum at a point of (a, b) then it must, by Theorem 6.1.4, be at a critical point of f; since f is differentiable on (a, b) the critical point is a stationary point.

(b) It could occur at a critical point which is not a stationary point: for example, $f(x) = |x|$ on $[-1, 1]$ has an absolute minimum at $x = 0$ but is not differentiable there.

38. Yes; by the Mean-Value Theorem there is a point c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a} = 0$.

39. **(a)** $f'(x) = -1/x^2 \neq 0$, no critical points; by inspection $M = -1/2$ at $x = -2$; $m = -1$ at $x = -1$

(b) $f'(x) = 3x^2 - 4x^3 = 0$ at $x = 0, 3/4$; $f(-1) = -2, f(0) = 0, f(3/4) = 27/256$, $f(3/2) = -27/16$, so $m = -2$ at $x = -1$, $M = 27/256$ at $x = 3/4$

(c) $f'(x) = \dfrac{x(7x-12)}{3(x-2)^{2/3}}$, critical points at $x = 12/7, 2$; $m = f(12/7) = \dfrac{144}{49}\left(-\dfrac{2}{7}\right)^{1/3} \approx -1.9356$ at $x = 12/7$, $M = 9$ at $x = 3$

(d) $\lim_{x\to 0^+} f(x) = \lim_{x\to+\infty} f(x) = +\infty$ and $f'(x) = \dfrac{e^x(x-2)}{x^3}$, stationary point at $x = 2$; by Theorem 6.1.5 $f(x)$ has an absolute minimum at $x = 2$, and $m = e^2/4$.

40. **(a)** $f'(x) = 2\dfrac{3-x^2}{(x^2+3)^2}$, critical point at $x = \sqrt{3}$. Since $\lim_{x\to 0^+} f(x) = 0$, $f(x)$ has no minimum, and $M = \sqrt{3}/3$ at $x = \sqrt{3}$.

(b) $f'(x) = 10x^3(x-2)$, critical points at $x = 0, 2$; $\lim_{x\to 3^-} f(x) = 88$, so $f(x)$ has no maximum; $m = -9$ at $x = 2$

(c) critical point at $x = 2$; $m = -3$ at $x = 3$, $M = 0$ at $x = 2$

42. $x = -2.11491, 0.25410, 1.86081$

43. $x = 2.3561945$

44. Let $a < x < x_0$. Then $\dfrac{f(x)-f(x_0)}{x-x_0} > 0$ since f is increasing on $[a,b]$. Similarly if $x_0 < x < b$ then $\dfrac{f(x)-f(x_0)}{x-x_0} > 0$. Thus, since the limit exists, $\lim_{x\to x_0} \dfrac{f(x)-f(x_0)}{x-x_0} \geq 0$.

45. **(a)** yes; $f'(0) = 0$

(b) no, f is not differentiable on $(-1, 1)$

(c) yes, $f'(\sqrt{\pi/2}) = 0$

46. **(a)** no, f is not differentiable on $(-2, 2)$

(b) yes, $\dfrac{f(3)-f(2)}{3-2} = -1 = f'(1+\sqrt{2})$

(c) $\lim_{x\to 1^-} f(x) = 2$, $\lim_{x\to 1^+} f(x) = 2$ so f is continuous on $[0, 2]$; $\lim_{x\to 1^-} f'(x) = \lim_{x\to 1^-} -2x = -2$ and $\lim_{x\to 1^+} f'(x) = \lim_{x\to 1^+} (-2/x^2) = -2$, so f is differentiable on $(0, 2)$; and $\dfrac{f(2)-f(0)}{2-0} = -1 = f'(\sqrt{2})$.

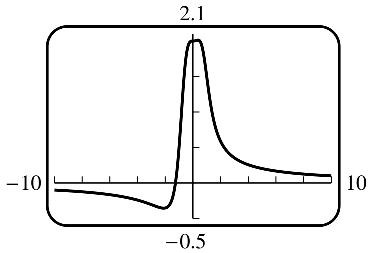
47. Let k be the amount of light admitted per unit area of clear glass. The total amount of light admitted by the entire window is
$T = k\cdot$ (area of clear glass) $+ \dfrac{1}{2}k\cdot$ (area of blue glass) $= 2krh + \dfrac{1}{4}\pi kr^2$.
But $P = 2h + 2r + \pi r$ which gives $2h = P - 2r - \pi r$ so

$$\begin{aligned} T &= kr(P - 2r - \pi r) + \frac{1}{4}\pi kr^2 = k\left[Pr - \left(2 + \pi - \frac{\pi}{4}\right)r^2\right] \\ &= k\left[Pr - \frac{8+3\pi}{4}r^2\right] \text{ for } 0 < r < \frac{P}{2+\pi}, \\ \frac{dT}{dr} &= k\left(P - \frac{8+3\pi}{2}r\right), \ \frac{dT}{dr} = 0 \text{ when } r = \frac{2P}{8+3\pi}. \end{aligned}$$

This is the only critical point and $d^2T/dr^2 < 0$ there so the most light is admitted when $r = 2P/(8+3\pi)$ ft.

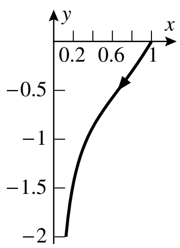
48. If one corner of the rectangle is at (x, y) with $x > 0$, $y > 0$, then $A = 4xy$, $y = 3\sqrt{1-(x/4)^2}$, $A = 12x\sqrt{1-(x/4)^2} = 3x\sqrt{16-x^2}$, $\dfrac{dA}{dx} = 6\dfrac{8-x^2}{\sqrt{16-x^2}}$, critical point at $x = 2\sqrt{2}$. Since $A = 0$ when $x = 0, 4$ and $A > 0$ otherwise, there is an absolute maximum $A = 24$ at $x = 2\sqrt{2}$.

49. **(a)**



(b) minimum: $(-2.111985,\ -0.355116)$
maximum: $(0.372591,\ 2.012931)$

50. **(a)**



(b) The distance between the boat and the origin is $\sqrt{x^2+y^2}$, where $y = (x^{10/3}-1)/(2x^{2/3})$. The minimum distance is 0.8247 mi when $x = 0.6598$ mi. The boat gets swept downstream.

(c) Use the equation of the path to obtain $dy/dt = (dy/dx)(dx/dt)$, $dx/dt = (dy/dt)/(dy/dx)$. Let $dy/dt = -4$ and find the value of dy/dx for the value of x obtained in Part (b) to get $dx/dt = -3$ mi/h.

51. Solve $\phi - 0.0167\sin\phi = 2\pi(90)/365$ to get $\phi = 1.565978$ so
$r = 150 \times 10^6(1 - 0.0167\cos\phi) = 149.988 \times 10^6$ km.

52. Solve $\phi - 0.0934\sin\phi = 2\pi(1)/1.88$ to get $\phi = 3.325078$ so
$r = 228 \times 10^6(1 - 0.0934\cos\phi) = 248.938 \times 10^6$ km.