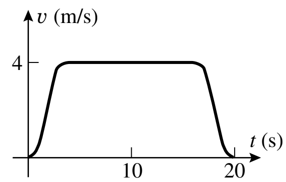
CHAPTER 3
The Derivative

EXERCISE SET 3.1

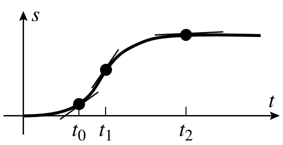
1. **(a)** $m_{\tan} = (50 - 10)/(15 - 5)$
$= 40/10$
$= 4$ m/s

(b)



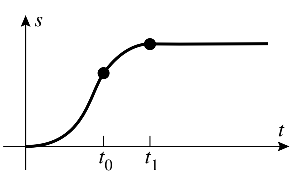
2. **(a)** $(10 - 10)/(3 - 0) = 0$ cm/s
(b) $t = 0$, $t = 2$, and $t = 4.2$ (horizontal tangent line)
(c) maximum: $t = 1$ (slope > 0) minimum: $t = 3$ (slope < 0)
(d) $(3 - 18)/(4 - 2) = -7.5$ cm/s (slope of estimated tangent line to curve at $t = 3$)

3. From the figure:



(a) The particle is moving faster at time t_0 because the slope of the tangent to the curve at t_0 is greater than that at t_2.
(b) The initial velocity is 0 because the slope of a horizontal line is 0.
(c) The particle is speeding up because the slope increases as t increases from t_0 to t_1.
(d) The particle is slowing down because the slope decreases as t increases from t_1 to t_2.

4.



5. It is a straight line with slope equal to the velocity.

6. **(a)** decreasing (slope of tangent line decreases with increasing time)
(b) increasing (slope of tangent line increases with increasing time)
(c) increasing (slope of tangent line increases with increasing time)
(d) decreasing (slope of tangent line decreases with increasing time)

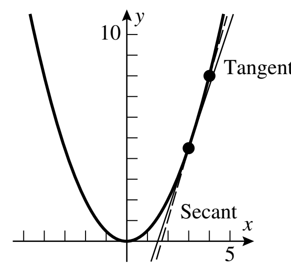
7. **(a)** $m_{\sec} = \dfrac{f(4) - f(3)}{4 - 3} = \dfrac{(4)^2/2 - (3)^2/2}{1} = \dfrac{7}{2}$

(b) $m_{\tan} = \lim_{x_1 \to 3} \dfrac{f(x_1) - f(3)}{x_1 - 3} = \lim_{x_1 \to 3} \dfrac{x_1^2/2 - 9/2}{x_1 - 3}$

$$= \lim_{x_1 \to 3} \frac{x_1^2 - 9}{2(x_1 - 3)} = \lim_{x_1 \to 3} \frac{(x_1 + 3)(x_1 - 3)}{2(x_1 - 3)} = \lim_{x_1 \to 3} \frac{x_1 + 3}{2} = 3$$

(c) $$m_{\tan} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1 \to x_0} \frac{x_1^2/2 - x_0^2/2}{x_1 - x_0} = \lim_{x_1 \to x_0} \frac{x_1^2 - x_0^2}{2(x_1 - x_0)} = \lim_{x_1 \to x_0} \frac{x_1 + x_0}{2} = x_0$$

(d)

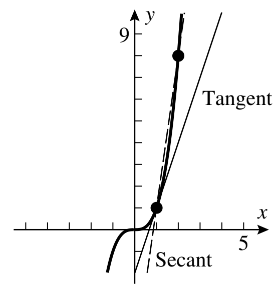


8. (a) $m_{\sec} = \dfrac{f(2) - f(1)}{2 - 1} = \dfrac{2^3 - 1^3}{1} = 7$

(b) $$m_{\tan} = \lim_{x_1 \to 1} \frac{f(x_1) - f(1)}{x_1 - 1} = \lim_{x_1 \to 1} \frac{x_1^3 - 1}{x_1 - 1} = \lim_{x_1 \to 1} \frac{(x_1 - 1)(x_1^2 + x_1 + 1)}{x_1 - 1} = \lim_{x_1 \to 1} (x_1^2 + x_1 + 1) = 3$$

(c) $$m_{\tan} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1 \to x_0} \frac{x_1^3 - x_0^3}{x_1 - x_0} = \lim_{x_1 \to x_0} (x_1^2 + x_1 x_0 + x_0^2) = 3x_0^2$$

(d)

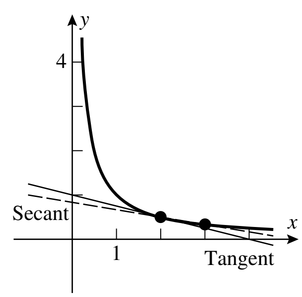


9. (a) $m_{\sec} = \dfrac{f(3) - f(2)}{3 - 2} = \dfrac{1/3 - 1/2}{1} = -\dfrac{1}{6}$

(b) $$m_{\tan} = \lim_{x_1 \to 2} \frac{f(x_1) - f(2)}{x_1 - 2} = \lim_{x_1 \to 2} \frac{1/x_1 - 1/2}{x_1 - 2} = \lim_{x_1 \to 2} \frac{2 - x_1}{2x_1(x_1 - 2)} = \lim_{x_1 \to 2} \frac{-1}{2x_1} = -\frac{1}{4}$$

(c) $$m_{\tan} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1 \to x_0} \frac{1/x_1 - 1/x_0}{x_1 - x_0} = \lim_{x_1 \to x_0} \frac{x_0 - x_1}{x_0 x_1 (x_1 - x_0)} = \lim_{x_1 \to x_0} \frac{-1}{x_0 x_1} = -\frac{1}{x_0^2}$$

(d)

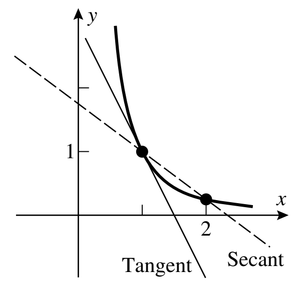


10. (a) $m_{\sec} = \dfrac{f(2) - f(1)}{2 - 1} = \dfrac{1/4 - 1}{1} = -\dfrac{3}{4}$

(b) $$m_{\tan} = \lim_{x_1 \to 1} \frac{f(x_1) - f(1)}{x_1 - 1} = \lim_{x_1 \to 1} \frac{1/x_1^2 - 1}{x_1 - 1} = \lim_{x_1 \to 1} \frac{1 - x_1^2}{x_1^2(x_1 - 1)} = \lim_{x_1 \to 1} \frac{-(x_1 + 1)}{x_1^2} = -2$$

(c) $$m_{\tan} = \lim_{x_1\to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1\to x_0} \frac{1/x_1^2 - 1/x_0^2}{x_1 - x_0} = \lim_{x_1\to x_0} \frac{x_0^2 - x_1^2}{x_0^2 x_1^2 (x_1 - x_0)} = \lim_{x_1\to x_0} \frac{-(x_1 + x_0)}{x_0^2 x_1^2} = -\frac{2}{x_0^3}$$

(d)



11. **(a)** $$m_{\tan} = \lim_{x_1\to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1\to x_0} \frac{(x_1^2 + 1) - (x_0^2 + 1)}{x_1 - x_0} = \lim_{x_1\to x_0} \frac{x_1^2 - x_0^2}{x_1 - x_0} = \lim_{x_1\to x_0} (x_1 + x_0) = 2x_0$$

(b) $m_{\tan} = 2(2) = 4$

12. **(a)** $$m_{\tan} = \lim_{x_1\to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1\to x_0} \frac{(x_1^2 + 3x_1 + 2) - (x_0^2 + 3x_0 + 2)}{x_1 - x_0} = \lim_{x_1\to x_0} \frac{(x_1^2 - x_0^2) + 3(x_1 - x_0)}{x_1 - x_0} = \lim_{x_1\to x_0} (x_1 + x_0 + 3) = 2x_0 + 3$$

(b) $m_{\tan} = 2(2) + 3 = 7$

13. **(a)** $$m_{\tan} = \lim_{x_1\to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1\to x_0} \frac{\sqrt{x_1} - \sqrt{x_0}}{x_1 - x_0} = \lim_{x_1\to x_0} \frac{1}{\sqrt{x_1} + \sqrt{x_0}} = \frac{1}{2\sqrt{x_0}}$$

(b) $m_{\tan} = \dfrac{1}{2\sqrt{1}} = \dfrac{1}{2}$

14. **(a)** $$m_{\tan} = \lim_{x_1\to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1\to x_0} \frac{1/\sqrt{x_1} - 1/\sqrt{x_0}}{x_1 - x_0} = \lim_{x_1\to x_0} \frac{\sqrt{x_0} - \sqrt{x_1}}{\sqrt{x_0}\,\sqrt{x_1}\,(x_1 - x_0)} = \lim_{x_1\to x_0} \frac{-1}{\sqrt{x_0}\,\sqrt{x_1}\,(\sqrt{x_1} + \sqrt{x_0})} = -\frac{1}{2x_0^{3/2}}$$

(b) $m_{\tan} = -\dfrac{1}{2(4)^{3/2}} = -\dfrac{1}{16}$

15. **(a)** 72°F at about 4:30 P.M.

(b) about $(67 - 43)/6 = 4$°F/h

(c) decreasing most rapidly at about 9 P.M.; rate of change of temperature is about -7°F/h (slope of estimated tangent line to curve at 9 P.M.)

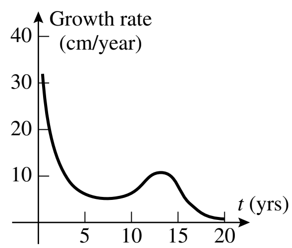
16. For $V = 10$ the slope of the tangent line is about -0.25 atm/L, for $V = 25$ the slope is about -0.04 atm/L.

17. **(a)** during the first year after birth

(b) about 6 cm/year (slope of estimated tangent line at age 5)

(c) the growth rate is greatest at about age 14; about 10 cm/year

(d)



18. **(a)** The rock will hit the ground when $16t^2 = 576$, $t^2 = 36$, $t = 6$ s (only $t \geq 0$ is meaningful)

(b) $v_{\text{ave}} = \dfrac{16(6)^2 - 16(0)^2}{6 - 0} = 96$ ft/s

(c) $v_{\text{ave}} = \dfrac{16(3)^2 - 16(0)^2}{3 - 0} = 48$ ft/s

(d) $v_{\text{inst}} = \lim\limits_{t_1 \to 6} \dfrac{16t_1^2 - 16(6)^2}{t_1 - 6} = \lim\limits_{t_1 \to 6} \dfrac{16(t_1^2 - 36)}{t_1 - 6}$

$= \lim\limits_{t_1 \to 6} 16(t_1 + 6) = 192$ ft/s

19. **(a)** $5(40)^3 = 320{,}000$ ft

(b) $v_{\text{ave}} = 320{,}000/40 = 8{,}000$ ft/s

(c) $5t^3 = 135$ when the rocket has gone 135 ft, so $t^3 = 27$, $t = 3$ s; $v_{\text{ave}} = 135/3 = 45$ ft/s.

(d) $v_{\text{inst}} = \lim\limits_{t_1 \to 40} \dfrac{5t_1^3 - 5(40)^3}{t_1 - 40} = \lim\limits_{t_1 \to 40} \dfrac{5(t_1^3 - 40^3)}{t_1 - 40}$

$= \lim\limits_{t_1 \to 40} 5(t_1^2 + 40t_1 + 1600) = 24,000$ ft/s

20. **(a)** $v_{\text{ave}} = \dfrac{[3(3)^2 + 3] - [3(1)^2 + 1]}{3 - 1} = 13$ mi/h

(b) $v_{\text{inst}} = \lim\limits_{t_1 \to 1} \dfrac{(3t_1^2 + t_1) - 4}{t_1 - 1} = \lim\limits_{t_1 \to 1} \dfrac{(3t_1 + 4)(t_1 - 1)}{t_1 - 1} = \lim\limits_{t_1 \to 1}(3t_1 + 4) = 7$ mi/h

21. **(a)** $v_{\text{ave}} = \dfrac{6(4)^4 - 6(2)^4}{4 - 2} = 720$ ft/min

(b) $v_{\text{inst}} = \lim\limits_{t_1 \to 2} \dfrac{6t_1^4 - 6(2)^4}{t_1 - 2} = \lim\limits_{t_1 \to 2} \dfrac{6(t_1^4 - 16)}{t_1 - 2}$

$= \lim\limits_{t_1 \to 2} \dfrac{6(t_1^2 + 4)(t_1^2 - 4)}{t_1 - 2} = \lim\limits_{t_1 \to 2} 6(t_1^2 + 4)(t_1 + 2) = 192$ ft/min

EXERCISE SET 3.2

1. $f'(1) = 2,\ f'(3) = 0,\ f'(5) = -2,\ f'(6) = -1/2$

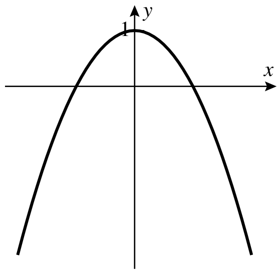
2. $f'(4) < f'(0) < f'(2) < 0 < f'(-3)$

3. **(b)** $m = f'(2) = 3$

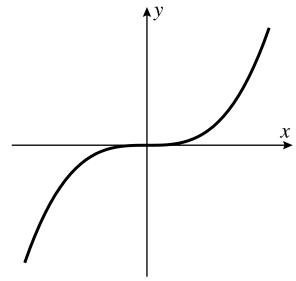
(c) the same, $f'(2) = 3$

4. $f'(-1) = m = \dfrac{4 - 3}{0 - (-1)} = 1$

5.



6.



7. $y-(-1)=5(x-3),\ y=5x-16$

8. $y-3=-4(x+2),\ y=-4x-5$

9. $f'(x)=\lim_{w\to x}\dfrac{f(w)-f(x)}{w-x}=\lim_{w\to x}\dfrac{3w^2-3x^2}{w-x}=\lim_{w\to x}3(w+x)=6x;\ f(3)=3(3)^2=27,\ f'(3)=18$ so $y-27=18(x-3),\ y=18x-27$

10. $f'(x)=\lim_{w\to x}\dfrac{f(w)-f(x)}{w-x}=\lim_{w\to x}\dfrac{w^4-x^4}{w-x}=\lim_{w\to x}(w^3+w^2x+wx^2+x^3)=4x^3;$ $f(-2)=(-2)^4=16,\ f'(-2)=-32$ so $y-16=-32(x+2),\ y=-32x-48$

11. $f'(x)=\lim_{w\to x}\dfrac{f(w)-f(x)}{w-x}=\lim_{w\to x}\dfrac{w^3-x^3}{w-x}=\lim_{w\to x}(w^2+wx+x^2)=3x^2;\ f(0)=0^3=0,$ $f'(0)=0$ so $y-0=(0)(x-0),\ y=0$

12. $f'(x)=\lim_{w\to x}\dfrac{f(w)-f(x)}{w-x}=\lim_{w\to x}\dfrac{2w^3+1-(2x^3+1)}{w-x}=\lim_{w\to x}2(w^2+wx+x^2)=6x^2;$ $f(-1)=2(-1)^3+1=-1,\ f'(-1)=6$ so $y+1=6(x+1),\ y=6x+5$

13. $f'(x)=\lim_{w\to x}\dfrac{f(w)-f(x)}{w-x}=\lim_{w\to x}\dfrac{\sqrt{w+1}-\sqrt{x+1}}{w-x}$

$$=\lim_{w\to x}\frac{\sqrt{w+1}-\sqrt{x+1}}{w-x}\,\frac{\sqrt{w+1}+\sqrt{x+1}}{\sqrt{w+1}+\sqrt{x+1}}=\lim_{w\to x}\frac{1}{\left(\sqrt{w+1}+\sqrt{x+1}\right)}=\frac{1}{2\sqrt{x+1}};$$

$f(8)=\sqrt{8+1}=3,\ f'(8)=\dfrac{1}{6}$ so $y-3=\dfrac{1}{6}(x-8),\ y=\dfrac{1}{6}x+\dfrac{5}{3}$

14. $f'(x)=\lim_{w\to x}\dfrac{f(w)-f(x)}{w-x}=\lim_{w\to x}\dfrac{\sqrt{2w+1}-\sqrt{2x+1}}{w-x}$

$$=\lim_{w\to x}\frac{2}{\sqrt{2w+1}+\sqrt{2x+1}}=\lim_{w\to x}\frac{2}{\sqrt{9+2h}+3}=\frac{1}{\sqrt{2x+1}}$$

$f(4)=\sqrt{2(4)+1}=\sqrt{9}=3, f'(4)=1/3$ so $y-3=\dfrac{1}{3}(x-4), y=\dfrac{1}{3}x+\dfrac{5}{3}$

15. $$f'(x)=\lim_{\Delta x\to 0}\frac{\dfrac{1}{x+\Delta x}-\dfrac{1}{x}}{\Delta x}=\lim_{\Delta x\to 0}\frac{\dfrac{x-(x+\Delta x)}{x(x+\Delta x)}}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{-\Delta x}{x\Delta x(x+\Delta x)}=\lim_{\Delta x\to 0}-\frac{1}{x(x+\Delta x)}=-\frac{1}{x^2}$$

16. $$f'(x) = \lim_{\Delta x\to 0} \frac{\dfrac{1}{(x+\Delta x)+1} - \dfrac{1}{x+1}}{\Delta x} = \lim_{\Delta x\to 0} \frac{\dfrac{(x+1)-(x+\Delta x+1)}{(x+1)(x+\Delta x+1)}}{\Delta x}$$
$$= \lim_{\Delta x\to 0} \frac{x+1-x-\Delta x-1}{\Delta x(x+1)(x+\Delta x+1)} = \lim_{\Delta x\to 0} \frac{-\Delta x}{\Delta x(x+1)(x+\Delta x+1)}$$
$$= \lim_{\Delta x\to 0} \frac{-1}{(x+1)(x+\Delta x+1)} = -\frac{1}{(x+1)^2}$$

17. $$f'(x) = \lim_{\Delta x\to 0} \frac{[a(x+\Delta x)^2+b]-[ax^2+b]}{\Delta x} = \lim_{\Delta x\to 0} \frac{ax^2+2ax\Delta x+a(\Delta x)^2+b-ax^2-b}{\Delta x}$$
$$= \lim_{\Delta x\to 0} \frac{2ax\Delta x+a(\Delta x)^2}{\Delta x} = \lim_{\Delta x\to 0}(2ax+a\Delta x) = 2ax$$

18. $$f'(x) = \lim_{\Delta x\to 0} \frac{(x+\Delta x)^2-(x+\Delta x)-(x^2-x)}{\Delta x} = \lim_{\Delta x\to 0} \frac{2x\Delta x+\Delta x^2-\Delta x}{\Delta x}$$
$$= \lim_{\Delta x\to 0}(2x-1+\Delta x) = 2x-1$$

19. $$f'(x) = \lim_{\Delta x\to 0} \frac{\dfrac{1}{\sqrt{x+\Delta x}} - \dfrac{1}{\sqrt{x}}}{\Delta x} = \lim_{\Delta x\to 0} \frac{\sqrt{x}-\sqrt{x+\Delta x}}{\Delta x\sqrt{x}\sqrt{x+\Delta x}}$$
$$= \lim_{\Delta x\to 0} \frac{x-(x+\Delta x)}{\Delta x\sqrt{x}\sqrt{x+\Delta x}(\sqrt{x}+\sqrt{x+\Delta x})} = \lim_{\Delta x\to 0} \frac{-1}{\sqrt{x}\sqrt{x+\Delta x}(\sqrt{x}+\sqrt{x+\Delta x})} = -\frac{1}{2x^{3/2}}$$

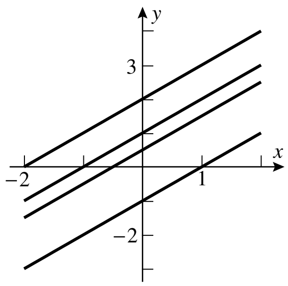
20. $$f'(x) = \lim_{\Delta x\to 0} \frac{\dfrac{1}{(x+\Delta x)^2} - \dfrac{1}{x^2}}{\Delta x} = \lim_{\Delta x\to 0} \frac{\dfrac{x^2-(x+\Delta x)^2}{x^2(x+\Delta x)^2}}{\Delta x}$$
$$= \lim_{\Delta x\to 0} \frac{x^2-x^2-2x\Delta x-\Delta x^2}{x^2\Delta x(x+\Delta x)^2} = \lim_{\Delta x\to 0} \frac{-2x\Delta x-\Delta x^2}{x^2\Delta x(x+\Delta x)^2} = \lim_{\Delta x\to 0} \frac{-2x-\Delta x}{x^2(x+\Delta x)^2} = -\frac{2}{x^3}$$

21. $$f'(t) = \lim_{h\to 0} \frac{f(t+h)-f(t)}{h} = \lim_{h\to 0} \frac{[4(t+h)^2+(t+h)]-[4t^2+t]}{h}$$
$$= \lim_{h\to 0} \frac{4t^2+8th+4h^2+t+h-4t^2-t}{h}$$
$$= \lim_{h\to 0} \frac{8th+4h^2+h}{h} = \lim_{h\to 0}(8t+4h+1) = 8t+1$$

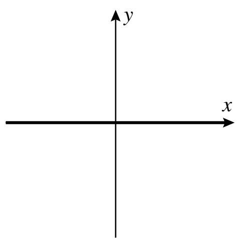
22. $$\frac{dV}{dr} = \lim_{h\to 0} \frac{\frac{4}{3}\pi(r+h)^3 - \frac{4}{3}\pi r^3}{h} = \lim_{h\to 0} \frac{\frac{4}{3}\pi(r^3+3r^2h+3rh^2+h^3-r^3)}{h}$$
$$= \lim_{h\to 0} \frac{4}{3}\pi(3r^2+3rh+h^2) = 4\pi r^2$$

23. **(a)** D **(b)** F **(c)** B **(d)** C **(e)** A **(f)** E

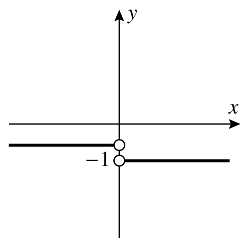
24. Any function of the form $f(x) = x + k$ has slope 1, and thus the derivative must be equal to 1 everywhere.



25. **(a)**



(b)



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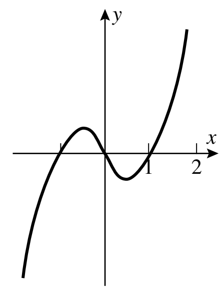
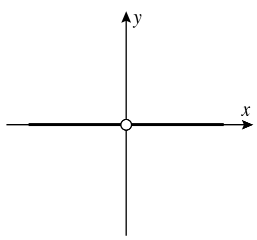


26. **(a)**



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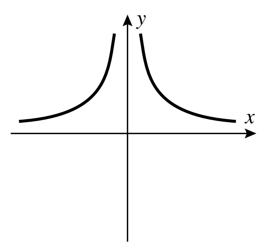

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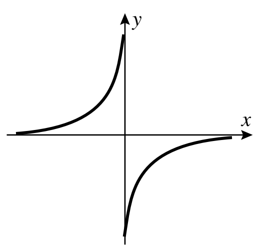


27. **(a)** $f(x) = x^2$ and $a = 3$ **(b)** $f(x) = \sqrt{x}$ and $a = 1$

28. **(a)** $f(x) = x^7$ and $a = 1$ **(b)** $f(x) = \cos x$ and $a = \pi$

29. $\dfrac{dy}{dx} = \lim\limits_{h\to 0} \dfrac{[4(x+h)^2+1]-[4x^2+1]}{h} = \lim\limits_{h\to 0} \dfrac{4x^2+8xh+4h^2+1-4x^2-1}{h} = \lim\limits_{h\to 0}(8x+4h) = 8x$

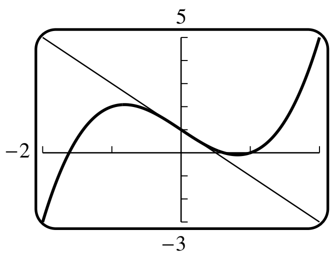
$\left.\dfrac{dy}{dx}\right|_{x=1} = 8(1) = 8$

30. $\dfrac{dy}{dx} = \lim_{h\to 0} \dfrac{\left(\dfrac{5}{x+h}+1\right) - \left(\dfrac{5}{x}+1\right)}{h} = \lim_{h\to 0} \dfrac{\dfrac{5}{x+h} - \dfrac{5}{x}}{h} = \lim_{h\to 0} \dfrac{\dfrac{5x-5(x+h)}{x(x+h)}}{h}$

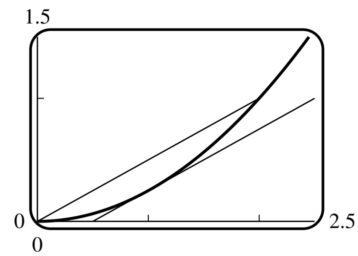
$= \lim_{h\to 0} \dfrac{5x-5x-5h}{hx(x+h)} = \lim_{h\to 0} \dfrac{-5}{x(x+h)} = -\dfrac{5}{x^2}$

$\left.\dfrac{dy}{dx}\right|_{x=-2} = -\dfrac{5}{(-2)^2} = -\dfrac{5}{4}$

31. $y = -2x + 1$



32.



33. (b)

h	0.5	0.1	0.01	0.001	0.0001	0.00001
$(f(1+h)-f(1))/h$	1.6569	1.4355	1.3911	1.3868	1.3863	1.3863

34. (b)

h	0.5	0.1	0.01	0.001	0.0001	0.00001
$(f(1+h)-f(1))/h$	0.50489	0.67060	0.70356	0.70675	0.70707	0.70710

35. (a) dollars/ft

(b) As you go deeper the price per foot may increase dramatically, so $f'(x)$ is roughly the price per additional foot.

(c) If each additional foot costs extra money (this is to be expected) then $f'(x)$ remains positive.

(d) From the approximation $1000 = f'(300) \approx \dfrac{f(301)-f(300)}{301-300}$ we see that $f(301) \approx f(300) + 1000$, so the extra foot will cost around \$1000.

36. (a) gallons/dollar

(b) The increase in the amount of paint that would be sold for one extra dollar.

(c) It should be negative since an increase in the price of paint would decrease the amount of paint sold.

(d) From $-100 = f'(10) \approx \dfrac{f(11)-f(10)}{11-10}$ we see that $f(11) \approx f(10) - 100$, so an increase of one dollar would decrease the amount of paint sold by around 100 gallons.

37. (a) $F \approx 200$ lb, $dF/d\theta \approx 50$ lb/rad **(b)** $\mu = (dF/d\theta)/F \approx 50/200 = 0.25$

38. (a) The slope of the tangent line $\approx \dfrac{10-2.2}{2050-1950} = 0.078$ billion, or in 2050 the world population was increasing at the rate of about 78 million per year.

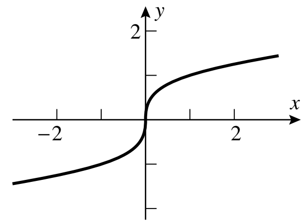
(b) $\dfrac{dN}{dt} =\approx \dfrac{0.078}{6} = 0.013 = 1.3$ %/year

39. (a) $T \approx 115°$F, $dT/dt \approx -3.35°$F/min

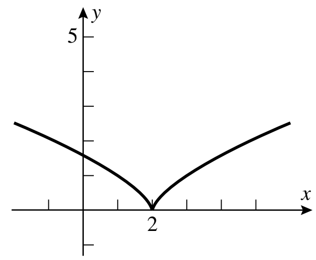
(b) $k = (dT/dt)/(T - T_0) \approx (-3.35)/(115-75) = -0.084$

41. $\lim_{x\to 0} f(x) = \lim_{x\to 0} \sqrt[3]{x} = 0 = f(0)$, so f is continuous at $x = 0$.

$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\sqrt[3]{h} - 0}{h} = \lim_{h\to 0} \frac{1}{h^{2/3}} = +\infty$, so $f'(0)$ does not exist.



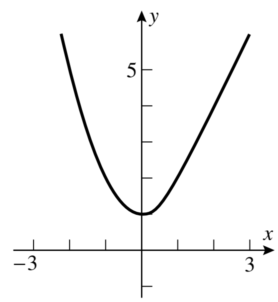
42. $\lim_{x\to 2} f(x) = \lim_{x\to 2} (x-2)^{2/3} = 0 = f(2)$ so f is continuous at $x = 2$. $\lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{h^{2/3} - 0}{h} = \lim_{h\to 0} \frac{1}{h^{1/3}}$ which does not exist so $f'(2)$ does not exist.



43. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$, so f is continuous at $x = 1$.

$\lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^-} \frac{[(1+h)^2 + 1] - 2}{h} = \lim_{h\to 0^-} (2+h) = 2$;

$\lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^+} \frac{2(1+h) - 2}{h} = \lim_{h\to 0^+} 2 = 2$, so $f'(1) = 2$.

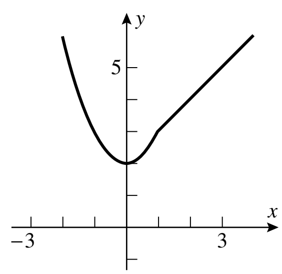


44. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$ so f is continuous at $x = 1$.

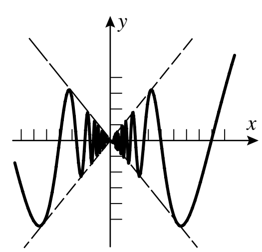
$\lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^-} \frac{[(1+h)^2 + 2] - 3}{h} = \lim_{h\to 0^-} (2+h) = 2$;

$\lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^+} \frac{[(1+h) + 2] - 3}{h} = \lim_{h\to 0^+} 1 = 1$,

so $f'(1)$ does not exist.



45. Since $-|x| \le x\sin(1/x) \le |x|$ it follows by the Squeezing Theorem (Theorem 2.6.2) that $\lim_{x\to 0} x\sin(1/x) = 0$. The derivative cannot exist: consider $\frac{f(x) - f(0)}{x} = \sin(1/x)$. This function oscillates between -1 and $+1$ and does not tend to zero as x tends to zero.



46. For continuity, compare with $\pm x^2$ to establish that the limit is zero. The differential quotient is $x\sin(1/x)$ and (see Exercise 45) this has a limit of zero at the origin.

47. f is continuous at $x = 1$ because it is differentiable there, thus $\lim_{h\to 0} f(1+h) = f(1)$ and so $f(1) = 0$ because $\lim_{h\to 0} \frac{f(1+h)}{h}$ exists; $f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{f(1+h)}{h} = 5$.

48. Let $x = y = 0$ to get $f(0) = f(0) + f(0) + 0$ so $f(0) = 0$. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$, but $f(x+h) = f(x) + f(h) + 5xh$ so $f(x+h) - f(x) = f(h) + 5xh$ and
$f'(x) = \lim_{h\to 0} \dfrac{f(h) + 5xh}{h} = \lim_{h\to 0} \left(\dfrac{f(h)}{h} + 5x\right) = 3 + 5x$.

49. $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{f(x)f(h) - f(x)}{h} = \lim_{h\to 0} \frac{f(x)[f(h) - 1]}{h}$$
$$= f(x) \lim_{h\to 0} \frac{f(h) - f(0)}{h} = f(x)f'(0) = f(x)$$

EXERCISE SET 3.3

1. $28x^6$ **2.** $-36x^{11}$ **3.** $24x^7 + 2$ **4.** $2x^3$

5. 0 **6.** $\sqrt{2}$ **7.** $-\frac{1}{3}(7x^6 + 2)$ **8.** $\frac{2}{5}x$

9. $3ax^2 + 2bx + c$ **10.** $\frac{1}{a}\left(2x + \frac{1}{b}\right)$ **11.** $24x^{-9} + 1/\sqrt{x}$ **12.** $-42x^{-7} - \frac{5}{2\sqrt{x}}$

13. $-3x^{-4} - 7x^{-8}$ **14.** $\frac{1}{2\sqrt{x}} - \frac{1}{x^2}$

15. $$f'(x) = (3x^2 + 6)\frac{d}{dx}\left(2x - \frac{1}{4}\right) + \left(2x - \frac{1}{4}\right)\frac{d}{dx}(3x^2 + 6) = (3x^2 + 6)(2) + \left(2x - \frac{1}{4}\right)(6x)$$
$$= 18x^2 - \frac{3}{2}x + 12$$

16. $$f'(x) = (2 - x - 3x^3)\frac{d}{dx}(7 + x^5) + (7 + x^5)\frac{d}{dx}(2 - x - 3x^3)$$
$$= (2 - x - 3x^3)(5x^4) + (7 + x^5)(-1 - 9x^2)$$
$$= -24x^7 - 6x^5 + 10x^4 - 63x^2 - 7$$

17. $$f'(x) = (x^3 + 7x^2 - 8)\frac{d}{dx}(2x^{-3} + x^{-4}) + (2x^{-3} + x^{-4})\frac{d}{dx}(x^3 + 7x^2 - 8)$$
$$= (x^3 + 7x^2 - 8)(-6x^{-4} - 4x^{-5}) + (2x^{-3} + x^{-4})(3x^2 + 14x)$$
$$= -15x^{-2} - 14x^{-3} + 48x^{-4} + 32x^{-5}$$

18. $$f'(x) = (x^{-1} + x^{-2})\frac{d}{dx}(3x^3 + 27) + (3x^3 + 27)\frac{d}{dx}(x^{-1} + x^{-2})$$
$$= (x^{-1} + x^{-2})(9x^2) + (3x^3 + 27)(-x^{-2} - 2x^{-3}) = 3 + 6x - 27x^{-2} - 54x^{-3}$$

19. $12x(3x^2 + 1)$ **20.** $f(x) = x^{10} + 4x^6 + 4x^2$, $f'(x) = 10x^9 + 24x^5 + 8x$

21. $$\frac{dy}{dx} = \frac{(5x - 3)\frac{d}{dx}(1) - (1)\frac{d}{dx}(5x - 3)}{(5x - 3)^2} = -\frac{5}{(5x - 3)^2};\quad y'(1) = -5/4$$

22. $\dfrac{dy}{dx} = \dfrac{(\sqrt{x}+2)\dfrac{d}{dx}(3) - 3\dfrac{d}{dx}(\sqrt{x}+2)}{(\sqrt{x}+2)^2} = -3/(2\sqrt{x}(\sqrt{x}+2)^2);\ y'(1) = -3/18 = -1/6$

23. $\dfrac{dx}{dt} = \dfrac{(2t+1)\dfrac{d}{dt}(3t) - (3t)\dfrac{d}{dt}(2t+1)}{(2t+1)^2} = \dfrac{(2t+1)(3) - (3t)(2)}{(2t+1)^2} = \dfrac{3}{(2t+1)^2}$

24. $\dfrac{dx}{dt} = \dfrac{(3t)\dfrac{d}{dt}(t^2+1) - (t^2+1)\dfrac{d}{dt}(3t)}{(3t)^2} = \dfrac{(3t)(2t) - (t^2+1)(3)}{9t^2} = \dfrac{t^2-1}{3t^2}$

25. $\dfrac{dy}{dx} = \dfrac{(x+3)\dfrac{d}{dx}(2x-1) - (2x-1)\dfrac{d}{dx}(x+3)}{(x+3)^2}$

$= \dfrac{(x+3)(2) - (2x-1)(1)}{(x+3)^2} = \dfrac{7}{(x+3)^2};\ \left.\dfrac{dy}{dx}\right|_{x=1} = \dfrac{7}{16}$

26. $\dfrac{dy}{dx} = \dfrac{(x^2-5)\dfrac{d}{dx}(4x+1) - (4x+1)\dfrac{d}{dx}(x^2-5)}{(x^2-5)^2}$

$= \dfrac{(x^2-5)(4) - (4x+1)(2x)}{(x^2-5)^2} = -\dfrac{4x^2+2x+20}{(x^2-5)^2};\ \left.\dfrac{dy}{dx}\right|_{x=1} = \dfrac{13}{8}$

27. $\dfrac{dy}{dx} = \left(\dfrac{3x+2}{x}\right)\dfrac{d}{dx}\left(x^{-5}+1\right) + \left(x^{-5}+1\right)\dfrac{d}{dx}\left(\dfrac{3x+2}{x}\right)$

$= \left(\dfrac{3x+2}{x}\right)\left(-5x^{-6}\right) + \left(x^{-5}+1\right)\left[\dfrac{x(3)-(3x+2)(1)}{x^2}\right]$

$= \left(\dfrac{3x+2}{x}\right)\left(-5x^{-6}\right) + \left(x^{-5}+1\right)\left(-\dfrac{2}{x^2}\right);$

$\left.\dfrac{dy}{dx}\right|_{x=1} = 5(-5) + 2(-2) = -29$

28. $\dfrac{dy}{dx} = (2x^7-x^2)\dfrac{d}{dx}\left(\dfrac{x-1}{x+1}\right) + \left(\dfrac{x-1}{x+1}\right)\dfrac{d}{dx}(2x^7-x^2)$

$= (2x^7-x^2)\left[\dfrac{(x+1)(1)-(x-1)(1)}{(x+1)^2}\right] + \left(\dfrac{x-1}{x+1}\right)(14x^6-2x)$

$= (2x^7-x^2)\cdot\dfrac{2}{(x+1)^2} + \left(\dfrac{x-1}{x+1}\right)(14x^6-2x);$

$\left.\dfrac{dy}{dx}\right|_{x=1} = (2-1)\dfrac{2}{4} + 0(14-2) = \dfrac{1}{2}$

29. $f'(1) \approx \dfrac{f(1.01)-f(1)}{0.01} = \dfrac{0.999699-(-1)}{0.01} = 0.0301$, and by differentiation, $f'(1) = 3(1)^2 - 3 = 0$

30. $f'(1) \approx \dfrac{f(1.01)-f(1)}{0.01} = \dfrac{1.01504-1}{0.01} = 1.504$, and by differentiation,

$f'(1) = \left.\left(\sqrt{x} + \dfrac{x}{2\sqrt{x}}\right)\right|_{x=1} = 1.5$

31. $f'(1) = 0$

32. $f'(1) = 1$

33. $32t$

34. 2π

35. $3\pi r^2$

36. $-2\alpha^{-2} + 1$

37. **(a)** $\dfrac{dV}{dr} = 4\pi r^2$ **(b)** $\left.\dfrac{dV}{dr}\right|_{r=5} = 4\pi(5)^2 = 100\pi$

38. $\dfrac{d}{d\lambda}\left[\dfrac{\lambda\lambda_0 + \lambda^6}{2 - \lambda_0}\right] = \dfrac{1}{2-\lambda_0}\dfrac{d}{d\lambda}(\lambda\lambda_0 + \lambda^6) = \dfrac{1}{2-\lambda_0}(\lambda_0 + 6\lambda^5) = \dfrac{\lambda_0 + 6\lambda^5}{2 - \lambda_0}$

39. **(a)** $g'(x) = \sqrt{x}f'(x) + \dfrac{1}{2\sqrt{x}}f(x)$, $g'(4) = (2)(-5) + \dfrac{1}{4}(3) = -37/4$

(b) $g'(x) = \dfrac{xf'(x) - f(x)}{x^2}$, $g'(4) = \dfrac{(4)(-5) - 3}{16} = -23/16$

40. **(a)** $g'(x) = 6x - 5f'(x)$, $g'(3) = 6(3) - 5(4) = -2$

(b) $g'(x) = \dfrac{2f(x) - (2x+1)f'(x)}{f^2(x)}$, $g'(3) = \dfrac{2(-2) - 7(4)}{(-2)^2} = -8$

41. **(a)** $F'(x) = 5f'(x) + 2g'(x), F'(2) = 5(4) + 2(-5) = 10$

(b) $F'(x) = f'(x) - 3g'(x), F'(2) = 4 - 3(-5) = 19$

(c) $F'(x) = f(x)g'(x) + g(x)f'(x), F'(2) = (-1)(-5) + (1)(4) = 9$

(d) $F'(x) = [g(x)f'(x) - f(x)g'(x)]/g^2(x), F'(2) = [(1)(4) - (-1)(-5)]/(1)^2 = -1$

42. **(a)** $F'(x) = 6f'(x) - 5g'(x), F'(\pi) = 6(-1) - 5(2) = -16$

(b) $F'(x) = f(x) + g(x) + x(f'(x) + g'(x)), F'(\pi) = 10 - 3 + \pi(-1 + 2) = 7 + \pi$

(c) $F'(x) = 2f(x)g'(x) + 2f'(x)g(x) = 2(20) + 2(3) = 46$

(d) $F'(x) = \dfrac{(4 + g(x))f'(x) - f(x)g'(x)}{(4 + g(x))^2} = \dfrac{(4-3)(-1) - 10(2)}{(4-3)^2} = -21$

43. $y - 2 = 5(x + 3)$, $y = 5x + 17$

44. $\dfrac{dy}{dx} = \dfrac{(1+x)(-1) - (1-x)(1)}{(1+x)^2} = -\dfrac{2}{(1+x)^2}$, $\left.\dfrac{dy}{dx}\right|_{x=2} = -\dfrac{2}{9}$ and $y = -\dfrac{1}{3}$ for $x = 2$ so an equation of the tangent line is $y - \left(-\dfrac{1}{3}\right) = -\dfrac{2}{9}(x - 2)$, or $y = -\dfrac{2}{9}x + \dfrac{1}{9}$.

45. **(a)** $dy/dx = 21x^2 - 10x + 1$, $d^2y/dx^2 = 42x - 10$

(b) $dy/dx = 24x - 2$, $d^2y/dx^2 = 24$

(c) $dy/dx = -1/x^2$, $d^2y/dx^2 = 2/x^3$

(d) $y = 35x^5 - 16x^3 - 3x$, $dy/dx = 175x^4 - 48x^2 - 3$, $d^2y/dx^2 = 700x^3 - 96x$

46. **(a)** $y' = 28x^6 - 15x^2 + 2$, $y'' = 168x^5 - 30x$

(b) $y' = 3$, $y'' = 0$

(c) $y' = \dfrac{2}{5x^2}$, $y'' = -\dfrac{4}{5x^3}$

(d) $y = 2x^4 + 3x^3 - 10x - 15$, $y' = 8x^3 + 9x^2 - 10$, $y'' = 24x^2 + 18x$

47. **(a)** $y' = -5x^{-6} + 5x^4$, $y'' = 30x^{-7} + 20x^3$, $y''' = -210x^{-8} + 60x^2$

(b) $y = x^{-1}$, $y' = -x^{-2}$, $y'' = 2x^{-3}$, $y''' = -6x^{-4}$

(c) $y' = 3ax^2 + b$, $y'' = 6ax$, $y''' = 6a$

48. **(a)** $dy/dx = 10x - 4$, $d^2y/dx^2 = 10$, $d^3y/dx^3 = 0$
(b) $dy/dx = -6x^{-3} - 4x^{-2} + 1$, $d^2y/dx^2 = 18x^{-4} + 8x^{-3}$, $d^3y/dx^3 = -72x^{-5} - 24x^{-4}$
(c) $dy/dx = 4ax^3 + 2bx$, $d^2y/dx^2 = 12ax^2 + 2b$, $d^3y/dx^3 = 24ax$

49. **(a)** $f'(x) = 6x$, $f''(x) = 6$, $f'''(x) = 0$, $f'''(2) = 0$
(b) $\dfrac{dy}{dx} = 30x^4 - 8x$, $\dfrac{d^2y}{dx^2} = 120x^3 - 8$, $\left.\dfrac{d^2y}{dx^2}\right|_{x=1} = 112$
(c) $\dfrac{d}{dx}\left[x^{-3}\right] = -3x^{-4}$, $\dfrac{d^2}{dx^2}\left[x^{-3}\right] = 12x^{-5}$, $\dfrac{d^3}{dx^3}\left[x^{-3}\right] = -60x^{-6}$, $\dfrac{d^4}{dx^4}\left[x^{-3}\right] = 360x^{-7}$,
$\left.\dfrac{d^4}{dx^4}\left[x^{-3}\right]\right|_{x=1} = 360$

50. **(a)** $y' = 16x^3 + 6x^2$, $y'' = 48x^2 + 12x$, $y''' = 96x + 12$, $y'''(0) = 12$
(b) $y = 6x^{-4}$, $\dfrac{dy}{dx} = -24x^{-5}$, $\dfrac{d^2y}{dx^2} = 120x^{-6}$, $\dfrac{d^3y}{dx^3} = -720x^{-7}$, $\dfrac{d^4y}{dx^4} = 5040x^{-8}$,
$\left.\dfrac{d^4y}{dx^4}\right|_{x=1} = 5040$

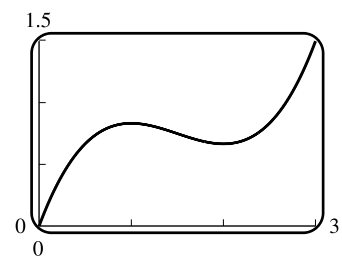
51. $y' = 3x^2 + 3$, $y'' = 6x$, and $y''' = 6$ so
$y''' + xy'' - 2y' = 6 + x(6x) - 2(3x^2 + 3) = 6 + 6x^2 - 6x^2 - 6 = 0$

52. $y = x^{-1}$, $y' = -x^{-2}$, $y'' = 2x^{-3}$ so
$x^3y'' + x^2y' - xy = x^3(2x^{-3}) + x^2(-x^{-2}) - x(x^{-1}) = 2 - 1 - 1 = 0$

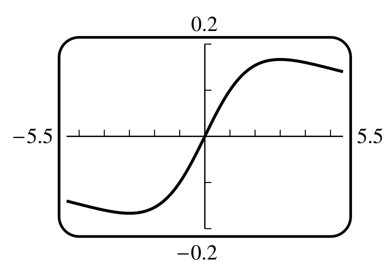
53. $F'(x) = xf'(x) + f(x)$, $F''(x) = xf''(x) + f'(x) + f'(x) = xf''(x) + 2f'(x)$

54. **(a)** $F'''(x) = xf'''(x) + 3f''(x)$
(b) Assume that $F^{n)}(x) = xf^{(n)}(x) + nf^{(n-1)}(x)$ for some n (for instance $n = 3$, as in part (a)). Then $F^{(n+1)}(x) = xf^{(n+1)}(x) + (1+n)f^{(n)}(x) = xf^{(n+1)}(x) + (n+1)f^{(n)}(x)$, which is an inductive proof.

55. The graph has a horizontal tangent at points where $\dfrac{dy}{dx} = 0$, but $\dfrac{dy}{dx} = x^2 - 3x + 2 = (x-1)(x-2) = 0$ if $x = 1, 2$. The corresponding values of y are $5/6$ and $2/3$ so the tangent line is horizontal at $(1, 5/6)$ and $(2, 2/3)$.



56. $\dfrac{dy}{dx} = \dfrac{9 - x^2}{(x^2+9)^2}$; $\dfrac{dy}{dx} = 0$ when $x^2 = 9$ so $x = \pm 3$.
The points are $(3, 1/6)$ and $(-3, -1/6)$.



57. The y-intercept is -2 so the point $(0,-2)$ is on the graph; $-2 = a(0)^2 + b(0) + c$, $c = -2$. The x-intercept is 1 so the point (1,0) is on the graph; $0 = a + b - 2$. The slope is $dy/dx = 2ax + b$; at $x = 0$ the slope is b so $b = -1$, thus $a = 3$. The function is $y = 3x^2 - x - 2$.

58. Let $P(x_0, y_0)$ be the point where $y = x^2 + k$ is tangent to $y = 2x$. The slope of the curve is $\dfrac{dy}{dx} = 2x$ and the slope of the line is 2 thus at P, $2x_0 = 2$ so $x_0 = 1$. But P is on the line, so $y_0 = 2x_0 = 2$. Because P is also on the curve we get $y_0 = x_0^2 + k$ so $k = y_0 - x_0^2 = 2 - (1)^2 = 1$.

59. The points $(-1, 1)$ and $(2, 4)$ are on the secant line so its slope is $(4-1)/(2+1) = 1$. The slope of the tangent line to $y = x^2$ is $y' = 2x$ so $2x = 1$, $x = 1/2$.

60. The points $(1, 1)$ and $(4, 2)$ are on the secant line so its slope is $1/3$. The slope of the tangent line to $y = \sqrt{x}$ is $y' = 1/(2\sqrt{x})$ so $1/(2\sqrt{x}) = 1/3$, $2\sqrt{x} = 3$, $x = 9/4$.

61. $y' = -2x$, so at any point (x_0, y_0) on $y = 1 - x^2$ the tangent line is $y - y_0 = -2x_0(x - x_0)$, or $y = -2x_0x + x_0^2 + 1$. The point $(2, 0)$ is to be on the line, so $0 = -4x_0 + x_0^2 + 1$, $x_0^2 - 4x_0 + 1 = 0$. Use the quadratic formula to get $x_0 = \dfrac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$.

62. Let $P_1(x_1, ax_1^2)$ and $P_2(x_2, ax_2^2)$ be the points of tangency. $y' = 2ax$ so the tangent lines at P_1 and P_2 are $y - ax_1^2 = 2ax_1(x - x_1)$ and $y - ax_2^2 = 2ax_2(x - x_2)$. Solve for x to get $x = \frac{1}{2}(x_1 + x_2)$ which is the x-coordinate of a point on the vertical line halfway between P_1 and P_2.

63. $y' = 3ax^2 + b$; the tangent line at $x = x_0$ is $y - y_0 = (3ax_0^2 + b)(x - x_0)$ where $y_0 = ax_0^3 + bx_0$. Solve with $y = ax^3 + bx$ to get

$$\begin{aligned}
(ax^3 + bx) - (ax_0^3 + bx_0) &= (3ax_0^2 + b)(x - x_0)\\
ax^3 + bx - ax_0^3 - bx_0 &= 3ax_0^2x - 3ax_0^3 + bx - bx_0\\
x^3 - 3x_0^2x + 2x_0^3 &= 0\\
(x - x_0)(x^2 + xx_0 - 2x_0^2) &= 0\\
(x - x_0)^2(x + 2x_0) &= 0, \text{ so } x = -2x_0.
\end{aligned}$$

64. Let (x_0, y_0) be the point of tangency. Refer to the solution to Exercise 65 to see that the endpoints of the line segment are at $(2x_0, 0)$ and $(0, 2y_0)$, so (x_0, y_0) is the midpoint of the segment.

65. $y' = -\dfrac{1}{x^2}$; the tangent line at $x = x_0$ is $y - y_0 = -\dfrac{1}{x_0^2}(x - x_0)$, or $y = -\dfrac{x}{x_0^2} + \dfrac{2}{x_0}$. The tangent line crosses the x-axis at $2x_0$, the y-axis at $2/x_0$, so that the area of the triangle is $\dfrac{1}{2}(2/x_0)(2x_0) = 2$.

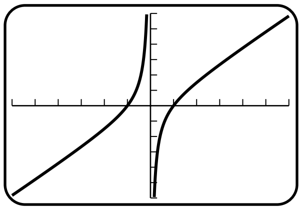
66. $f'(x) = 3ax^2 + 2bx + c$; there is a horizontal tangent where $f'(x) = 0$. Use the quadratic formula on $3ax^2 + 2bx + c = 0$ to get $x = (-b \pm \sqrt{b^2 - 3ac})/(3a)$ which gives two real solutions, one real solution, or none if

(a) $b^2 - 3ac > 0$ **(b)** $b^2 - 3ac = 0$ **(c)** $b^2 - 3ac < 0$

67. $F = GmMr^{-2}$, $\dfrac{dF}{dr} = -2GmMr^{-3} = -\dfrac{2GmM}{r^3}$

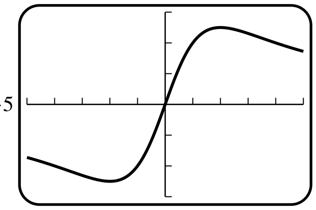
68. $dR/dT = 0.04124 - 3.558 \times 10^{-5}T$ which decreases as T increases from 0 to 700. When $T = 0$, $dR/dT = 0.04124\,\Omega/^\circ\mathrm{C}$; when $T = 700$, $dR/dT = 0.01633\,\Omega/^\circ\mathrm{C}$. The resistance is most sensitive to temperature changes at $T = 0^\circ\mathrm{C}$, least sensitive at $T = 700^\circ\mathrm{C}$.

69. $f'(x) = 1 + 1/x^2 > 0$ for all $x \neq 0$



70. $f'(x) = -5\dfrac{x^2 - 4}{(x^2 + 4)^2}$;

$f'(x) > 0$ when $x^2 < 4$, i.e. on $-2 < x < 2$



71. $(f \cdot g \cdot h)' = [(f \cdot g) \cdot h]' = (f \cdot g)h' + h(f \cdot g)' = (f \cdot g)h' + h[fg' + f'g] = fgh' + fg'h + f'gh$

72. $(f_1 f_2 \cdots f_n)' = (f_1' f_2 \cdots f_n) + (f_1 f_2' \cdots f_n) + \cdots + (f_1 f_2 \cdots f_n')$

73. **(a)** $2(1 + x^{-1})(x^{-3} + 7) + (2x + 1)(-x^{-2})(x^{-3} + 7) + (2x + 1)(1 + x^{-1})(-3x^{-4})$

(b) $(x^7 + 2x - 3)^3 = (x^7 + 2x - 3)(x^7 + 2x - 3)(x^7 + 2x - 3)$ so

$$\begin{aligned}\frac{d}{dx}(x^7 + 2x - 3)^3 &= (7x^6 + 2)(x^7 + 2x - 3)(x^7 + 2x - 3)\\ &\quad + (x^7 + 2x - 3)(7x^6 + 2)(x^7 + 2x - 3)\\ &\quad + (x^7 + 2x - 3)(x^7 + 2x - 3)(7x^6 + 2)\\ &= 3(7x^6 + 2)(x^7 + 2x - 3)^2\end{aligned}$$

74. **(a)** $-5x^{-6}(x^2 + 2x)(4 - 3x)(2x^9 + 1) + x^{-5}(2x + 2)(4 - 3x)(2x^9 + 1)$
$+x^{-5}(x^2 + 2x)(-3)(2x^9 + 1) + x^{-5}(x^2 + 2x)(4 - 3x)(18x^8)$

(b) $(x^2 + 1)^{50} = (x^2 + 1)(x^2 + 1) \cdots (x^2 + 1)$, where $(x^2 + 1)$ occurs 50 times so

$$\begin{aligned}\frac{d}{dx}(x^2 + 1)^{50} &= [(2x)(x^2 + 1) \cdots (x^2 + 1)] + [(x^2 + 1)(2x) \cdots (x^2 + 1)]\\ &\quad + \cdots + [(x^2 + 1)(x^2 + 1) \cdots (2x)]\\ &= 2x(x^2 + 1)^{49} + 2x(x^2 + 1)^{49} + \cdots + 2x(x^2 + 1)^{49}\\ &= 100x(x^2 + 1)^{49} \text{ because } 2x(x^2 + 1)^{49} \text{ occurs 50 times.}\end{aligned}$$

75. f is continuous at 1 because $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$; also $\lim_{x \to 1^-} f'(x) = \lim_{x \to 1^-} (2x + 1) = 3$ and $\lim_{x \to 1^+} f'(x) = \lim_{x \to 1^+} 3 = 3$ so f is differentiable at 1.

76. f is not continuous at $x = 9$ because $\lim_{x \to 9^-} f(x) = -63$ and $\lim_{x \to 9^+} f(x) = 36$.
f cannot be differentiable at $x = 9$, for if it were, then f would also be continuous, which it is not.

77. f is continuous at 1 because $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$, also $\lim_{x \to 1^-} f'(x) = \lim_{x \to 1^-} 2x = 2$ and $\lim_{x \to 1^+} f'(x) = \lim_{x \to 1^+} \dfrac{1}{2\sqrt{x}} = \dfrac{1}{2}$ so f is not differentiable at 1.

78. f is continuous at 1/2 because $\lim_{x \to 1/2^-} f(x) = \lim_{x \to 1/2^+} f(x) = f(1/2)$, also
$\lim_{x \to 1/2^-} f'(x) = \lim_{x \to 1/2^-} 3x^2 = 3/4$ and $\lim_{x \to 1/2^+} f'(x) = \lim_{x \to 1/2^+} 3x/2 = 3/4$ so $f'(1/2) = 3/4$, and f is differentiable at $x = 1/2$.

79. **(a)** $f(x) = 3x - 2$ if $x \geq 2/3$, $f(x) = -3x + 2$ if $x < 2/3$ so f is differentiable everywhere except perhaps at 2/3. f is continuous at 2/3, also $\lim_{x\to 2/3^-} f'(x) = \lim_{x\to 2/3^-}(-3) = -3$ and $\lim_{x\to 2/3^+} f'(x) = \lim_{x\to 2/3^+}(3) = 3$ so f is not differentiable at $x = 2/3$.

(b) $f(x) = x^2 - 4$ if $|x| \geq 2$, $f(x) = -x^2 + 4$ if $|x| < 2$ so f is differentiable everywhere except perhaps at ± 2. f is continuous at -2 and 2, also $\lim_{x\to 2^-} f'(x) = \lim_{x\to 2^-}(-2x) = -4$ and $\lim_{x\to 2^+} f'(x) = \lim_{x\to 2^+}(2x) = 4$ so f is not differentiable at $x = 2$. Similarly, f is not differentiable at $x = -2$.

80. **(a)** $f'(x) = -(1)x^{-2}$, $f''(x) = (2\cdot 1)x^{-3}$, $f'''(x) = -(3\cdot 2\cdot 1)x^{-4}$

$$f^{(n)}(x) = (-1)^n \frac{n(n-1)(n-2)\cdots 1}{x^{n+1}}$$

(b) $f'(x) = -2x^{-3}$, $f''(x) = (3\cdot 2)x^{-4}$, $f'''(x) = -(4\cdot 3\cdot 2)x^{-5}$

$$f^{(n)}(x) = (-1)^n \frac{(n+1)(n)(n-1)\cdots 2}{x^{n+2}}$$

81. **(a)** $$\frac{d^2}{dx^2}[cf(x)] = \frac{d}{dx}\left[\frac{d}{dx}[cf(x)]\right] = \frac{d}{dx}\left[c\frac{d}{dx}[f(x)]\right] = c\frac{d}{dx}\left[\frac{d}{dx}[f(x)]\right] = c\frac{d^2}{dx^2}[f(x)]$$

$$\frac{d^2}{dx^2}[f(x)+g(x)] = \frac{d}{dx}\left[\frac{d}{dx}[f(x)+g(x)]\right] = \frac{d}{dx}\left[\frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]\right] = \frac{d^2}{dx^2}[f(x)] + \frac{d^2}{dx^2}[g(x)]$$

(b) yes, by repeated application of the procedure illustrated in Part (a)

82. $(f\cdot g)' = fg' + gf', (f\cdot g)'' = fg'' + g'f' + gf'' + f'g' = f''g + 2f'g' + fg''$

83. **(a)** $f'(x) = nx^{n-1}$, $f''(x) = n(n-1)x^{n-2}$, $f'''(x) = n(n-1)(n-2)x^{n-3}, \ldots,$ $f^{(n)}(x) = n(n-1)(n-2)\cdots 1$

(b) from Part (a), $f^{(k)}(x) = k(k-1)(k-2)\cdots 1$ so $f^{(k+1)}(x) = 0$ thus $f^{(n)}(x) = 0$ if $n > k$

(c) from Parts (a) and (b), $f^{(n)}(x) = a_n n(n-1)(n-2)\cdots 1$

84. $\lim_{h\to 0} \frac{f'(2+h) - f'(2)}{h} = f''(2)$; $f'(x) = 8x^7 - 2$, $f''(x) = 56x^6$, so $f''(2) = 56(2^6) = 3584$.

85. **(a)** If a function is differentiable at a point then it is continuous at that point, thus f' is continuous on (a, b) and consequently so is f.

(b) f and all its derivatives up to $f^{(n-1)}(x)$ are continuous on (a, b)

EXERCISE SET 3.4

1. $f'(x) = -2\sin x - 3\cos x$

2. $f'(x) = \sin x(-\sin x) + \cos x(\cos x) = \cos^2 x - \sin^2 x = \cos 2x$

3. $f'(x) = \dfrac{x(\cos x) - (\sin x)(1)}{x^2} = \dfrac{x\cos x - \sin x}{x^2}$

4. $f'(x) = x^2(-\sin x) + (\cos x)(2x) = -x^2 \sin x + 2x \cos x$

5. $f'(x) = x^3(\cos x) + (\sin x)(3x^2) - 5(-\sin x) = x^3 \cos x + (3x^2 + 5)\sin x$

6. $f(x) = \dfrac{\cot x}{x}$ (because $\dfrac{\cos x}{\sin x} = \cot x$), $f'(x) = \dfrac{x(-\csc^2 x) - (\cot x)(1)}{x^2} = -\dfrac{x\csc^2 x + \cot x}{x^2}$

7. $f'(x) = \sec x \tan x - \sqrt{2}\sec^2 x$

8. $f'(x) = (x^2+1)\sec x \tan x + (\sec x)(2x) = (x^2+1)\sec x \tan x + 2x \sec x$

9. $f'(x) = \sec x(\sec^2 x) + (\tan x)(\sec x \tan x) = \sec^3 x + \sec x \tan^2 x$

10. $f'(x) = \dfrac{(1+\tan x)(\sec x \tan x) - (\sec x)(\sec^2 x)}{(1+\tan x)^2} = \dfrac{\sec x \tan x + \sec x \tan^2 x - \sec^3 x}{(1+\tan x)^2}$

$= \dfrac{\sec x(\tan x + \tan^2 x - \sec^2 x)}{(1+\tan x)^2} = \dfrac{\sec x(\tan x - 1)}{(1+\tan x)^2}$

11. $f'(x) = (\csc x)(-\csc^2 x) + (\cot x)(-\csc x \cot x) = -\csc^3 x - \csc x \cot^2 x$

12. $f'(x) = 1 + 4\csc x \cot x - 2\csc^2 x$

13. $f'(x) = \dfrac{(1+\csc x)(-\csc^2 x) - \cot x(0 - \csc x \cot x)}{(1+\csc x)^2} = \dfrac{\csc x(-\csc x - \csc^2 x + \cot^2 x)}{(1+\csc x)^2}$ but

$1 + \cot^2 x = \csc^2 x$ (identity) thus $\cot^2 x - \csc^2 x = -1$ so

$f'(x) = \dfrac{\csc x(-\csc x - 1)}{(1+\csc x)^2} = -\dfrac{\csc x}{1+\csc x}$

14. $f'(x) = \dfrac{\tan x(-\csc x \cot x) - \csc x(\sec^2 x)}{\tan^2 x} = -\dfrac{\csc x(1+\sec^2 x)}{\tan^2 x}$

15. $f(x) = \sin^2 x + \cos^2 x = 1$ (identity) so $f'(x) = 0$

16. $f(x) = \dfrac{1}{\cot x} = \tan x$, so $f'(x) = \sec^2 x$

17. $f(x) = \dfrac{\tan x}{1 + x\tan x}$ (because $\sin x \sec x = (\sin x)(1/\cos x) = \tan x$),

$f'(x) = \dfrac{(1+x\tan x)(\sec^2 x) - \tan x[x(\sec^2 x) + (\tan x)(1)]}{(1+x\tan x)^2}$

$= \dfrac{\sec^2 x - \tan^2 x}{(1+x\tan x)^2} = \dfrac{1}{(1+x\tan x)^2}$ (because $\sec^2 x - \tan^2 x = 1$)

18. $f(x) = \dfrac{(x^2+1)\cot x}{3 - \cot x}$ (because $\cos x \csc x = (\cos x)(1/\sin x) = \cot x$),

$f'(x) = \dfrac{(3-\cot x)[2x\cot x - (x^2+1)\csc^2 x] - (x^2+1)\cot x \csc^2 x}{(3-\cot x)^2}$

$= \dfrac{6x\cot x - 2x\cot^2 x - 3(x^2+1)\csc^2 x}{(3-\cot x)^2}$

19. $dy/dx = -x\sin x + \cos x$, $d^2y/dx^2 = -x\cos x - \sin x - \sin x = -x\cos x - 2\sin x$

20. $dy/dx = -\csc x\cot x$, $d^2y/dx^2 = -[(\csc x)(-\csc^2 x) + (\cot x)(-\csc x\cot x)] = \csc^3 x + \csc x\cot^2 x$

21. $dy/dx = x(\cos x) + (\sin x)(1) - 3(-\sin x) = x\cos x + 4\sin x$,
$d^2y/dx^2 = x(-\sin x) + (\cos x)(1) + 4\cos x = -x\sin x + 5\cos x$

22. $dy/dx = x^2(-\sin x) + (\cos x)(2x) + 4\cos x = -x^2\sin x + 2x\cos x + 4\cos x$,
$d^2y/dx^2 = -[x^2(\cos x) + (\sin x)(2x)] + 2[x(-\sin x) + \cos x] - 4\sin x = (2 - x^2)\cos x - 4(x+1)\sin x$

23. $dy/dx = (\sin x)(-\sin x) + (\cos x)(\cos x) = \cos^2 x - \sin^2 x$,
$d^2y/dx^2 = (\cos x)(-\sin x) + (\cos x)(-\sin x) - [(\sin x)(\cos x) + (\sin x)(\cos x)] = -4\sin x\cos x$

24. $dy/dx = \sec^2 x$; $d^2y/dx^2 = 2\sec^2 x\tan x$

25. Let $f(x) = \tan x$, then $f'(x) = \sec^2 x$.
(a) $f(0) = 0$ and $f'(0) = 1$ so $y - 0 = (1)(x - 0)$, $y = x$.
(b) $f\left(\frac{\pi}{4}\right) = 1$ and $f'\left(\frac{\pi}{4}\right) = 2$ so $y - 1 = 2\left(x - \frac{\pi}{4}\right)$, $y = 2x - \frac{\pi}{2} + 1$.
(c) $f\left(-\frac{\pi}{4}\right) = -1$ and $f'\left(-\frac{\pi}{4}\right) = 2$ so $y + 1 = 2\left(x + \frac{\pi}{4}\right)$, $y = 2x + \frac{\pi}{2} - 1$.

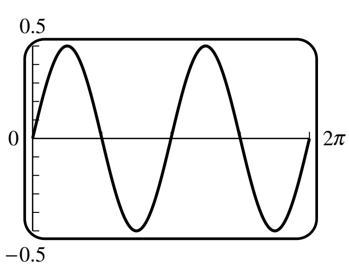
26. Let $f(x) = \sin x$, then $f'(x) = \cos x$.
(a) $f(0) = 0$ and $f'(0) = 1$ so $y - 0 = (1)(x - 0)$, $y = x$
(b) $f(\pi) = 0$ and $f'(\pi) = -1$ so $y - 0 = (-1)(x - \pi)$, $y = -x + \pi$
(c) $f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ and $f'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ so $y - \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right)$, $y = \frac{1}{\sqrt{2}}x - \frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}}$

27. **(a)** If $y = x\sin x$ then $y' = \sin x + x\cos x$ and $y'' = 2\cos x - x\sin x$ so $y'' + y = 2\cos x$.
(b) If $y = x\sin x$ then $y' = \sin x + x\cos x$ and $y'' = 2\cos x - x\sin x$ so $y'' + y = 2\cos x$; differentiate twice more to get $y^{(4)} + y'' = -2\cos x$.

28. **(a)** If $y = \cos x$ then $y' = -\sin x$ and $y'' = -\cos x$ so $y'' + y = (-\cos x) + (\cos x) = 0$;
if $y = \sin x$ then $y' = \cos x$ and $y'' = -\sin x$ so $y'' + y = (-\sin x) + (\sin x) = 0$.
(b) $y' = A\cos x - B\sin x$, $y'' = -A\sin x - B\cos x$ so
$y'' + y = (-A\sin x - B\cos x) + (A\sin x + B\cos x) = 0$.

29. **(a)** $f'(x) = \cos x = 0$ at $x = \pm\pi/2, \pm 3\pi/2$.
(b) $f'(x) = 1 - \sin x = 0$ at $x = -3\pi/2, \pi/2$.
(c) $f'(x) = \sec^2 x \geq 1$ always, so no horizontal tangent line.
(d) $f'(x) = \sec x\tan x = 0$ when $\sin x = 0$, $x = \pm 2\pi, \pm\pi, 0$

30. **(a)**



(b) $y = \sin x\cos x = (1/2)\sin 2x$ and $y' = \cos 2x$. So $y' = 0$ when $2x = (2n+1)\pi/2$ for $n = 0, 1, 2, 3$ or $x = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$

31. $x = 10\sin\theta$, $dx/d\theta = 10\cos\theta$; if $\theta = 60°$, then
$dx/d\theta = 10(1/2) = 5$ ft/rad $= \pi/36$ ft/deg ≈ 0.087 ft/deg

32. $s = 3800\csc\theta, ds/d\theta = -3800\csc\theta\cot\theta$; if $\theta = 30°$, then
$ds/d\theta = -3800(2)(\sqrt{3}) = -7600\sqrt{3}$ ft/rad $= -380\sqrt{3}\pi/9$ ft/deg ≈ -230 ft/deg

33. $D = 50\tan\theta, dD/d\theta = 50\sec^2\theta$; if $\theta = 45°$, then
$dD/d\theta = 50(\sqrt{2})^2 = 100$ m/rad $= 5\pi/9$ m/deg ≈ 1.75 m/deg

34. **(a)** From the right triangle shown, $\sin\theta = r/(r+h)$ so $r + h = r\csc\theta$, $h = r(\csc\theta - 1)$.
(b) $dh/d\theta = -r\csc\theta\cot\theta$; if $\theta = 30°$, then
$dh/d\theta = -6378(2)(\sqrt{3}) \approx -22,094$ km/rad ≈ -386 km/deg

35. **(a)** $\dfrac{d^4}{dx^4}\sin x = \sin x$, so $\dfrac{d^{4k}}{dx^{4k}}\sin x = \sin x$; $\dfrac{d^{87}}{dx^{87}}\sin x = \dfrac{d^3}{dx^3}\dfrac{d^{4\cdot 21}}{dx^{4\cdot 21}}\sin x = \dfrac{d^3}{dx^3}\sin x = -\cos x$

(b) $\dfrac{d^{100}}{dx^{100}}\cos x = \dfrac{d^{4k}}{dx^{4k}}\cos x = \cos x$

36. $\dfrac{d}{dx}[x\sin x] = x\cos x + \sin x$ $\qquad \dfrac{d^2}{dx^2}[x\sin x] = -x\sin x + 2\cos x$

$\dfrac{d^3}{dx^3}[x\sin x] = -x\cos x - 3\sin x$ $\qquad \dfrac{d^4}{dx^4}[x\sin x] = x\sin x - 4\cos x$

By mathematical induction one can show

$\dfrac{d^{4k}}{dx^{4k}}[x\sin x] = x\sin x - (4k)\cos x;$ $\qquad \dfrac{d^{4k+1}}{dx^{4k+1}}[x\sin x] = x\cos x + (4k+1)\sin x;$

$\dfrac{d^{4k+2}}{dx^{4k+2}}[x\sin x] = -x\sin x + (4k+2)\cos x;$ $\qquad \dfrac{d^{4k+3}}{dx^{4k+3}}[x\sin x] = -x\cos x - (4k+3)\sin x;$

Since $17 = 4\cdot 4 + 1$, $\dfrac{d^{17}}{dx^{17}}[x\sin x] = x\cos x + 17\sin x$

37. **(a)** all x **(b)** all x
(c) $x \ne \pi/2 + n\pi$, $n = 0, \pm1, \pm2, \ldots$ **(d)** $x \ne n\pi$, $n = 0, \pm1, \pm2, \ldots$
(e) $x \ne \pi/2 + n\pi$, $n = 0, \pm1, \pm2, \ldots$ **(f)** $x \ne n\pi$, $n = 0, \pm1, \pm2, \ldots$
(g) $x \ne (2n+1)\pi$, $n = 0, \pm1, \pm2, \ldots$ **(h)** $x \ne n\pi/2$, $n = 0, \pm1, \pm2, \ldots$
(i) all x

38. **(a)** $$\frac{d}{dx}[\cos x] = \lim_{h\to 0}\frac{\cos(x+h) - \cos x}{h} = \lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h}$$
$$= \lim_{h\to 0}\left[\cos x\left(\frac{\cos h - 1}{h}\right) - \sin x\left(\frac{\sin h}{h}\right)\right] = (\cos x)(0) - (\sin x)(1) = -\sin x$$

(b) $$\frac{d}{dx}[\sec x] = \frac{d}{dx}\left[\frac{1}{\cos x}\right] = \frac{\cos x(0) - (1)(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \sec x\tan x$$

(c) $$\frac{d}{dx}[\cot x] = \frac{d}{dx}\left[\frac{\cos x}{\sin x}\right] = \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x}$$
$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x$$

(d) $$\frac{d}{dx}[\csc x] = \frac{d}{dx}\left[\frac{1}{\sin x}\right] = \frac{(\sin x)(0) - (1)(\cos x)}{\sin^2 x} = -\frac{\cos x}{\sin^2 x} = -\csc x\cot x$$

39. $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, and $f^{(4)}(x) = \cos x$ with higher order derivatives repeating this pattern, so $f^{(n)}(x) = \sin x$ for $n = 3, 7, 11, \ldots$

40. **(a)** $\lim\limits_{h\to 0} \dfrac{\tan h}{h} = \lim\limits_{h\to 0} \dfrac{\left(\dfrac{\sin h}{\cos h}\right)}{h} = \lim\limits_{h\to 0} \dfrac{\left(\dfrac{\sin h}{h}\right)}{\cos h} = \dfrac{1}{1} = 1$

(b)
$$\frac{d}{dx}[\tan x] = \lim_{h\to 0} \frac{\tan(x+h) - \tan x}{h} = \lim_{h\to 0} \frac{\dfrac{\tan x + \tan h}{1 - \tan x \tan h} - \tan x}{h}$$
$$= \lim_{h\to 0} \frac{\tan x + \tan h - \tan x + \tan^2 x \tan h}{h(1 - \tan x \tan h)} = \lim_{h\to 0} \frac{\tan h(1 + \tan^2 x)}{h(1 - \tan x \tan h)}$$
$$= \lim_{h\to 0} \frac{\tan h \sec^2 x}{h(1 - \tan x \tan h)} = \sec^2 x \lim_{h\to 0} \frac{\dfrac{\tan h}{h}}{1 - \tan x \tan h}$$
$$= \sec^2 x \frac{\lim\limits_{h\to 0} \dfrac{\tan h}{h}}{\lim\limits_{h\to 0}(1 - \tan x \tan h)} = \sec^2 x$$

41. $\lim\limits_{x\to 0} \dfrac{\tan(x+y) - \tan y}{x} = \lim\limits_{h\to 0} \dfrac{\tan(y+h) - \tan y}{h} = \dfrac{d}{dy}(\tan y) = \sec^2 y$

43. Let t be the radian measure, then $h = \dfrac{180}{\pi} t$ and $\cos h = \cos t$, $\sin h = \sin t$.

(a) $\lim\limits_{h\to 0} \dfrac{\cos h - 1}{h} = \lim\limits_{t\to 0} \dfrac{\cos t - 1}{180t/\pi} = \dfrac{\pi}{180} \lim\limits_{t\to 0} \dfrac{\cos t - 1}{t} = 0$

(b) $\lim\limits_{h\to 0} \dfrac{\sin h}{h} = \lim\limits_{t\to 0} \dfrac{\sin t}{180t/\pi} = \dfrac{\pi}{180} \lim\limits_{t\to 0} \dfrac{\sin t}{t} = \dfrac{\pi}{180}$

(c) $\dfrac{d}{dx}[\sin x] = \sin x \lim\limits_{h\to 0} \dfrac{\cos h - 1}{h} + \cos x \lim\limits_{h\to 0} \dfrac{\sin h}{h} = (\sin x)(0) + (\cos x)(\pi/180) = \dfrac{\pi}{180} \cos x$

EXERCISE SET 3.5

1. $(f \circ g)'(x) = f'(g(x))g'(x)$ so $(f \circ g)'(0) = f'(g(0))g'(0) = f'(0)(3) = (2)(3) = 6$

2. $(f \circ g)'(2) = f'(g(2))g'(2) = 5(-3) = -15$

3. **(a)** $(f \circ g)(x) = f(g(x)) = (2x-3)^5$ and $(f \circ g)'(x) = f'(g(x)g'(x) = 5(2x-3)^4(2) = 10(2x-3)^4$
(b) $(g \circ f)(x) = g(f(x)) = 2x^5 - 3$ and $(g \circ f)'(x) = g'(f(x))f'(x) = 2(5x^4) = 10x^4$

4. **(a)** $(f \circ g)(x) = 5\sqrt{4 + \cos x}$ and $(f \circ g)'(x) = f'(g(x))g'(x) = \dfrac{5}{2\sqrt{4 + \cos x}}(-\sin x)$

(b) $(g \circ f)(x) = 4 + \cos(5\sqrt{x})$ and $(g \circ f)'(x) = g'(f(x))f'(x) = -\sin(5\sqrt{x})\dfrac{5}{2\sqrt{x}}$

5. **(a)** $F'(x) = f'(g(x))g'(x) = f'(g(3))g'(3) = -1(7) = -7$
(b) $G'(x) = g'(f(x))f'(x) = g'(f(3))f'(3) = 4(-2) = -8$

6. **(a)** $F'(x) = f'(g(x))g'(x)$, $F'(-1) = f'(g(-1))g'(-1) = f'(2)(-3) = (4)(-3) = -12$
(b) $G'(x) = g'(f(x))f'(x), G'(-1) = g'(f(-1))f'(-1) = -5(3) = -15$

7. $f'(x) = 37(x^3 + 2x)^{36} \frac{d}{dx}(x^3 + 2x) = 37(x^3 + 2x)^{36}(3x^2 + 2)$

8. $f'(x) = 6(3x^2 + 2x - 1)^5 \frac{d}{dx}(3x^2 + 2x - 1) = 6(3x^2 + 2x - 1)^5(6x + 2) = 12(3x^2 + 2x - 1)^5(3x + 1)$

9. $f'(x) = -2\left(x^3 - \frac{7}{x}\right)^{-3} \frac{d}{dx}\left(x^3 - \frac{7}{x}\right) = -2\left(x^3 - \frac{7}{x}\right)^{-3}\left(3x^2 + \frac{7}{x^2}\right)$

10. $f(x) = (x^5 - x + 1)^{-9}$,
$f'(x) = -9(x^5 - x + 1)^{-10} \frac{d}{dx}(x^5 - x + 1) = -9(x^5 - x + 1)^{-10}(5x^4 - 1) = -\frac{9(5x^4 - 1)}{(x^5 - x + 1)^{10}}$

11. $f(x) = 4(3x^2 - 2x + 1)^{-3}$,
$f'(x) = -12(3x^2 - 2x + 1)^{-4} \frac{d}{dx}(3x^2 - 2x + 1) = -12(3x^2 - 2x + 1)^{-4}(6x - 2) = \frac{24(1 - 3x)}{(3x^2 - 2x + 1)^4}$

12. $f'(x) = \frac{1}{2\sqrt{x^3 - 2x + 5}} \frac{d}{dx}(x^3 - 2x + 5) = \frac{3x^2 - 2}{2\sqrt{x^3 - 2x + 5}}$

13. $f'(x) = \frac{1}{2\sqrt{4 + 3\sqrt{x}}} \frac{d}{dx}(4 + 3\sqrt{x}) = \frac{3}{4\sqrt{x}\sqrt{4 + 3\sqrt{x}}}$

14. $f'(x) = 3\sin^2 x \frac{d}{dx}(\sin x) = 3\sin^2 x \cos x$

15. $f'(x) = \cos(x^3) \frac{d}{dx}(x^3) = 3x^2 \cos(x^3)$

16. $f'(x) = 2\cos(3\sqrt{x}) \frac{d}{dx}[\cos(3\sqrt{x})] = -2\cos(3\sqrt{x})\sin(3\sqrt{x}) \frac{d}{dx}(3\sqrt{x}) = -\frac{3\cos(3\sqrt{x})\sin(3\sqrt{x})}{\sqrt{x}}$

17. $f'(x) = 20\cos^4 x \frac{d}{dx}(\cos x) = 20\cos^4 x(-\sin x) = -20\cos^4 x \sin x$

18. $f'(x) = -\csc(x^3)\cot(x^3) \frac{d}{dx}(x^3) = -3x^2 \csc(x^3)\cot(x^3)$

19. $f'(x) = \cos(1/x^2) \frac{d}{dx}(1/x^2) = -\frac{2}{x^3}\cos(1/x^2)$

20. $f'(x) = 4\tan^3(x^3) \frac{d}{dx}[\tan(x^3)] = 4\tan^3(x^3)\sec^2(x^3) \frac{d}{dx}(x^3) = 12x^2 \tan^3(x^3)\sec^2(x^3)$

21. $f'(x) = 4\sec(x^7) \frac{d}{dx}[\sec(x^7)] = 4\sec(x^7)\sec(x^7)\tan(x^7) \frac{d}{dx}(x^7) = 28x^6 \sec^2(x^7)\tan(x^7)$

22. $f'(x) = 3\cos^2\left(\frac{x}{x+1}\right) \frac{d}{dx}\cos\left(\frac{x}{x+1}\right) = 3\cos^2\left(\frac{x}{x+1}\right)\left[-\sin\left(\frac{x}{x+1}\right)\right] \frac{(x+1)(1) - x(1)}{(x+1)^2}$
$= -\frac{3}{(x+1)^2}\cos^2\left(\frac{x}{x+1}\right)\sin\left(\frac{x}{x+1}\right)$

23. $f'(x) = \frac{1}{2\sqrt{\cos(5x)}} \frac{d}{dx}[\cos(5x)] = -\frac{5\sin(5x)}{2\sqrt{\cos(5x)}}$

24. $f'(x) = \frac{1}{2\sqrt{3x - \sin^2(4x)}} \frac{d}{dx}[3x - \sin^2(4x)] = \frac{3 - 8\sin(4x)\cos(4x)}{2\sqrt{3x - \sin^2(4x)}}$

25. $f'(x) = -3\left[x+\csc(x^3+3)\right]^{-4}\dfrac{d}{dx}\left[x+\csc(x^3+3)\right]$

$= -3\left[x+\csc(x^3+3)\right]^{-4}\left[1-\csc(x^3+3)\cot(x^3+3)\dfrac{d}{dx}(x^3+3)\right]$

$= -3\left[x+\csc(x^3+3)\right]^{-4}\left[1-3x^2\csc(x^3+3)\cot(x^3+3)\right]$

26. $f'(x) = -4\left[x^4-\sec(4x^2-2)\right]^{-5}\dfrac{d}{dx}\left[x^4-\sec(4x^2-2)\right]$

$= -4\left[x^4-\sec(4x^2-2)\right]^{-5}\left[4x^3-\sec(4x^2-2)\tan(4x^2-2)\dfrac{d}{dx}(4x^2-2)\right]$

$= -16x\left[x^4-\sec(4x^2-2)\right]^{-5}\left[x^2-2\ \sec(4x^2-2)\tan(4x^2-2)\right]$

27. $\dfrac{dy}{dx} = x^3(2\sin 5x)\dfrac{d}{dx}(\sin 5x)+3x^2\sin^2 5x = 10x^3\sin 5x\cos 5x+3x^2\sin^2 5x$

28. $\dfrac{dy}{dx} = \sqrt{x}\left[3\tan^2(\sqrt{x})\sec^2(\sqrt{x})\dfrac{1}{2\sqrt{x}}\right]+\dfrac{1}{2\sqrt{x}}\tan^3(\sqrt{x}) = \dfrac{3}{2}\tan^2(\sqrt{x})\sec^2(\sqrt{x})+\dfrac{1}{2\sqrt{x}}\tan^3(\sqrt{x})$

29. $\dfrac{dy}{dx} = x^5\sec\left(\dfrac{1}{x}\right)\tan\left(\dfrac{1}{x}\right)\dfrac{d}{dx}\left(\dfrac{1}{x}\right)+\sec\left(\dfrac{1}{x}\right)(5x^4)$

$= x^5\sec\left(\dfrac{1}{x}\right)\tan\left(\dfrac{1}{x}\right)\left(-\dfrac{1}{x^2}\right)+5x^4\sec\left(\dfrac{1}{x}\right)$

$= -x^3\sec\left(\dfrac{1}{x}\right)\tan\left(\dfrac{1}{x}\right)+5x^4\sec\left(\dfrac{1}{x}\right)$

30. $\dfrac{dy}{dx} = \dfrac{\sec(3x+1)\cos x-3\sin x\sec(3x+1)\tan(3x+1)}{\sec^2(3x+1)} = \cos x\cos(3x+1)-3\sin x\sin(3x+1)$

31. $\dfrac{dy}{dx} = -\sin(\cos x)\dfrac{d}{dx}(\cos x) = -\sin(\cos x)(-\sin x) = \sin(\cos x)\sin x$

32. $\dfrac{dy}{dx} = \cos(\tan 3x)\dfrac{d}{dx}(\tan 3x) = 3\sec^2 3x\cos(\tan 3x)$

33. $\dfrac{dy}{dx} = 3\cos^2(\sin 2x)\dfrac{d}{dx}[\cos(\sin 2x)] = 3\cos^2(\sin 2x)[-\sin(\sin 2x)]\dfrac{d}{dx}(\sin 2x)$

$= -6\cos^2(\sin 2x)\sin(\sin 2x)\cos 2x$

34. $\dfrac{dy}{dx} = \dfrac{(1-\cot x^2)(-2x\csc x^2\cot x^2)-(1+\csc x^2)(2x\csc^2 x^2)}{(1-\cot x^2)^2} = -2x\csc x^2\ \dfrac{1+\cot x^2+\csc x^2}{(1-\cot x^2)^2}$

35. $\dfrac{dy}{dx} = (5x+8)^{13}12(x^3+7x)^{11}\dfrac{d}{dx}(x^3+7x)+(x^3+7x)^{12}13(5x+8)^{12}\dfrac{d}{dx}(5x+8)$

$= 12(5x+8)^{13}(x^3+7x)^{11}(3x^2+7)+65(x^3+7x)^{12}(5x+8)^{12}$

36. $\dfrac{dy}{dx} = (2x-5)^2 3(x^2+4)^2(2x)+(x^2+4)^3 2(2x-5)(2)$

$= 6x(2x-5)^2(x^2+4)^2+4(2x-5)(x^2+4)^3$

$= 2(2x-5)(x^2+4)^2(8x^2-15x+8)$

37. $\dfrac{dy}{dx} = 3\left[\dfrac{x-5}{2x+1}\right]^2 \dfrac{d}{dx}\left[\dfrac{x-5}{2x+1}\right] = 3\left[\dfrac{x-5}{2x+1}\right]^2 \cdot \dfrac{11}{(2x+1)^2} = \dfrac{33(x-5)^2}{(2x+1)^4}$

38. $\dfrac{dy}{dx} = 17\left(\dfrac{1+x^2}{1-x^2}\right)^{16} \dfrac{d}{dx}\left(\dfrac{1+x^2}{1-x^2}\right) = 17\left(\dfrac{1+x^2}{1-x^2}\right)^{16} \dfrac{(1-x^2)(2x)-(1+x^2)(-2x)}{(1-x^2)^2}$

$= 17\left(\dfrac{1+x^2}{1-x^2}\right)^{16} \dfrac{4x}{(1-x^2)^2} = \dfrac{68x(1+x^2)^{16}}{(1-x^2)^{18}}$

39. $\dfrac{dy}{dx} = \dfrac{(4x^2-1)^8(3)(2x+3)^2(2)-(2x+3)^3(8)(4x^2-1)^7(8x)}{(4x^2-1)^{16}}$

$= \dfrac{2(2x+3)^2(4x^2-1)^7[3(4x^2-1)-32x(2x+3)]}{(4x^2-1)^{16}} = -\dfrac{2(2x+3)^2(52x^2+96x+3)}{(4x^2-1)^9}$

40. $\dfrac{dy}{dx} = 12[1+\sin^3(x^5)]^{11}\dfrac{d}{dx}[1+\sin^3(x^5)]$

$= 12[1+\sin^3(x^5)]^{11} 3\sin^2(x^5)\dfrac{d}{dx}\sin(x^5) = 180x^4[1+\sin^3(x^5)]^{11}\sin^2(x^5)\cos(x^5)$

41. $\dfrac{dy}{dx} = 5\left[x\sin 2x + \tan^4(x^7)\right]^4 \dfrac{d}{dx}\left[x \sin 2x \tan^4(x^7)\right]$

$= 5\left[x\sin 2x + \tan^4(x^7)\right]^4 \left[x\cos 2x \dfrac{d}{dx}(2x) + \sin 2x + 4\tan^3(x^7)\dfrac{d}{dx}\tan(x^7)\right]$

$= 5\left[x\sin 2x + \tan^4(x^7)\right]^4 \left[2x\cos 2x + \sin 2x + 28x^6\tan^3(x^7)\sec^2(x^7)\right]$

42. $\dfrac{dy}{dx} = 4\tan^3\left(2+\dfrac{(7-x)\sqrt{3x^2+5}}{x^3+\sin x}\right)\sec^2\left(2+\dfrac{(7-x)\sqrt{3x^2+5}}{x^3+\sin x}\right)$

$\times\left(-\dfrac{\sqrt{3x^2+5}}{x^3+\sin x} + 3\dfrac{(7-x)x}{\sqrt{3x^2+5}\,(x^3+\sin x)} - \dfrac{(7-x)\sqrt{3x^2+5}\,(3x^2+\cos x)}{(x^3+\sin x)^2}\right)$

43. $\dfrac{dy}{dx} = \cos 3x - 3x\sin 3x$; if $x = \pi$ then $\dfrac{dy}{dx} = -1$ and $y = -\pi$, so the equation of the tangent line is $y + \pi = -(x-\pi), y = x$

44. $\dfrac{dy}{dx} = 3x^2\cos(1+x^3)$; if $x = -3$ then $y = -\sin 26$, $\dfrac{dy}{dx} = -27\cos 26$, so the equation of the tangent line is $y + \sin 26 = -27(\cos 26)(x+3), y = -27(\cos 26)x - 81\cos 26 - \sin 26$

45. $\dfrac{dy}{dx} = -3\sec^3(\pi/2 - x)\tan(\pi/2 - x)$; if $x = -\pi/2$ then $\dfrac{dy}{dx} = 0, y = -1$ so the equation of the tangent line is $y + 1 = 0, y = -1$

46. $\dfrac{dy}{dx} = 3\left(x - \dfrac{1}{x}\right)^2\left(1+\dfrac{1}{x^2}\right)$; if $x = 2$ then $y = \dfrac{27}{8}, \dfrac{dy}{dx} = 3\dfrac{9}{4}\dfrac{5}{4} = \dfrac{135}{16}$ so the equation of the tangent line is $y - 27/8 = (135/16)(x-2), y = \dfrac{135}{16}x - \dfrac{108}{8}$

47. $\dfrac{dy}{dx} = \sec^2(4x^2)\dfrac{d}{dx}(4x^2) = 8x\sec^2(4x^2)$, $\left.\dfrac{dy}{dx}\right|_{x=\sqrt{\pi}} = 8\sqrt{\pi}\sec^2(4\pi) = 8\sqrt{\pi}$. When $x = \sqrt{\pi}$, $y = \tan(4\pi) = 0$, so the equation of the tangent line is $y = 8\sqrt{\pi}(x - \sqrt{\pi}) = 8\sqrt{\pi}x - 8\pi$.

48. $\dfrac{dy}{dx} = 12\cot^3 x\dfrac{d}{dx}\cot x = -12\cot^3 x\csc^2 x$, $\left.\dfrac{dy}{dx}\right|_{x=\pi/4} = -24$. When $x = \pi/4, y = 3$, so the equation of the tangent line is $y - 3 = -24(x - \pi/4)$, or $y = -24x + 3 + 6\pi$.

49. $\dfrac{dy}{dx} = 2x\sqrt{5 - x^2} + \dfrac{x^2}{2\sqrt{5 - x^2}}(-2x)$, $\left.\dfrac{dy}{dx}\right|_{x=1} = 4 - 1/2 = 7/2$. When $x = 1, y = 2$, so the equation of the tangent line is $y - 2 = (7/2)(x - 1)$, or $y = \dfrac{7}{2}x - \dfrac{3}{2}$.

50. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 - x^2}} - \dfrac{x}{2}(1 - x^2)^{3/2}(-2x)$, $\left.\dfrac{dy}{dx}\right|_{x=0} = 1$. When $x = 0, y = 0$, so the equation of the tangent line is $y = x$.

51. $$\begin{aligned}\frac{dy}{dx} &= x(-\sin(5x))\frac{d}{dx}(5x) + \cos(5x) - 2\sin x\frac{d}{dx}(\sin x)\\ &= -5x\sin(5x) + \cos(5x) - 2\sin x\cos x = -5x\sin(5x) + \cos(5x) - \sin(2x),\\ \frac{d^2y}{dx^2} &= -5x\cos(5x)\frac{d}{dx}(5x) - 5\sin(5x) - \sin(5x)\frac{d}{dx}(5x) - \cos(2x)\frac{d}{dx}(2x)\\ &= -25x\cos(5x) - 10\sin(5x) - 2\cos(2x)\end{aligned}$$

52. $$\begin{aligned}\frac{dy}{dx} &= \cos(3x^2)\frac{d}{dx}(3x^2) = 6x\cos(3x^2),\\ \frac{d^2y}{dx^2} &= 6x(-\sin(3x^2))\frac{d}{dx}(3x^2) + 6\cos(3x^2) = -36x^2\sin(3x^2) + 6\cos(3x^2)\end{aligned}$$

53. $\dfrac{dy}{dx} = \dfrac{(1 - x) + (1 + x)}{(1 - x)^2} = \dfrac{2}{(1 - x)^2} = 2(1 - x)^{-2}$ and $\dfrac{d^2y}{dx^2} = -2(2)(-1)(1 - x)^{-3} = 4(1 - x)^{-3}$

54. $$\begin{aligned}\frac{dy}{dx} &= x\sec^2\left(\frac{1}{x}\right)\frac{d}{dx}\left(\frac{1}{x}\right) + \tan\left(\frac{1}{x}\right) = -\frac{1}{x}\sec^2\left(\frac{1}{x}\right) + \tan\left(\frac{1}{x}\right),\\ \frac{d^2y}{dx^2} &= -\frac{2}{x}\sec\left(\frac{1}{x}\right)\frac{d}{dx}\sec\left(\frac{1}{x}\right) + \frac{1}{x^2}\sec^2\left(\frac{1}{x}\right) + \sec^2\left(\frac{1}{x}\right)\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{2}{x^3}\sec^2\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right)\end{aligned}$$

so $y - \dfrac{27}{8} = \dfrac{135}{16}(x - 2)$, $y = \dfrac{135}{16}x - \dfrac{27}{2}$

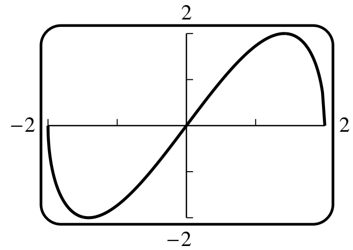
55. $y = \cot^3(\pi - \theta) = -\cot^3\theta$ so $dy/dx = 3\cot^2\theta\csc^2\theta$

56. $6\left(\dfrac{au + b}{cu + d}\right)^5 \dfrac{ad - bc}{(cu + d)^2}$

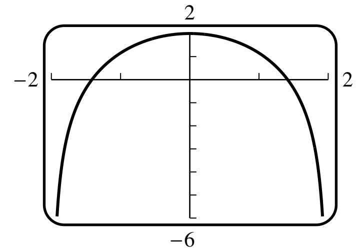
57. $$\begin{aligned}\frac{d}{d\omega}[a\cos^2\pi\omega + b\sin^2\pi\omega] &= -2\pi a\cos\pi\omega\sin\pi\omega + 2\pi b\sin\pi\omega\cos\pi\omega\\ &= \pi(b - a)(2\sin\pi\omega\cos\pi\omega) = \pi(b - a)\sin 2\pi\omega\end{aligned}$$

58. $2\csc^2(\pi/3 - y)\cot(\pi/3 - y)$

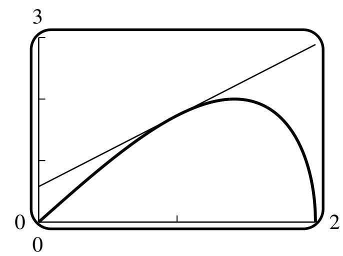
59. **(a)**



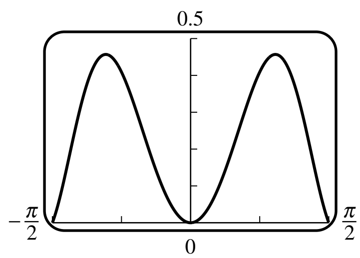
(c) $f'(x) = x\dfrac{-x}{\sqrt{4-x^2}} + \sqrt{4-x^2} = \dfrac{4-2x^2}{\sqrt{4-x^2}}$



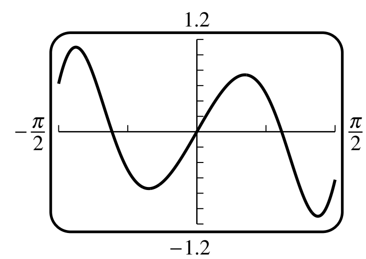
(d) $f(1) = \sqrt{3}$ and $f'(1) = \dfrac{2}{\sqrt{3}}$ so the tangent line has the equation $y - \sqrt{3} = \dfrac{2}{\sqrt{3}}(x-1)$.



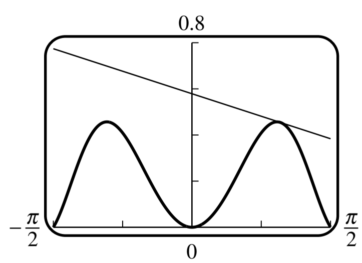
60. **(a)**



(c) $f'(x) = 2x\cos(x^2)\cos x - \sin x \sin(x^2)$



(d) $f(1) = \sin 1 \cos 1$ and $f'(1) = 2\cos^2 1 - \sin^2 1$, so the tangent line has the equation $y - \sin 1 \cos 1 = (2\cos^2 1 - \sin^2 1)(x-1)$.



61. **(a)** $dy/dt = -A\omega \sin \omega t, d^2y/dt^2 = -A\omega^2 \cos \omega t = -\omega^2 y$

(b) one complete oscillation occurs when ωt increases over an interval of length 2π, or if t increases over an interval of length $2\pi/\omega$

(c) $f = 1/T$

(d) amplitude $= 0.6$ cm, $T = 2\pi/15$ s/oscillation, $f = 15/(2\pi)$ oscillations/s

62. $dy/dt = 3A\cos 3t, d^2y/dt^2 = -9A\sin 3t$, so $-9A\sin 3t + 2A\sin 3t = 4\sin 3t$,
$-7A\sin 3t = 4\sin 3t, -7A = 4, A = -4/7$

63. **(a)** $p \approx 10$ lb/in^2, $dp/dh \approx -2$ lb/in^2/mi

(b) $\dfrac{dp}{dt} = \dfrac{dp}{dh}\dfrac{dh}{dt} \approx (-2)(0.3) = -0.6$ lb/in^2/s

64. **(a)** $F = \dfrac{45}{\cos\theta + 0.3\sin\theta}, \dfrac{dF}{d\theta} = -\dfrac{45(-\sin\theta + 0.3\cos\theta)}{(\cos\theta + 0.3\sin\theta)^2}$;
if $\theta = 30°$, then $dF/d\theta \approx 10.5$ lb/rad ≈ 0.18 lb/deg

(b) $\dfrac{dF}{dt} = \dfrac{dF}{d\theta}\dfrac{d\theta}{dt} \approx (0.18)(-0.5) = -0.09$ lb/s

65. With $u = \sin x$, $\dfrac{d}{dx}(|\sin x|) = \dfrac{d}{dx}(|u|) = \dfrac{d}{du}(|u|)\dfrac{du}{dx} = \dfrac{d}{du}(|u|)\cos x = \begin{cases} \cos x, & u > 0 \\ -\cos x, & u < 0 \end{cases}$

$$= \begin{cases} \cos x, & \sin x > 0 \\ -\cos x, & \sin x < 0 \end{cases} = \begin{cases} \cos x, & 0 < x < \pi \\ -\cos x, & -\pi < x < 0 \end{cases}$$

66. $\dfrac{d}{dx}(\cos x) = \dfrac{d}{dx}[\sin(\pi/2 - x)] = -\cos(\pi/2 - x) = -\sin x$

67. **(a)** For $x \neq 0, |f(x)| \leq |x|$, and $\lim\limits_{x\to 0}|x| = 0$, so by the Squeezing Theorem, $\lim\limits_{x\to 0} f(x) = 0$.

(b) If $f'(0)$ were to exist, then the limit $\dfrac{f(x) - f(0)}{x - 0} = \sin(1/x)$ would have to exist, but it doesn't.

(c) for $x \neq 0$, $f'(x) = x\left(\cos\dfrac{1}{x}\right)\left(-\dfrac{1}{x^2}\right) + \sin\dfrac{1}{x} = -\dfrac{1}{x}\cos\dfrac{1}{x} + \sin\dfrac{1}{x}$

(d) $\lim\limits_{x\to 0}\dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x\to 0}\sin(1/x)$, which does not exist, thus $f'(0)$ does not exist.

68. **(a)** $-x^2 \leq x^2\sin(1/x) \leq x^2$, so by the Squeezing Theorem $\lim\limits_{x\to 0} f(x) = 0$.

(b) $f'(0) = \lim\limits_{x\to 0}\dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x\to 0} x\sin(1/x) = 0$ by Exercise 67, Part a.

(c) For $x \neq 0, f'(x) = 2x\sin(1/x) + x^2\cos(1/x)(-1/x^2) = 2x\sin(1/x) - \cos(1/x)$

(d) If $f'(x)$ were continuous at $x = 0$ then so would $\cos(1/x) = f'(x) - 2x\sin(1/x)$ be, since $2x\sin(1/x)$ is continuous there. But $\cos(1/x)$ oscillates at $x = 0$.

69. **(a)** $g'(x) = 3[f(x)]^2 f'(x)$, $g'(2) = 3[f(2)]^2 f'(2) = 3(1)^2(7) = 21$
(b) $h'(x) = f'(x^3)(3x^2)$, $h'(2) = f'(8)(12) = (-3)(12) = -36$

70. $F'(x) = f'(g(x))g'(x) = \sqrt{3(x^2-1)+4}(2x) = 2x\sqrt{3x^2+1}$

71. $F'(x) = f'(g(x))g'(x) = f'(\sqrt{3x-1})\dfrac{3}{2\sqrt{3x-1}} = \dfrac{\sqrt{3x-1}}{(3x-1)+1}\dfrac{3}{2\sqrt{3x-1}} = \dfrac{1}{2x}$

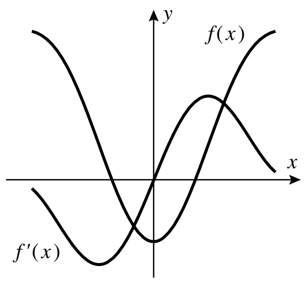
72. $\dfrac{d}{dx}[f(x^2)] = f'(x^2)(2x)$, thus $f'(x^2)(2x) = x^2$ so $f'(x^2) = x/2$ if $x \neq 0$

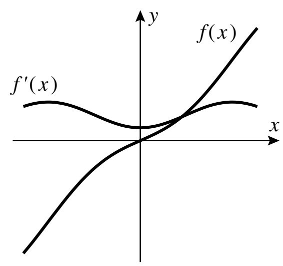
73. $\dfrac{d}{dx}[f(3x)] = f'(3x)\dfrac{d}{dx}(3x) = 3f'(3x) = 6x$, so $f'(3x) = 2x$. Let $u = 3x$ to get $f'(u) = \dfrac{2}{3}u$;
$\dfrac{d}{dx}[f(x)] = f'(x) = \dfrac{2}{3}x$.

74. **(a)** If $f(-x) = f(x)$, then $\dfrac{d}{dx}[f(-x)] = \dfrac{d}{dx}[f(x)]$, $f'(-x)(-1) = f'(x)$, $f'(-x) = -f'(x)$ so f' is odd.

(b) If $f(-x) = -f(x)$, then $\dfrac{d}{dx}[f(-x)] = -\dfrac{d}{dx}[f(x)]$, $f'(-x)(-1) = -f'(x)$, $f'(-x) = f'(x)$ so f' is even.

75. For an even function, the graph is symmetric about the y-axis; the slope of the tangent line at $(a, f(a))$ is the negative of the slope of the tangent line at $(-a, f(-a))$. For an odd function, the graph is symmetric about the origin; the slope of the tangent line at $(a, f(a))$ is the same as the slope of the tangent line at $(-a, f(-a))$.





76. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dv}\dfrac{dv}{dw}\dfrac{dw}{dx}$

77. $$\frac{d}{dx}[f(g(h(x)))] = \frac{d}{dx}[f(g(u))], \quad u = h(x)$$
$$= \frac{d}{du}[f(g(u))]\frac{du}{dx} = f'(g(u))g'(u)\frac{du}{dx} = f'(g(h(x)))g'(h(x))h'(x)$$

EXERCISE SET 3.6

1. $y = (2x-5)^{1/3}$; $dy/dx = \dfrac{2}{3}(2x-5)^{-2/3}$

2. $dy/dx = \dfrac{1}{3}\left[2+\tan(x^2)\right]^{-2/3}\sec^2(x^2)(2x) = \dfrac{2}{3}x\sec^2(x^2)\left[2+\tan(x^2)\right]^{-2/3}$

3. $dy/dx = \dfrac{3}{2}\left[\dfrac{x-1}{x+2}\right]^{1/2}\dfrac{d}{dx}\left[\dfrac{x-1}{x+2}\right] = \dfrac{9}{2(x+2)^2}\left[\dfrac{x-1}{x+2}\right]^{1/2}$

4. $dy/dx = \dfrac{1}{2}\left[\dfrac{x^2+1}{x^2-5}\right]^{-1/2}\dfrac{d}{dx}\left[\dfrac{x^2+1}{x^2-5}\right] = \dfrac{1}{2}\left[\dfrac{x^2+1}{x^2-5}\right]^{-1/2}\dfrac{-12x}{(x^2-5)^2} = -\dfrac{6x}{(x^2-5)^2}\left[\dfrac{x^2+1}{x^2-5}\right]^{-1/2}$

5. $dy/dx = x^3\left(-\dfrac{2}{3}\right)(5x^2+1)^{-5/3}(10x) + 3x^2(5x^2+1)^{-2/3} = \dfrac{1}{3}x^2(5x^2+1)^{-5/3}(25x^2+9)$

6. $dy/dx = \dfrac{x^2\dfrac{4}{3}(3-2x)^{1/3}(-2) - (3-2x)^{4/3}(2x)}{x^4} = \dfrac{2(3-2x)^{1/3}(2x-9)}{3x^3}$

7. $dy/dx = \dfrac{5}{2}[\sin(3/x)]^{3/2}[\cos(3/x)](-3/x^2) = -\dfrac{15[\sin(3/x)]^{3/2}\cos(3/x)}{2x^2}$

8. $dy/dx = -\frac{1}{2}\left[\cos(x^3)\right]^{-3/2}\left[-\sin(x^3)\right](3x^2) = \frac{3}{2}x^2\sin(x^3)\left[\cos(x^3)\right]^{-3/2}$

9. **(a)** $3x^2 + x\frac{dy}{dx} + y - 2 = 0,\ \frac{dy}{dx} = \frac{2 - 3x^2 - y}{x}$

(b) $y = \frac{1 + 2x - x^3}{x} = \frac{1}{x} + 2 - x^2$ so $\frac{dy}{dx} = -\frac{1}{x^2} - 2x$

(c) from Part (a), $\frac{dy}{dx} = \frac{2 - 3x^2 - y}{x} = \frac{2 - 3x^2 - (1/x + 2 - x^2)}{x} = -2x - \frac{1}{x^2}$

10. **(a)** $\frac{1}{2}y^{-1/2}\frac{dy}{dx} - e^x = 0$ or $\frac{dy}{dx} = 2e^x\sqrt{y}$

(b) $y = (2 + e^x)^2 = 2 + 4e^x + e^{2x}$ so $\frac{dy}{dx} = 4e^x + 2e^{2x}$

(c) from Part (a), $\frac{dy}{dx} = 2e^x\sqrt{y} = 2e^x(2 + e^x) = 4e^x + 2e^{2x}$

11. $2x + 2y\frac{dy}{dx} = 0$ so $\frac{dy}{dx} = -\frac{x}{y}$

12. $3x^2 - 3y^2\frac{dy}{dx} = 6(x\frac{dy}{dx} + y),\ -(3y^2 + 6x)\frac{dy}{dx} = 6y - 3x^2$ so $\frac{dy}{dx} = \frac{x^2 - 2y}{y^2 + 2x}$

13. $x^2\frac{dy}{dx} + 2xy + 3x(3y^2)\frac{dy}{dx} + 3y^3 - 1 = 0$

$(x^2 + 9xy^2)\frac{dy}{dx} = 1 - 2xy - 3y^3$ so $\frac{dy}{dx} = \frac{1 - 2xy - 3y^3}{x^2 + 9xy^2}$

14. $x^3(2y)\frac{dy}{dx} + 3x^2y^2 - 5x^2\frac{dy}{dx} - 10xy + 1 = 0$

$(2x^3y - 5x^2)\frac{dy}{dx} = 10xy - 3x^2y^2 - 1$ so $\frac{dy}{dx} = \frac{10xy - 3x^2y^2 - 1}{2x^3y - 5x^2}$

15. $-\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{x^2} = 0$ so $\frac{dy}{dx} = -\frac{y^2}{x^2}$

16. $2x = \frac{(x - y)(1 + dy/dx) - (x + y)(1 - dy/dx)}{(x - y)^2}$,

$2x(x - y)^2 = -2y + 2x\frac{dy}{dx}$ so $\frac{dy}{dx} = \frac{x(x - y)^2 + y}{x}$

17. $\cos(x^2y^2)\left[x^2(2y)\frac{dy}{dx} + 2xy^2\right] = 1,\ \frac{dy}{dx} = \frac{1 - 2xy^2\cos(x^2y^2)}{2x^2y\cos(x^2y^2)}$

18. $2x = \frac{(1 + \csc y)(-\csc^2 y)(dy/dx) - (\cot y)(-\csc y\cot y)(dy/dx)}{(1 + \csc y)^2}$,

$2x(1 + \csc y)^2 = -\csc y(\csc y + \csc^2 y - \cot^2 y)\frac{dy}{dx}$,

but $\csc^2 y - \cot^2 y = 1$, so $\frac{dy}{dx} = -\frac{2x(1 + \csc y)}{\csc y}$

19. $3\tan^2(xy^2 + y)\sec^2(xy^2 + y)\left(2xy\frac{dy}{dx} + y^2 + \frac{dy}{dx}\right) = 1$

so $\frac{dy}{dx} = \frac{1 - 3y^2\tan^2(xy^2 + y)\sec^2(xy^2 + y)}{3(2xy + 1)\tan^2(xy^2 + y)\sec^2(xy^2 + y)}$

20. $\dfrac{(1+\sec y)[3xy^2(dy/dx)+y^3]-xy^3(\sec y\tan y)(dy/dx)}{(1+\sec y)^2}=4y^3\dfrac{dy}{dx}$,

multiply through by $(1+\sec y)^2$ and solve for $\dfrac{dy}{dx}$ to get

$\dfrac{dy}{dx}=\dfrac{y(1+\sec y)}{4y(1+\sec y)^2-3x(1+\sec y)+xy\sec y\tan y}$

21. $\dfrac{dy}{dx}=\dfrac{3x}{4y}$, $\dfrac{d^2y}{dx^2}=\dfrac{(4y)(3)-(3x)(4dy/dx)}{16y^2}=\dfrac{12y-12x(3x/(4y))}{16y^2}=\dfrac{12y^2-9x^2}{16y^3}=\dfrac{-3(3x^2-4y^2)}{16y^3}$,

but $3x^2-4y^2=7$ so $\dfrac{d^2y}{dx^2}=\dfrac{-3(7)}{16y^3}=-\dfrac{21}{16y^3}$

22. $\dfrac{dy}{dx}=-\dfrac{x^2}{y^2}$, $\dfrac{d^2y}{dx^2}=-\dfrac{y^2(2x)-x^2(2ydy/dx)}{y^4}=-\dfrac{2xy^2-2x^2y(-x^2/y^2)}{y^4}=-\dfrac{2x(y^3+x^3)}{y^5}$,

but $x^3+y^3=1$ so $\dfrac{d^2y}{dx^2}=-\dfrac{2x}{y^5}$

23. $\dfrac{dy}{dx}=-\dfrac{y}{x}$, $\dfrac{d^2y}{dx^2}=-\dfrac{x(dy/dx)-y(1)}{x^2}=-\dfrac{x(-y/x)-y}{x^2}=\dfrac{2y}{x^2}$

24. $\dfrac{dy}{dx}=\dfrac{y}{y-x}$,

$$\frac{d^2y}{dx^2}=\frac{(y-x)(dy/dx)-y(dy/dx-1)}{(y-x)^2}=\frac{(y-x)\left(\dfrac{y}{y-x}\right)-y\left(\dfrac{y}{y-x}-1\right)}{(y-x)^2}$$

$=\dfrac{y^2-2xy}{(y-x)^3}$ but $y^2-2xy=-3$, so $\dfrac{d^2y}{dx^2}=-\dfrac{3}{(y-x)^3}$

25. $\dfrac{dy}{dx}=(1+\cos y)^{-1}$, $\dfrac{d^2y}{dx^2}=-(1+\cos y)^{-2}(-\sin y)\dfrac{dy}{dx}=\dfrac{\sin y}{(1+\cos y)^3}$

26. $\dfrac{dy}{dx}=\dfrac{\cos y}{1+x\sin y}$,

$$\frac{d^2y}{dx^2}=\frac{(1+x\sin y)(-\sin y)(dy/dx)-(\cos y)[(x\cos y)(dy/dx)+\sin y]}{(1+x\sin y)^2}$$

$$=-\frac{2\sin y\cos y+(x\cos y)(2\sin^2 y+\cos^2 y)}{(1+x\sin y)^3},$$

but $x\cos y=y$, $2\sin y\cos y=\sin 2y$, and $\sin^2 y+\cos^2 y=1$ so

$$\frac{d^2y}{dx^2}=-\frac{\sin 2y+y(\sin^2 y+1)}{(1+x\sin y)^3}$$

27. By implicit differentiation, $2x+2y(dy/dx)=0$, $\dfrac{dy}{dx}=-\dfrac{x}{y}$; at $(1/\sqrt{2},1/\sqrt{2})$, $\dfrac{dy}{dx}=-1$; at $(1/\sqrt{2},-1/\sqrt{2})$, $\dfrac{dy}{dx}=+1$. Directly, at the upper point $y=\sqrt{1-x^2}$, $\dfrac{dy}{dx}=\dfrac{-x}{\sqrt{1-x^2}}=-1$ and at the lower point $y=-\sqrt{1-x^2}$, $\dfrac{dy}{dx}=\dfrac{x}{\sqrt{1-x^2}}=+1$.

28. If $y^2-x+1=0$, then $y=\sqrt{x-1}$ goes through the point $(10,3)$ so $dy/dx=1/(2\sqrt{x-1})$. By implicit differentiation $dy/dx=1/(2y)$. In both cases, $dy/dx|_{(10,3)}=1/6$. Similarly $y=-\sqrt{x-1}$ goes through $(10,-3)$ so $dy/dx=-1/(2\sqrt{x-1})=-1/6$ which yields $dy/dx=1/(2y)=-1/6$.

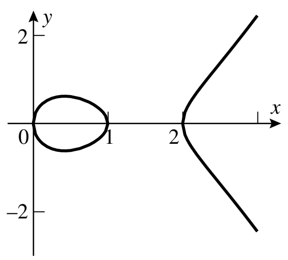
29. $4x^3 + 4y^3\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -\dfrac{x^3}{y^3} = -\dfrac{1}{15^{3/4}} \approx -0.1312$.

30. $3y^2\dfrac{dy}{dx} + x^2\dfrac{dy}{dx} + 2xy + 2x - 6y\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -2x\dfrac{y+1}{3y^2 + x^2 - 6y} = 0$ at $x = 0$

31. $4(x^2+y^2)\left(2x + 2y\dfrac{dy}{dx}\right) = 25\left(2x - 2y\dfrac{dy}{dx}\right)$,
$\dfrac{dy}{dx} = \dfrac{x[25 - 4(x^2+y^2)]}{y[25 + 4(x^2+y^2)]}$; at $(3,1)$ $\dfrac{dy}{dx} = -9/13$

32. $\dfrac{2}{3}\left(x^{-1/3} + y^{-1/3}\dfrac{dy}{dx}\right) = 0$, $\dfrac{dy}{dx} = -\dfrac{y^{1/3}}{x^{1/3}} = \sqrt{3}$ at $(-1, 3\sqrt{3})$

34. **(a)**

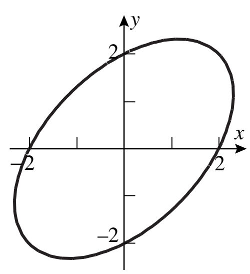


(b) $2y\dfrac{dy}{dx} = (x-a)(x-b) + x(x-b) + x(x-a) = 3x^2 - 2(a+b)x + ab$. If $\dfrac{dy}{dx} = 0$ then $3x^2 - 2(a+b)x + ab = 0$. By the Quadratic Formula

$$x = \frac{2(a+b) \pm \sqrt{4(a+b)^2 - 4\cdot 3ab}}{6} = \frac{1}{3}\left[a + b \pm (a^2 + b^2 - ab)^{1/2}\right].$$

(c) $y = \pm\sqrt{x(x-a)(x-b)}$. The square root is only defined for nonnegative arguments, so it is necessary that all three of the factors x, $x-a$, $x-b$ be nonnegative, or that two of them be nonpositive. If, for example, $0 < a < b$ then the function is defined on the disjoint intervals $0 < x < a$ and $b < x < +\infty$, so there are two parts.

35. **(a)**



(b) ± 1.1547

(c) Implicit differentiation yields $2x - x\dfrac{dy}{dx} - y + 2y\dfrac{dy}{dx} = 0$. Solve for $\dfrac{dy}{dx} = \dfrac{y - 2x}{2y - x}$. If $\dfrac{dy}{dx} = 0$ then $y - 2x = 0$ or $y = 2x$. Thus $4 = x^2 - xy + y^2 = x^2 - 2x^2 + 4x^2 = 3x^2$, $x = \pm\dfrac{2}{\sqrt{3}}$.

36. $\dfrac{1}{2}u^{-1/2}\dfrac{du}{dv} + \dfrac{1}{2}v^{-1/2} = 0$ so $\dfrac{du}{dv} = -\dfrac{\sqrt{u}}{\sqrt{v}}$

37. $4a^3\dfrac{da}{dt} - 4t^3 = 6\left(a^2 + 2at\dfrac{da}{dt}\right)$, solve for $\dfrac{da}{dt}$ to get $\dfrac{da}{dt} = \dfrac{2t^3 + 3a^2}{2a^3 - 6at}$

38. $1 = (\cos x)\dfrac{dx}{dy}$ so $\dfrac{dx}{dy} = \dfrac{1}{\cos x} = \sec x$

39. $2a^2\omega\dfrac{d\omega}{d\lambda} + 2b^2\lambda = 0$ so $\dfrac{d\omega}{d\lambda} = -\dfrac{b^2\lambda}{a^2\omega}$

40. Let $P(x_0, y_0)$ be the required point. The slope of the line $4x - 3y + 1 = 0$ is $4/3$ so the slope of the tangent to $y^2 = 2x^3$ at P must be $-3/4$. By implicit differentiation $dy/dx = 3x^2/y$, so at P, $3x_0^2/y_0 = -3/4$, or $y_0 = -4x_0^2$. But $y_0^2 = 2x_0^3$ because P is on the curve $y^2 = 2x^3$. Elimination of y_0 gives $16x_0^4 = 2x_0^3$, $x_0^3(8x_0 - 1) = 0$, so $x_0 = 0$ or $1/8$. From $y_0 = -4x_0^2$ it follows that $y_0 = 0$ when $x_0 = 0$, and $y_0 = -1/16$ when $x_0 = 1/8$. It does not follow, however, that $(0,0)$ is a solution because $dy/dx = 3x^2/y$ (the slope of the curve as determined by implicit differentiation) is valid only if $y \neq 0$. Further analysis shows that the curve is tangent to the x-axis at $(0,0)$, so the point $(1/8, -1/16)$ is the only solution.

41. The point (1,1) is on the graph, so $1 + a = b$. The slope of the tangent line at (1,1) is $-4/3$; use implicit differentiation to get $\dfrac{dy}{dx} = -\dfrac{2xy}{x^2 + 2ay}$ so at (1,1), $-\dfrac{2}{1+2a} = -\dfrac{4}{3}$, $1 + 2a = 3/2$, $a = 1/4$ and hence $b = 1 + 1/4 = 5/4$.

42. Use implicit differentiation to get $dy/dx = (y - 3x^2)/(3y^2 - x)$, so $dy/dx = 0$ if $y = 3x^2$. Substitute this into $x^3 - xy + y^3 = 0$ to obtain $27x^6 - 2x^3 = 0, x^3 = 2/27$, $x = \sqrt[3]{2}/3$ and hence $y = \sqrt[3]{4}/3$.

43. Let $P(x_0, y_0)$ be a point where a line through the origin is tangent to the curve $x^2 - 4x + y^2 + 3 = 0$. Implicit differentiation applied to the equation of the curve gives $dy/dx = (2 - x)/y$. At P the slope of the curve must equal the slope of the line so $(2 - x_0)/y_0 = y_0/x_0$, or $y_0^2 = 2x_0 - x_0^2$. But $x_0^2 - 4x_0 + y_0^2 + 3 = 0$ because (x_0, y_0) is on the curve, and elimination of y_0^2 in the latter two equations gives $x_0^2 - 4x_0 + (2x_0 - x_0^2) + 3 = 0$, $x_0 = 3/2$ which when substituted into $y_0^2 = 2x_0 - x_0^2$ yields $y_0^2 = 3/4$, so $y_0 = \pm\sqrt{3}/2$. The slopes of the lines are $(\pm\sqrt{3}/2)/(3/2) = \pm\sqrt{3}/3$ and their equations are $y = (\sqrt{3}/3)x$ and $y = -(\sqrt{3}/3)x$.

44. By implicit differentiation, $dy/dx = k/(2y)$ so the slope of the tangent to $y^2 = kx$ at (x_0, y_0) is $k/(2y_0)$ if $y_0 \neq 0$. The tangent line in this case is $y - y_0 = \dfrac{k}{2y_0}(x - x_0)$, or $2y_0y - 2y_0^2 = kx - kx_0$. But $y_0^2 = kx_0$ because (x_0, y_0) is on the curve $y^2 = kx$, so the equation of the tangent line becomes $2y_0y - 2kx_0 = kx - kx_0$ which gives $y_0y = k(x + x_0)/2$. If $y_0 = 0$, then $x_0 = 0$; the graph of $y^2 = kx$ has a vertical tangent at $(0,0)$ so its equation is $x = 0$, but $y_0y = k(x + x_0)/2$ gives the same result when $x_0 = y_0 = 0$.

45. By the chain rule, $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx}$. Use implicit differentiation on $2y^3t + t^3y = 1$ to get $\dfrac{dy}{dt} = -\dfrac{2y^3 + 3t^2y}{6ty^2 + t^3}$, but $\dfrac{dt}{dx} = \dfrac{1}{\cos t}$ so $\dfrac{dy}{dx} = -\dfrac{2y^3 + 3t^2y}{(6ty^2 + t^3)\cos t}$.

46. $2x^3y\dfrac{dy}{dt} + 3x^2y^2\dfrac{dx}{dt} + \dfrac{dy}{dt} = 0$, $\dfrac{dy}{dt} = -\dfrac{3x^2y^2}{2x^3y + 1}\dfrac{dx}{dt}$

47. $2xy\dfrac{dy}{dt} = y^2\dfrac{dx}{dt} = 3(\cos 3x)\dfrac{dx}{dt}$, $\dfrac{dy}{dt} = \dfrac{3\cos 3x - y^2}{2xy}\dfrac{dx}{dt}$

48. **(a)** $f'(x) = \dfrac{4}{3}x^{1/3}$, $f''(x) = \dfrac{4}{9}x^{-2/3}$

(b) $f'(x) = \dfrac{7}{3}x^{4/3}$, $f''(x) = \dfrac{28}{9}x^{1/3}$, $f'''(x) = \dfrac{28}{27}x^{-2/3}$

(c) generalize parts (a) and (b) with $k = (n - 1) + 1/3 = n - 2/3$

49. $y' = rx^{r-1}, y'' = r(r-1)x^{r-2}$ so $3x^2\left[r(r-1)x^{r-2}\right] + 4x\left(rx^{r-1}\right) - 2x^r = 0$,
$3r(r-1)x^r + 4rx^r - 2x^r = 0, (3r^2 + r - 2)x^r = 0$,
$3r^2 + r - 2 = 0$, $(3r-2)(r+1) = 0$; $r = -1, 2/3$

50. $y' = rx^{r-1}, y'' = r(r-1)x^{r-2}$ so $16x^2\left[r(r-1)x^{r-2}\right] + 24x\left(rx^{r-1}\right) + x^r = 0$,
$16r(r-1)x^r + 24rx^r + x^r = 0, (16r^2 + 8r + 1)x^r = 0$,
$16r^2 + 8r + 1 = 0, (4r+1)^2 = 0$; $r = -1/4$

51. We shall find when the curves intersect and check that the slopes are negative reciprocals. For the intersection solve the simultaneous equations $x^2 + (y-c)^2 = c^2$ and $(x-k)^2 + y^2 = k^2$ to obtain $cy = kx = \frac{1}{2}(x^2 + y^2)$. Thus $x^2 + y^2 = cy + kx$, or $y^2 - cy = -x^2 + kx$, and $\frac{y-c}{x} = -\frac{x-k}{y}$. Differentiating the two families yields (black) $\frac{dy}{dx} = -\frac{x}{y-c}$, and (gray) $\frac{dy}{dx} = -\frac{x-k}{y}$. But it was proven that these quantities are negative reciprocals of each other.

52. Differentiating, we get the equations (black) $x\frac{dy}{dx} + y = 0$ and (gray) $2x - 2y\frac{dy}{dx} = 0$. The first says the (black) slope is $= -\frac{y}{x}$ and the second says the (gray) slope is $\frac{x}{y}$, and these are negative reciprocals of each other.

EXERCISE SET 3.7

1. $\frac{dy}{dt} = 3\frac{dx}{dt}$

(a) $\frac{dy}{dt} = 3(2) = 6$

(b) $-1 = 3\frac{dx}{dt}, \frac{dx}{dt} = -\frac{1}{3}$

2. $\frac{dx}{dt} + 4\frac{dy}{dt} = 0$

(a) $1 + 4\frac{dy}{dt} = 0$ so $\frac{dy}{dt} = -\frac{1}{4}$ when $x = 2$.

(b) $\frac{dx}{dt} + 4(4) = 0$ so $\frac{dx}{dt} = -16$ when $x = 3$.

3. $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

(a) $2\frac{1}{2}(1) + 2\frac{\sqrt{3}}{2}\frac{dy}{dt} = 0$, so $\frac{dy}{dt} = -\frac{1}{\sqrt{3}}$.

(b) $2\frac{\sqrt{2}}{2}\frac{dx}{dt} + 2\frac{\sqrt{2}}{2}(-2) = 0$, so $\frac{dx}{dt} = 2$.

4. $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2\frac{dx}{dt}$

(a) $-2 + 2\frac{dy}{dt} = -2, \frac{dy}{dt} = 0$

(b) $\frac{2+\sqrt{2}}{2}\frac{dx}{dt} + \frac{\sqrt{2}}{2}(3) = \frac{dx}{dt}, (2 + \sqrt{2} - 2)\frac{dx}{dt} + 3\sqrt{2} = 0, \frac{dx}{dt} = -3$

5. **(b)** $A = x^2$

(c) $\frac{dA}{dt} = 2x\frac{dx}{dt}$

Find $\left.\frac{dA}{dt}\right|_{x=3}$ given that $\left.\frac{dx}{dt}\right|_{x=3} = 2$. From Part (c), $\left.\frac{dA}{dt}\right|_{x=3} = 2(3)(2) = 12$ ft^2/min.

6. **(b)** $A = \pi r^2$ **(c)** $\dfrac{dA}{dt} = 2\pi r \dfrac{dr}{dt}$

(d) Find $\left.\dfrac{dA}{dt}\right|_{r=5}$ given that $\left.\dfrac{dr}{dt}\right|_{r=5} = 2$. From Part (c), $\left.\dfrac{dA}{dt}\right|_{r=5} = 2\pi(5)(2) = 20\pi$ cm^2/s.

7. **(a)** $V = \pi r^2 h$, so $\dfrac{dV}{dt} = \pi\left(r^2\dfrac{dh}{dt} + 2rh\dfrac{dr}{dt}\right)$.

(b) Find $\left.\dfrac{dV}{dt}\right|_{\substack{h=6,\\ r=10}}$ given that $\left.\dfrac{dh}{dt}\right|_{\substack{h=6,\\ r=10}} = 1$ and $\left.\dfrac{dr}{dt}\right|_{\substack{h=6,\\ r=10}} = -1$. From Part (a),

$\left.\dfrac{dV}{dt}\right|_{\substack{h=6,\\ r=10}} = \pi[10^2(1) + 2(10)(6)(-1)] = -20\pi$ in^3/s; the volume is decreasing.

8. **(a)** $\ell^2 = x^2 + y^2$, so $\dfrac{d\ell}{dt} = \dfrac{1}{\ell}\left(x\dfrac{dx}{dt} + y\dfrac{dy}{dt}\right)$.

(b) Find $\left.\dfrac{d\ell}{dt}\right|_{\substack{x=3,\\ y=4}}$ given that $\dfrac{dx}{dt} = \dfrac{1}{2}$ and $\dfrac{dy}{dt} = -\dfrac{1}{4}$.

From Part (a) and the fact that $\ell = 5$ when $x = 3$ and $y = 4$,

$\left.\dfrac{d\ell}{dt}\right|_{\substack{x=3,\\ y=4}} = \dfrac{1}{5}\left[3\left(\dfrac{1}{2}\right) + 4\left(-\dfrac{1}{4}\right)\right] = \dfrac{1}{10}$ ft/s; the diagonal is increasing.

9. **(a)** $\tan\theta = \dfrac{y}{x}$, so $\sec^2\theta\dfrac{d\theta}{dt} = \dfrac{x\dfrac{dy}{dt} - y\dfrac{dx}{dt}}{x^2}$, $\dfrac{d\theta}{dt} = \dfrac{\cos^2\theta}{x^2}\left(x\dfrac{dy}{dt} - y\dfrac{dx}{dt}\right)$

(b) Find $\left.\dfrac{d\theta}{dt}\right|_{\substack{x=2,\\ y=2}}$ given that $\left.\dfrac{dx}{dt}\right|_{\substack{x=2,\\ y=2}} = 1$ and $\left.\dfrac{dy}{dt}\right|_{\substack{x=2,\\ y=2}} = -\dfrac{1}{4}$.

When $x = 2$ and $y = 2$, $\tan\theta = 2/2 = 1$ so $\theta = \dfrac{\pi}{4}$ and $\cos\theta = \cos\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$. Thus from Part (a), $\left.\dfrac{d\theta}{dt}\right|_{\substack{x=2,\\ y=2}} = \dfrac{(1/\sqrt{2})^2}{2^2}\left[2\left(-\dfrac{1}{4}\right) - 2(1)\right] = -\dfrac{5}{16}$ rad/s; θ is decreasing.

10. Find $\left.\dfrac{dz}{dt}\right|_{\substack{x=1,\\ y=2}}$ given that $\left.\dfrac{dx}{dt}\right|_{\substack{x=1,\\ y=2}} = -2$ and $\left.\dfrac{dy}{dt}\right|_{\substack{x=1,\\ y=2}} = 3$.

$\dfrac{dz}{dt} = 2x^3y\dfrac{dy}{dt} + 3x^2y^2\dfrac{dx}{dt}$, $\left.\dfrac{dz}{dt}\right|_{\substack{x=1,\\ y=2}} = (4)(3) + (12)(-2) = -12$ units/s; z is decreasing

11. Let A be the area swept out, and θ the angle through which the minute hand has rotated. Find $\dfrac{dA}{dt}$ given that $\dfrac{d\theta}{dt} = \dfrac{\pi}{30}$ rad/min; $A = \dfrac{1}{2}r^2\theta = 8\theta$, so $\dfrac{dA}{dt} = 8\dfrac{d\theta}{dt} = \dfrac{4\pi}{15}$ in^2/min.

12. Let r be the radius and A the area enclosed by the ripple. We want $\left.\dfrac{dA}{dt}\right|_{t=10}$ given that $\dfrac{dr}{dt} = 3$. We know that $A = \pi r^2$, so $\dfrac{dA}{dt} = 2\pi r\dfrac{dr}{dt}$. Because r is increasing at the constant rate of 3 ft/s, it follows that $r = 30$ ft after 10 seconds so $\left.\dfrac{dA}{dt}\right|_{t=10} = 2\pi(30)(3) = 180\pi$ ft^2/s.

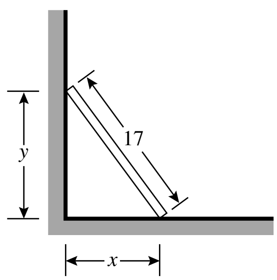
13. Find $\left.\dfrac{dr}{dt}\right|_{A=9}$ given that $\dfrac{dA}{dt} = 6$. From $A = \pi r^2$ we get $\dfrac{dA}{dt} = 2\pi r\dfrac{dr}{dt}$ so $\dfrac{dr}{dt} = \dfrac{1}{2\pi r}\dfrac{dA}{dt}$. If $A = 9$ then $\pi r^2 = 9$, $r = 3/\sqrt{\pi}$ so $\left.\dfrac{dr}{dt}\right|_{A=9} = \dfrac{1}{2\pi(3/\sqrt{\pi})}(6) = 1/\sqrt{\pi}$ mi/h.

14. The volume V of a sphere of radius r is given by $V = \dfrac{4}{3}\pi r^3$ or, because $r = \dfrac{D}{2}$ where D is the diameter, $V = \dfrac{4}{3}\pi\left(\dfrac{D}{2}\right)^3 = \dfrac{1}{6}\pi D^3$. We want $\left.\dfrac{dD}{dt}\right|_{r=1}$ given that $\dfrac{dV}{dt} = 3$. From $V = \dfrac{1}{6}\pi D^3$ we get $\dfrac{dV}{dt} = \dfrac{1}{2}\pi D^2\dfrac{dD}{dt}$, $\dfrac{dD}{dt} = \dfrac{2}{\pi D^2}\dfrac{dV}{dt}$, so $\left.\dfrac{dD}{dt}\right|_{r=1} = \dfrac{2}{\pi(2)^2}(3) = \dfrac{3}{2\pi}$ ft/min.

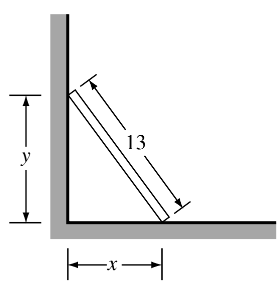
15. Find $\left.\dfrac{dV}{dt}\right|_{r=9}$ given that $\dfrac{dr}{dt} = -15$. From $V = \dfrac{4}{3}\pi r^3$ we get $\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$ so

$\left.\dfrac{dV}{dt}\right|_{r=9} = 4\pi(9)^2(-15) = -4860\pi$. Air must be removed at the rate of 4860π cm^3/min.

16. Let x and y be the distances shown in the diagram. We want to find $\left.\dfrac{dy}{dt}\right|_{y=8}$ given that $\dfrac{dx}{dt} = 5$. From $x^2+y^2 = 17^2$ we get $2x\dfrac{dx}{dt}+2y\dfrac{dy}{dt} = 0$, so $\dfrac{dy}{dt} = -\dfrac{x}{y}\dfrac{dx}{dt}$. When $y = 8$, $x^2 + 8^2 = 17^2$, $x^2 = 289 - 64 = 225$, $x = 15$ so $\left.\dfrac{dy}{dt}\right|_{y=8} = -\dfrac{15}{8}(5) = -\dfrac{75}{8}$ ft/s; the top of the ladder is moving down the wall at a rate of 75/8 ft/s.

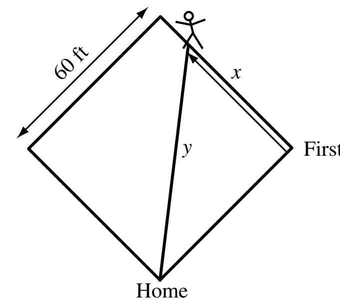


17. Find $\left.\dfrac{dx}{dt}\right|_{y=5}$ given that $\dfrac{dy}{dt} = -2$. From $x^2 + y^2 = 13^2$ we get $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$ so $\dfrac{dx}{dt} = -\dfrac{y}{x}\dfrac{dy}{dt}$. Use $x^2 + y^2 = 169$ to find that $x = 12$ when $y = 5$ so $\left.\dfrac{dx}{dt}\right|_{y=5} = -\dfrac{5}{12}(-2) = \dfrac{5}{6}$ ft/s.

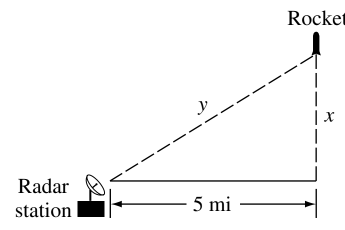


18. Let θ be the acute angle, and x the distance of the bottom of the plank from the wall. Find $\left.\dfrac{d\theta}{dt}\right|_{x=2}$ given that $\left.\dfrac{dx}{dt}\right|_{x=2} = -\dfrac{1}{2}$ ft/s. The variables θ and x are related by the equation $\cos\theta = \dfrac{x}{10}$ so $-\sin\theta\dfrac{d\theta}{dt} = \dfrac{1}{10}\dfrac{dx}{dt}$, $\dfrac{d\theta}{dt} = -\dfrac{1}{10\sin\theta}\dfrac{dx}{dt}$. When $x = 2$, the top of the plank is $\sqrt{10^2 - 2^2} = \sqrt{96}$ ft above the ground so $\sin\theta = \sqrt{96}/10$ and $\left.\dfrac{d\theta}{dt}\right|_{x=2} = -\dfrac{1}{\sqrt{96}}\left(-\dfrac{1}{2}\right) = \dfrac{1}{2\sqrt{96}} \approx 0.051$ rad/s.

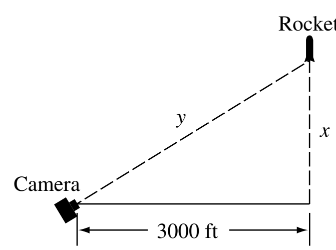
19. Let x denote the distance from first base and y the distance from home plate. Then $x^2 + 60^2 = y^2$ and $2x\dfrac{dx}{dt} = 2y\dfrac{dy}{dt}$. When $x = 50$ then $y = 10\sqrt{61}$ so $\dfrac{dy}{dt} = \dfrac{x}{y}\dfrac{dx}{dt} = \dfrac{50}{10\sqrt{61}}(25) = \dfrac{125}{\sqrt{61}}$ ft/s.



20. Find $\left.\dfrac{dx}{dt}\right|_{x=4}$ given that $\left.\dfrac{dy}{dt}\right|_{x=4} = 2000$. From $x^2 + 5^2 = y^2$ we get $2x\dfrac{dx}{dt} = 2y\dfrac{dy}{dt}$ so $\dfrac{dx}{dt} = \dfrac{y}{x}\dfrac{dy}{dt}$. Use $x^2 + 25 = y^2$ to find that $y = \sqrt{41}$ when $x = 4$ so $\left.\dfrac{dx}{dt}\right|_{x=4} = \dfrac{\sqrt{41}}{4}(2000) = 500\sqrt{41}$ mi/h.



21. Find $\left.\dfrac{dy}{dt}\right|_{x=4000}$ given that $\left.\dfrac{dx}{dt}\right|_{x=4000} = 880$. From $y^2 = x^2 + 3000^2$ we get $2y\dfrac{dy}{dt} = 2x\dfrac{dx}{dt}$ so $\dfrac{dy}{dt} = \dfrac{x}{y}\dfrac{dx}{dt}$. If $x = 4000$, then $y = 5000$ so $\left.\dfrac{dy}{dt}\right|_{x=4000} = \dfrac{4000}{5000}(880) = 704$ ft/s.



22. Find $\left.\dfrac{dx}{dt}\right|_{\phi=\pi/4}$ given that $\left.\dfrac{d\phi}{dt}\right|_{\phi=\pi/4} = 0.2$. But $x = 3000\tan\phi$ so $\dfrac{dx}{dt} = 3000(\sec^2\phi)\dfrac{d\phi}{dt}$, $\left.\dfrac{dx}{dt}\right|_{\phi=\pi/4} = 3000\left(\sec^2\dfrac{\pi}{4}\right)(0.2) = 1200$ ft/s.

23. **(a)** If x denotes the altitude, then $r - x = 3960$, the radius of the Earth. $\theta = 0$ at perigee, so $r = 4995/1.12 \approx 4460$; the altitude is $x = 4460 - 3960 = 500$ miles. $\theta = \pi$ at apogee, so $r = 4995/0.88 \approx 5676$; the altitude is $x = 5676 - 3960 = 1716$ miles.

(b) If $\theta = 120°$, then $r = 4995/0.94 \approx 5314$; the altitude is $5314 - 3960 = 1354$ miles. The rate of change of the altitude is given by

$$\frac{dx}{dt} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{4995(0.12\sin\theta)}{(1 + 0.12\cos\theta)^2}\frac{d\theta}{dt}.$$

Use $\theta = 120°$ and $d\theta/dt = 2.7°/\text{min} = (2.7)(\pi/180)$ rad/min to get $dr/dt \approx 27.7$ mi/min.

24. **(a)** Let x be the horizontal distance shown in the figure. Then $x = 4000\cot\theta$ and

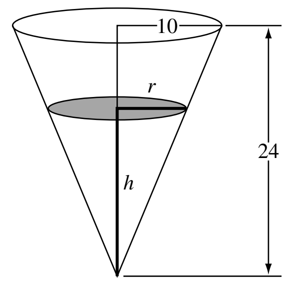
$\dfrac{dx}{dt} = -4000\csc^2\theta\dfrac{d\theta}{dt}$, so $\dfrac{d\theta}{dt} = -\dfrac{\sin^2\theta}{4000}\dfrac{dx}{dt}$. Use $\theta = 30°$ and

$dx/dt = 300$ mi/h $= 300(5280/3600)$ ft/s $= 440$ ft/s to get

$d\theta/dt = -0.0275$ rad/s $\approx -1.6°$/s; θ is decreasing at the rate of $1.6°$/s.

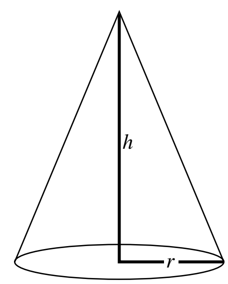
(b) Let y be the distance between the observation point and the aircraft. Then $y = 4000 \csc\theta$ so $dy/dt = -4000(\csc\theta\cot\theta)(d\theta/dt)$. Use $\theta = 30°$ and $d\theta/dt = -0.0275$ rad/s to get $dy/dt \approx 381$ ft/s.

25. Find $\left.\dfrac{dh}{dt}\right|_{h=16}$ given that $\dfrac{dV}{dt} = 20$. The volume of water in the tank at a depth h is $V = \dfrac{1}{3}\pi r^2 h$. Use similar triangles (see figure) to get $\dfrac{r}{h} = \dfrac{10}{24}$ so $r = \dfrac{5}{12}h$ thus $V = \dfrac{1}{3}\pi\left(\dfrac{5}{12}h\right)^2 h = \dfrac{25}{432}\pi h^3$, $\dfrac{dV}{dt} = \dfrac{25}{144}\pi h^2\dfrac{dh}{dt}$; $\dfrac{dh}{dt} = \dfrac{144}{25\pi h^2}\dfrac{dV}{dt}$, $\left.\dfrac{dh}{dt}\right|_{h=16} = \dfrac{144}{25\pi(16)^2}(20) = \dfrac{9}{20\pi}$ ft/min.



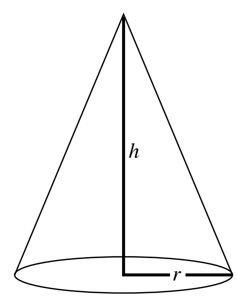
26. Find $\left.\dfrac{dh}{dt}\right|_{h=6}$ given that $\dfrac{dV}{dt} = 8$. $V = \dfrac{1}{3}\pi r^2 h$, but $r = \dfrac{1}{2}h$ so $V = \dfrac{1}{3}\pi\left(\dfrac{h}{2}\right)^2 h = \dfrac{1}{12}\pi h^3$, $\dfrac{dV}{dt} = \dfrac{1}{4}\pi h^2\dfrac{dh}{dt}$,

$\dfrac{dh}{dt} = \dfrac{4}{\pi h^2}\dfrac{dV}{dt}$, $\left.\dfrac{dh}{dt}\right|_{h=6} = \dfrac{4}{\pi(6)^2}(8) = \dfrac{8}{9\pi}$ ft/min.



27. Find $\left.\dfrac{dV}{dt}\right|_{h=10}$ given that $\dfrac{dh}{dt} = 5$. $V = \dfrac{1}{3}\pi r^2 h$, but $r = \dfrac{1}{2}h$ so $V = \dfrac{1}{3}\pi\left(\dfrac{h}{2}\right)^2 h = \dfrac{1}{12}\pi h^3$, $\dfrac{dV}{dt} = \dfrac{1}{4}\pi h^2\dfrac{dh}{dt}$,

$\left.\dfrac{dV}{dt}\right|_{h=10} = \dfrac{1}{4}\pi(10)^2(5) = 125\pi$ ft^3/min.



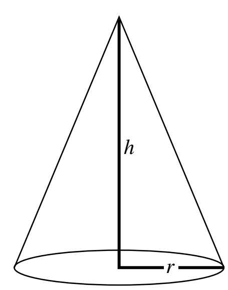
28. Let r and h be as shown in the figure. If C is the circumference of the base, then we want to find $\left.\dfrac{dC}{dt}\right|_{h=8}$ given that $\dfrac{dV}{dt} = 10$.

It is given that $r = \dfrac{1}{2}h$, thus $C = 2\pi r = \pi h$ so

$$\frac{dC}{dt} = \pi\frac{dh}{dt} \qquad (1)$$

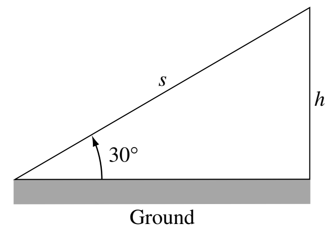
Use $V = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{12}\pi h^3$ to get $\dfrac{dV}{dt} = \dfrac{1}{4}\pi h^2\dfrac{dh}{dt}$, so

$$\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt} \qquad (2)$$



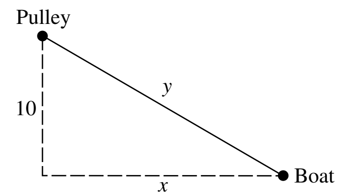
Substitution of (2) into (1) gives $\dfrac{dC}{dt} = \dfrac{4}{h^2}\dfrac{dV}{dt}$ so $\left.\dfrac{dC}{dt}\right|_{h=8} = \dfrac{4}{64}(10) = \dfrac{5}{8}$ ft/min.

29. With s and h as shown in the figure, we want to find $\frac{dh}{dt}$ given that $\frac{ds}{dt} = 500$. From the figure, $h = s\sin 30^\circ = \frac{1}{2}s$ so $\frac{dh}{dt} = \frac{1}{2}\frac{ds}{dt} = \frac{1}{2}(500) = 250$ mi/h.

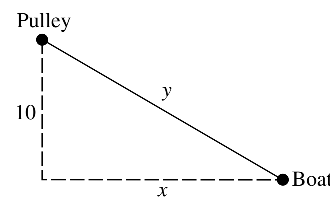


30. Find $\left.\frac{dx}{dt}\right|_{y=125}$ given that $\frac{dy}{dt} = -20$. From $x^2 + 10^2 = y^2$ we get $2x\frac{dx}{dt} = 2y\frac{dy}{dt}$ so $\frac{dx}{dt} = \frac{y}{x}\frac{dy}{dt}$. Use $x^2 + 100 = y^2$ to find that $x = \sqrt{15,525} = 15\sqrt{69}$ when $y = 125$ so $\left.\frac{dx}{dt}\right|_{y=125} = \frac{125}{15\sqrt{69}}(-20) = -\frac{500}{3\sqrt{69}}$.

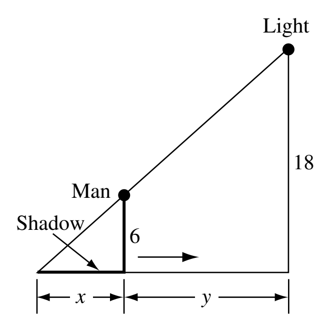
The boat is approaching the dock at the rate of $\frac{500}{3\sqrt{69}}$ ft/min.



31. Find $\frac{dy}{dt}$ given that $\left.\frac{dx}{dt}\right|_{y=125} = -12$. From $x^2+10^2 = y^2$ we get $2x\frac{dx}{dt} = 2y\frac{dy}{dt}$ so $\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$. Use $x^2 + 100 = y^2$ to find that $x = \sqrt{15,525} = 15\sqrt{69}$ when $y = 125$ so $\frac{dy}{dt} = \frac{15\sqrt{69}}{125}(-12) = -\frac{36\sqrt{69}}{25}$. The rope must be pulled at the rate of $\frac{36\sqrt{69}}{25}$ ft/min.



32. **(a)** Let x and y be as shown in the figure. It is required to find $\frac{dx}{dt}$, given that $\frac{dy}{dt} = -3$. By similar triangles, $\frac{x}{6} = \frac{x+y}{18}$, $18x = 6x + 6y$, $12x = 6y$, $x = \frac{1}{2}y$, so $\frac{dx}{dt} = \frac{1}{2}\frac{dy}{dt} = \frac{1}{2}(-3) = -\frac{3}{2}$ ft/s.

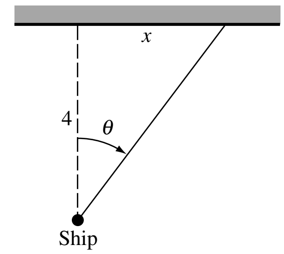


(b) The tip of the shadow is $z = x + y$ feet from the street light, thus the rate at which it is moving is given by $\frac{dz}{dt} = \frac{dx}{dt} + \frac{dy}{dt}$. In part (a) we found that $\frac{dx}{dt} = -\frac{3}{2}$ when $\frac{dy}{dt} = -3$ so $\frac{dz}{dt} = (-3/2) + (-3) = -9/2$ ft/s; the tip of the shadow is moving at the rate of 9/2 ft/s toward the street light.

33. Find $\left.\dfrac{dx}{dt}\right|_{\theta=\pi/4}$ given that $\dfrac{d\theta}{dt} = \dfrac{2\pi}{10} = \dfrac{\pi}{5}$ rad/s.

Then $x = 4\tan\theta$ (see figure) so $\dfrac{dx}{dt} = 4\sec^2\theta\dfrac{d\theta}{dt}$,

$\left.\dfrac{dx}{dt}\right|_{\theta=\pi/4} = 4\left(\sec^2\dfrac{\pi}{4}\right)\left(\dfrac{\pi}{5}\right) = 8\pi/5$ km/s.



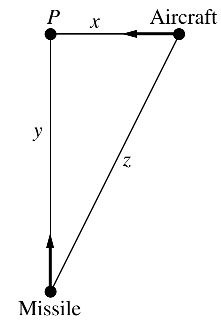
34. If x, y, and z are as shown in the figure, then we want $\left.\dfrac{dz}{dt}\right|_{\substack{x=2,\\y=4}}$ given that $\dfrac{dx}{dt} = -600$ and $\left.\dfrac{dy}{dt}\right|_{\substack{x=2,\\y=4}} = -1200$. But

$z^2 = x^2 + y^2$ so $2z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt}$, $\dfrac{dz}{dt} = \dfrac{1}{z}\left(x\dfrac{dx}{dt} + y\dfrac{dy}{dt}\right)$.

When $x = 2$ and $y = 4$, $z^2 = 2^2 + 4^2 = 20$, $z = \sqrt{20} = 2\sqrt{5}$ so

$\left.\dfrac{dz}{dt}\right|_{\substack{x=2,\\y=4}} = \dfrac{1}{2\sqrt{5}}[2(-600) + 4(-1200)] = -\dfrac{3000}{\sqrt{5}} = -600\sqrt{5}$ mi/h;

the distance between missile and aircraft is decreasing at the rate of $600\sqrt{5}$ mi/h.

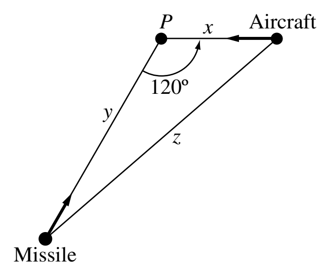


35. We wish to find $\left.\dfrac{dz}{dt}\right|_{\substack{x=2,\\y=4}}$ given $\dfrac{dx}{dt} = -600$ and $\left.\dfrac{dy}{dt}\right|_{\substack{x=2,\\y=4}} = -1200$ (see figure). From the law of cosines,

$z^2 = x^2 + y^2 - 2xy\cos 120° = x^2 + y^2 - 2xy(-1/2)$

$= x^2 + y^2 + xy$, so $2z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} + x\dfrac{dy}{dt} + y\dfrac{dx}{dt}$,

$\dfrac{dz}{dt} = \dfrac{1}{2z}\left[(2x + y)\dfrac{dx}{dt} + (2y + x)\dfrac{dy}{dt}\right]$.



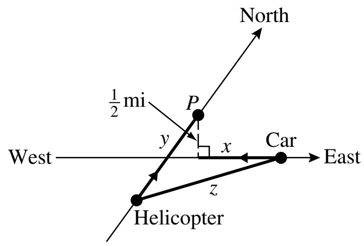
When $x = 2$ and $y = 4$, $z^2 = 2^2 + 4^2 + (2)(4) = 28$, so $z = \sqrt{28} = 2\sqrt{7}$, thus

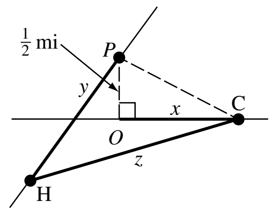
$\left.\dfrac{dz}{dt}\right|_{\substack{x=2,\\y=4}} = \dfrac{1}{2(2\sqrt{7})}[(2(2) + 4)(-600) + (2(4) + 2)(-1200)] = -\dfrac{4200}{\sqrt{7}} = -600\sqrt{7}$ mi/h;

the distance between missile and aircraft is decreasing at the rate of $600\sqrt{7}$ mi/h.

36. **(a)** Let P be the point on the helicopter's path that lies directly above the car's path. Let x, y, and z be the distances shown in the first figure. Find $\left.\dfrac{dz}{dt}\right|_{\substack{x=2,\\y=0}}$ given that $\dfrac{dx}{dt} = -75$ and $\dfrac{dy}{dt} = 100$. In order to find an equation relating x, y, and z, first draw the line segment that joins the point P to the car, as shown in the second figure. Because triangle OPC is a right triangle, it follows that PC has length $\sqrt{x^2 + (1/2)^2}$; but triangle HPC is also a right triangle so $z^2 = \left(\sqrt{x^2 + (1/2)^2}\right)^2 + y^2 = x^2 + y^2 + 1/4$ and $2z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} + 0$, $\dfrac{dz}{dt} = \dfrac{1}{z}\left(x\dfrac{dx}{dt} + y\dfrac{dy}{dt}\right)$. Now, when $x = 2$ and $y = 0$, $z^2 = (2)^2 + (0)^2 + 1/4 = 17/4$,

$$z = \sqrt{17}/2 \text{ so } \left.\frac{dz}{dt}\right|_{\substack{x=2,\\y=0}} = \frac{1}{(\sqrt{17}/2)}[2(-75)+0(100)] = -300/\sqrt{17} \text{ mi/h}$$





(b) decreasing, because $\frac{dz}{dt} < 0$.

37. **(a)** We want $\left.\frac{dy}{dt}\right|_{\substack{x=1,\\y=2}}$ given that $\left.\frac{dx}{dt}\right|_{\substack{x=1,\\y=2}} = 6$. For convenience, first rewrite the equation as

$$xy^3 = \frac{8}{5} + \frac{8}{5}y^2 \text{ then } 3xy^2\frac{dy}{dt} + y^3\frac{dx}{dt} = \frac{16}{5}y\frac{dy}{dt},\ \frac{dy}{dt} = \frac{y^3}{\frac{16}{5}y - 3xy^2}\frac{dx}{dt} \text{ so}$$

$$\left.\frac{dy}{dt}\right|_{\substack{x=1,\\y=2}} = \frac{2^3}{\frac{16}{5}(2) - 3(1)2^2}(6) = -60/7 \text{ units/s.}$$

(b) falling, because $\frac{dy}{dt} < 0$

38. Find $\left.\frac{dx}{dt}\right|_{(2,5)}$ given that $\left.\frac{dy}{dt}\right|_{(2,5)} = 2$. Square and rearrange to get $x^3 = y^2 - 17$

so $3x^2\frac{dx}{dt} = 2y\frac{dy}{dt},\ \frac{dx}{dt} = \frac{2y}{3x^2}\frac{dy}{dt},\ \left.\frac{dx}{dt}\right|_{(2,5)} = \left(\frac{5}{6}\right)(2) = \frac{5}{3}$ units/s.

39. The coordinates of P are $(x, 2x)$, so the distance between P and the point $(3,0)$ is

$D = \sqrt{(x-3)^2 + (2x-0)^2} = \sqrt{5x^2 - 6x + 9}$. Find $\left.\frac{dD}{dt}\right|_{x=3}$ given that $\left.\frac{dx}{dt}\right|_{x=3} = -2$.

$\frac{dD}{dt} = \frac{5x-3}{\sqrt{5x^2-6x+9}}\frac{dx}{dt}$, so $\left.\frac{dD}{dt}\right|_{x=3} = \frac{12}{\sqrt{36}}(-2) = -4$ units/s.

40. **(a)** Let D be the distance between P and $(2,0)$. Find $\left.\frac{dD}{dt}\right|_{x=3}$ given that $\left.\frac{dx}{dt}\right|_{x=3} = 4$.

$D = \sqrt{(x-2)^2 + y^2} = \sqrt{(x-2)^2 + x} = \sqrt{x^2 - 3x + 4}$ so $\frac{dD}{dt} = \frac{2x-3}{2\sqrt{x^2-3x+4}}$;

$\left.\frac{dD}{dt}\right|_{x=3} = \frac{3}{2\sqrt{4}} = \frac{3}{4}$ units/s.

(b) Let θ be the angle of inclination. Find $\left.\frac{d\theta}{dt}\right|_{x=3}$ given that $\left.\frac{dx}{dt}\right|_{x=3} = 4$.

$\tan\theta = \frac{y}{x-2} = \frac{\sqrt{x}}{x-2}$ so $\sec^2\theta\frac{d\theta}{dt} = -\frac{x+2}{2\sqrt{x}(x-2)^2}\frac{dx}{dt},\ \frac{d\theta}{dt} = -\cos^2\theta\frac{x+2}{2\sqrt{x}(x-2)^2}\frac{dx}{dt}$.

When $x = 3$, $D = 2$ so $\cos\theta = \frac{1}{2}$ and $\left.\frac{d\theta}{dt}\right|_{x=3} = -\frac{1}{4}\frac{5}{2\sqrt{3}}(4) = -\frac{5}{2\sqrt{3}}$ rad/s.

41. Solve $\dfrac{dx}{dt} = 3\dfrac{dy}{dt}$ given $y = x/(x^2+1)$. Then $y(x^2+1) = x$. Differentiating with respect to x, $(x^2+1)\dfrac{dy}{dx} + y(2x) = 1$. But $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1}{3}$ so $(x^2+1)\dfrac{1}{3} + 2xy = 1$, $x^2+1+6xy = 3$, $x^2+1+6x^2/(x^2+1) = 3$, $(x^2+1)^2+6x^2-3x^2-3 = 0$, $x^4+5x^2-3 = 0$. By the binomial theorem applied to x^2 we obtain $x^2 = (-5 \pm \sqrt{25+12})/2$. The minus sign is spurious since x^2 cannot be negative, so $x^2 = (\sqrt{33}-5)/2$, $x \approx \pm 0.6101486081, y = \pm 0.4446235604$.

42. $32x\dfrac{dx}{dt} + 18y\dfrac{dy}{dt} = 0$; if $\dfrac{dy}{dt} = \dfrac{dx}{dt} \neq 0$, then $(32x+18y)\dfrac{dx}{dt} = 0$, $32x + 18y = 0$, $y = -\dfrac{16}{9}x$ so $16x^2 + 9\dfrac{256}{81}x^2 = 144$, $\dfrac{400}{9}x^2 = 144$, $x^2 = \dfrac{81}{25}$, $x = \pm\dfrac{9}{5}$. If $x = \dfrac{9}{5}$, then $y = -\dfrac{16}{9}\dfrac{9}{5} = -\dfrac{16}{5}$. Similarly, if $x = -\dfrac{9}{5}$, then $y = \dfrac{16}{5}$. The points are $\left(\dfrac{9}{5}, -\dfrac{16}{5}\right)$ and $\left(-\dfrac{9}{5}, \dfrac{16}{5}\right)$.

43. Find $\left.\dfrac{dS}{dt}\right|_{s=10}$ given that $\left.\dfrac{ds}{dt}\right|_{s=10} = -2$. From $\dfrac{1}{s} + \dfrac{1}{S} = \dfrac{1}{6}$ we get $-\dfrac{1}{s^2}\dfrac{ds}{dt} - \dfrac{1}{S^2}\dfrac{dS}{dt} = 0$, so $\dfrac{dS}{dt} = -\dfrac{S^2}{s^2}\dfrac{ds}{dt}$. If $s = 10$, then $\dfrac{1}{10} + \dfrac{1}{S} = \dfrac{1}{6}$ which gives $S = 15$. So $\left.\dfrac{dS}{dt}\right|_{s=10} = -\dfrac{225}{100}(-2) = 4.5$ cm/s. The image is moving away from the lens.

44. Suppose that the reservoir has height H and that the radius at the top is R. At any instant of time let h and r be the corresponding dimensions of the cone of water (see figure). We want to show that $\dfrac{dh}{dt}$ is constant and independent of H and R, given that $\dfrac{dV}{dt} = -kA$ where V is the volume of water, A is the area of a circle of radius r, and k is a positive constant. The volume of a cone of radius r and height h is $V = \dfrac{1}{3}\pi r^2 h$. By similar triangles $\dfrac{r}{h} = \dfrac{R}{H}$, $r = \dfrac{R}{H}h$ thus $V = \dfrac{1}{3}\pi\left(\dfrac{R}{H}\right)^2 h^3$ so

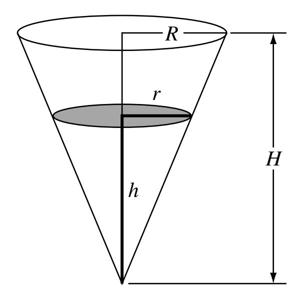
$$\frac{dV}{dt} = \pi\left(\frac{R}{H}\right)^2 h^2\frac{dh}{dt} \qquad (1)$$

But it is given that $\dfrac{dV}{dt} = -kA$ or, because

$A = \pi r^2 = \pi\left(\dfrac{R}{H}\right)^2 h^2$, $\dfrac{dV}{dt} = -k\pi\left(\dfrac{R}{H}\right)^2 h^2$,

which when substituted into equation (1) gives

$-k\pi\left(\dfrac{R}{H}\right)^2 h^2 = \pi\left(\dfrac{R}{H}\right)^2 h^2\dfrac{dh}{dt}$, $\dfrac{dh}{dt} = -k$.



45. Let r be the radius, V the volume, and A the surface area of a sphere. Show that $\dfrac{dr}{dt}$ is a constant given that $\dfrac{dV}{dt} = -kA$, where k is a positive constant. Because $V = \dfrac{4}{3}\pi r^3$,

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} \qquad (1)$$

But it is given that $\dfrac{dV}{dt} = -kA$ or, because $A = 4\pi r^2$, $\dfrac{dV}{dt} = -4\pi r^2 k$ which when substituted into equation (1) gives $-4\pi r^2 k = 4\pi r^2\dfrac{dr}{dt}$, $\dfrac{dr}{dt} = -k$.

46. Let x be the distance between the tips of the minute and hour hands, and α and β the angles shown in the figure. Because the minute hand makes one revolution in 60 minutes, $\dfrac{d\alpha}{dt} = \dfrac{2\pi}{60} = \pi/30$ rad/min; the hour hand makes one revolution in 12 hours (720 minutes), thus $\dfrac{d\beta}{dt} = \dfrac{2\pi}{720} = \pi/360$ rad/min. We want to find $\left.\dfrac{dx}{dt}\right|_{\substack{\alpha=2\pi,\\ \beta=3\pi/2}}$ given that $\dfrac{d\alpha}{dt} = \pi/30$ and $\dfrac{d\beta}{dt} = \pi/360$.

Using the law of cosines on the triangle shown in the figure,

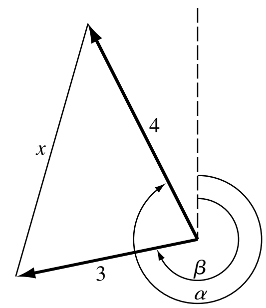
$x^2 = 3^2 + 4^2 - 2(3)(4)\cos(\alpha - \beta) = 25 - 24\cos(\alpha - \beta)$, so

$2x\dfrac{dx}{dt} = 0 + 24\sin(\alpha - \beta)\left(\dfrac{d\alpha}{dt} - \dfrac{d\beta}{dt}\right)$,

$\dfrac{dx}{dt} = \dfrac{12}{x}\left(\dfrac{d\alpha}{dt} - \dfrac{d\beta}{dt}\right)\sin(\alpha - \beta)$. When $\alpha = 2\pi$ and $\beta = 3\pi/2$,

$x^2 = 25 - 24\cos(2\pi - 3\pi/2) = 25$, $x = 5$; so

$\left.\dfrac{dx}{dt}\right|_{\substack{\alpha=2\pi,\\ \beta=3\pi/2}} = \dfrac{12}{5}(\pi/30 - \pi/360)\sin(2\pi - 3\pi/2) = \dfrac{11\pi}{150}$ in/min.

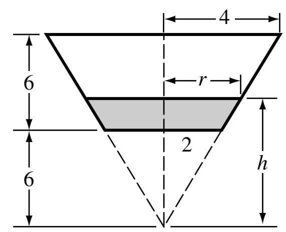


47. Extend sides of cup to complete the cone and let V_0 be the volume of the portion added, then (see figure)

$V = \dfrac{1}{3}\pi r^2 h - V_0$ where $\dfrac{r}{h} = \dfrac{4}{12} = \dfrac{1}{3}$ so $r = \dfrac{1}{3}h$ and

$V = \dfrac{1}{3}\pi\left(\dfrac{h}{3}\right)^2 h - V_0 = \dfrac{1}{27}\pi h^3 - V_0$, $\dfrac{dV}{dt} = \dfrac{1}{9}\pi h^2\dfrac{dh}{dt}$,

$\dfrac{dh}{dt} = \dfrac{9}{\pi h^2}\dfrac{dV}{dt}$, $\left.\dfrac{dh}{dt}\right|_{h=9} = \dfrac{9}{\pi(9)^2}(20) = \dfrac{20}{9\pi}$ cm/s.



EXERCISE SET 3.8

1. **(a)** $f(x) \approx f(1) + f'(1)(x-1) = 1 + 3(x-1)$

(b) $f(1 + \Delta x) \approx f(1) + f'(1)\Delta x = 1 + 3\Delta x$

(c) From Part (a), $(1.02)^3 \approx 1 + 3(0.02) = 1.06$. From Part (b), $(1.02)^3 \approx 1 + 3(0.02) = 1.06$.

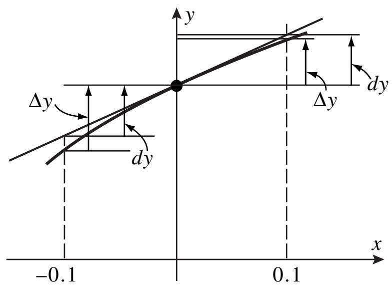
2. **(a)** $f(x) \approx f(2) + f'(2)(x-2) = 1/2 + (-1/2^2)(x-2) = (1/2) - (1/4)(x-2)$

(b) $f(2 + \Delta x) \approx f(2) + f'(2)\Delta x = 1/2 - (1/4)\Delta x$

(c) From Part (a), $1/2.05 \approx 0.5 - 0.25(0.05) = 0.4875$, and from Part (b), $1/2.05 \approx 0.5 - 0.25(0.05) = 0.4875$.

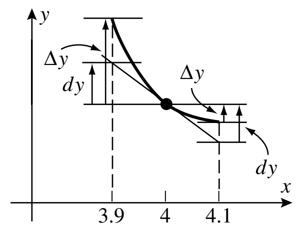
3. **(a)** $f(x) \approx f(x_0) + f'(x_0)(x - x_0) = 1 + (1/(2\sqrt{1})(x-0) = 1 + (1/2)x$, so with $x_0 = 0$ and $x = -0.1$, we have $\sqrt{0.9} = f(-0.1) \approx 1 + (1/2)(-0.1) = 1 - 0.05 = 0.95$. With $x = 0.1$ we have $\sqrt{1.1} = f(0.1) \approx 1 + (1/2)(0.1) = 1.05$.

(b)



4. (a) $f(x) \approx f(x_0)+f'(x_0)(x-x_0) = 1/2-\left[1/(2\cdot 4^{3/2})\right](x-4) = 1/2-(x-4)/16$, so with $x_0 = 4$ and $x = 3.9$ we have $1/\sqrt{3.9} = f(3.9) \approx 0.5-(-0.1)/16 = 0.50625$. If $x_0 = 4$ and $x = 4.1$ then $1/\sqrt{4.1} = f(4.1) \approx 0.5-(0.1)/16 = 0.49375$

(b)



5. $f(x) = (1+x)^{15}$ and $x_0 = 0$. Thus $(1+x)^{15} \approx f(x_0)+f'(x_0)(x-x_0) = 1+15(1)^{14}(x-0) = 1+15x$.

6. $f(x) = \dfrac{1}{\sqrt{1-x}}$ and $x_0 = 0$, so $\dfrac{1}{\sqrt{1-x}} \approx f(x_0)+f'(x_0)(x-x_0) = 1+\dfrac{1}{2(1-0)^{3/2}}(x-0) = 1+x/2$

7. $\tan x \approx \tan(0)+\sec^2(0)(x-0) = x$

8. $\dfrac{1}{1+x} \approx 1+\dfrac{-1}{(1+0)^2}(x-0) = 1-x$

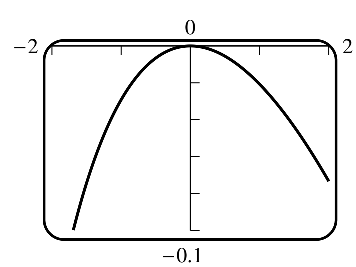
9. $x^4 \approx (1)^4+4(1)^3(x-1)$. Set $\Delta x = x-1$; then $x = \Delta x+1$ and $(1+\Delta x)^4 = 1+4\Delta x$.

10. $\sqrt{x} \approx \sqrt{1}+\dfrac{1}{2\sqrt{1}}(x-1)$, and $x = 1+\Delta x$, so $\sqrt{1+\Delta x} \approx 1+\Delta x/2$

11. $\dfrac{1}{2+x} \approx \dfrac{1}{2+1}-\dfrac{1}{(2+1)^2}(x-1)$, and $2+x = 3+\Delta x$, so $\dfrac{1}{3+\Delta x} \approx \dfrac{1}{3}-\dfrac{1}{9}\Delta x$

12. $(4+x)^3 \approx (4+1)^3+3(4+1)^2(x-1)$ so, with $4+x = 5+\Delta x$ we get $(5+\Delta x)^3 \approx 125+75\Delta x$

13. $f(x) = \sqrt{x+3}$ and $x_0 = 0$, so
$\sqrt{x+3} \approx \sqrt{3}+\dfrac{1}{2\sqrt{3}}(x-0) = \sqrt{3}+\dfrac{1}{2\sqrt{3}}x$, and
$\left|f(x)-\left(\sqrt{3}+\dfrac{1}{2\sqrt{3}}x\right)\right| < 0.1$ if $|x| < 1.692$.

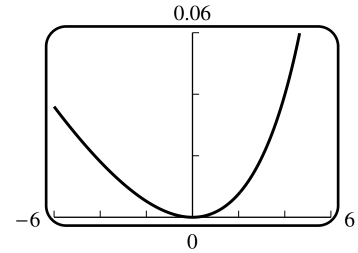


$\left|f(x)-\left(\sqrt{3}+\frac{1}{2\sqrt{3}}x\right)\right|$

14. $f(x) = \dfrac{1}{\sqrt{9-x}}$ so

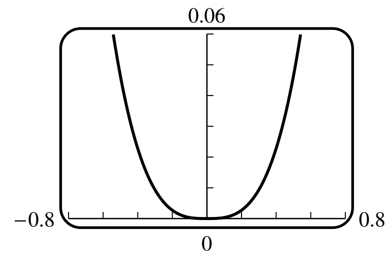
$$\frac{1}{\sqrt{9-x}} \approx \frac{1}{\sqrt{9}} + \frac{1}{2(9-0)^{3/2}}(x-0) = \frac{1}{3} + \frac{1}{54}x,$$

and $\left|f(x) - \left(\dfrac{1}{3} + \dfrac{1}{54}x\right)\right| < 0.1$ if $|x| < 5.5114$



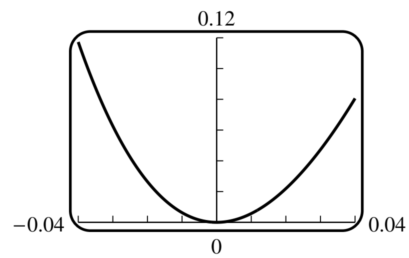
$\left|f(x) - \left(\frac{1}{3} + \frac{1}{54}x\right)\right|$

15. $\tan x \approx \tan 0 + (\sec^2 0)(x-0) = x,$
and $|\tan x - x| < 0.1$ if $|x| < 0.6316$



$|f(x) - x|$

16. $\dfrac{1}{(1+2x)^5} \approx \dfrac{1}{(1+2\cdot 0)^5} + \dfrac{-5(2)}{(1+2\cdot 0)^6}(x-0) = 1 - 10x,$
and $|f(x) - (1-10x)| < 0.1$ if $|x| < 0.0372$



$|f(x) - (1 - 10x)|$

17. **(a)** The local linear approximation $\sin x \approx x$ gives $\sin 1^\circ = \sin(\pi/180) \approx \pi/180 = 0.0174533$ and a calculator gives $\sin 1^\circ = 0.0174524$. The relative error $|\sin(\pi/180) - (\pi/180)|/(\sin \pi/180) = 0.000051$ is very small, so for such a small value of x the approximation is very good.

(b) Use $x_0 = 45^\circ$ (this assumes you know, or can approximate, $\sqrt{2}/2$).

(c) $44^\circ = \dfrac{44\pi}{180}$ radians, and $45^\circ = \dfrac{45\pi}{180} = \dfrac{\pi}{4}$ radians. With $x = \dfrac{44\pi}{180}$ and $x_0 = \dfrac{\pi}{4}$ we obtain $\sin 44^\circ = \sin\dfrac{44\pi}{180} \approx \sin\dfrac{\pi}{4} + \left(\cos\dfrac{\pi}{4}\right)\left(\dfrac{44\pi}{180} - \dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}\left(\dfrac{-\pi}{180}\right) = 0.694765$. With a calculator, $\sin 44^\circ = 0.694658$.

18. **(a)** $\tan x \approx \tan 0 + \sec^2 0(x-0) = x$, so $\tan 2^\circ = \tan(2\pi/180) \approx 2\pi/180 = 0.034907$, and with a calculator $\tan 2^\circ = 0.034921$

(b) use $x_0 = \pi/3$ because we know $\tan 60^\circ = \tan(\pi/3) = \sqrt{3}$

(c) with $x_0 = \dfrac{\pi}{3} = \dfrac{60\pi}{180}$ and $x = \dfrac{61\pi}{180}$ we have

$$\tan 61^\circ = \tan\frac{61\pi}{180} \approx \tan\frac{\pi}{3} + \left(\sec^2\frac{\pi}{3}\right)\left(\frac{61\pi}{180} - \frac{\pi}{3}\right) = \sqrt{3} + 4\frac{\pi}{180} = 1.8019,$$

and with a calculator $\tan 61^\circ = 1.8040$

19. $f(x) = x^4$, $f'(x) = 4x^3$, $x_0 = 3$, $\Delta x = 0.02$; $(3.02)^4 \approx 3^4 + (108)(0.02) = 81 + 2.16 = 83.16$

20. $f(x) = x^3$, $f'(x) = 3x^2$, $x_0 = 2$, $\Delta x = -0.03$; $(1.97)^3 \approx 2^3 + (12)(-0.03) = 8 - 0.36 = 7.64$

21. $f(x) = \sqrt{x}$, $f'(x) = \dfrac{1}{2\sqrt{x}}$, $x_0 = 64$, $\Delta x = 1$; $\sqrt{65} \approx \sqrt{64} + \dfrac{1}{16}(1) = 8 + \dfrac{1}{16} = 8.0625$

22. $f(x) = \sqrt{x}$, $f'(x) = \dfrac{1}{2\sqrt{x}}$, $x_0 = 25$, $\Delta x = -1$; $\sqrt{24} \approx \sqrt{25} + \dfrac{1}{10}(-1) = 5 - 0.1 = 4.9$

23. $f(x) = \sqrt{x}$, $f'(x) = \dfrac{1}{2\sqrt{x}}$, $x_0 = 81$, $\Delta x = -0.1$; $\sqrt{80.9} \approx \sqrt{81} + \dfrac{1}{18}(-0.1) \approx 8.9944$

24. $f(x) = \sqrt{x}$, $f'(x) = \dfrac{1}{2\sqrt{x}}$, $x_0 = 36$, $\Delta x = 0.03$; $\sqrt{36.03} \approx \sqrt{36} + \dfrac{1}{12}(0.03) = 6 + 0.0025 = 6.0025$

25. $f(x) = \sin x$, $f'(x) = \cos x$, $x_0 = 0$, $\Delta x = 0.1$; $\sin 0.1 \approx \sin 0 + (\cos 0)(0.1) = 0.1$

26. $f(x) = \tan x$, $f'(x) = \sec^2 x$, $x_0 = 0$, $\Delta x = 0.2$; $\tan 0.2 \approx \tan 0 + (\sec^2 0)(0.2) = 0.2$

27. $f(x) = \cos x$, $f'(x) = -\sin x$, $x_0 = \pi/6$, $\Delta x = \pi/180$;
$\cos 31^\circ \approx \cos 30^\circ + \left(-\dfrac{1}{2}\right)\left(\dfrac{\pi}{180}\right) = \dfrac{\sqrt{3}}{2} - \dfrac{\pi}{360} \approx 0.8573$

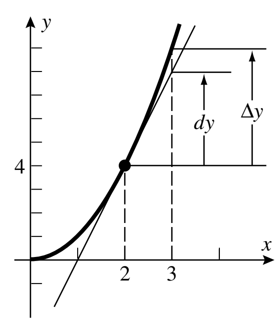
28. **(a)** Let $f(x) = (1+x)^k$ and $x_0 = 0$. Then $(1+x)^k \approx 1^k + k(1)^{k-1}(x-0) = 1 + kx$. Set $k = 37$ and $x = 0.001$ to obtain $(1.001)^{37} \approx 1.037$.

(b) With a calculator $(1.001)^{37} = 1.03767$.

(c) The approximation is $(1.1)^{37} \approx 1 + 37(0.1) = 4.7$, and the calculator value is 34.004. The error is due to the relative largeness of $f'(1)\Delta x = 37(0.1) = 3.7$.

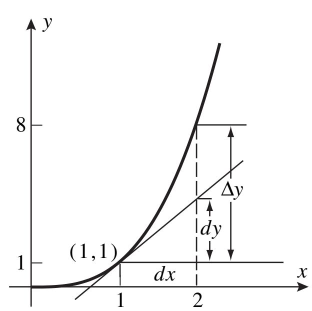
29. **(a)** $dy = f'(x)dx = 2xdx = 4(1) = 4$ and
$\Delta y = (x + \Delta x)^2 - x^2 = (2+1)^2 - 2^2 = 5$

(b)



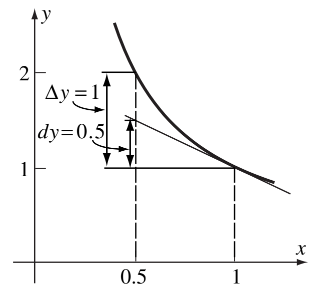
30. **(a)** $dy = 3x^2 dx = 3(1)^2(1) = 3$ and
$\Delta y = (x + \Delta x)^3 - x^3 = (1+1)^3 - 1^3 = 7$

(b)



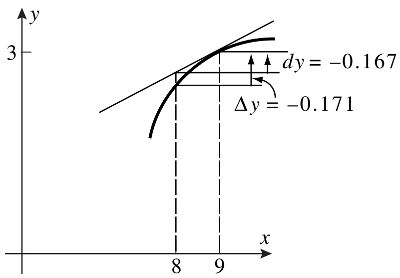
31. **(a)** $dy = (-1/x^2)dx = (-1)(-0.5) = 0.5$ and
$\Delta y = 1/(x + \Delta x) - 1/x$
$= 1/(1 - 0.5) - 1/1 = 2 - 1 = 1$

(b)



32. **(a)** $dy = (1/2\sqrt{x})dx = (1/(2 \cdot 3))(-1) = -1/6 \approx -0.167$ and
$\Delta y = \sqrt{x + \Delta x} - \sqrt{x} = \sqrt{9 + (-1)} - \sqrt{9} = \sqrt{8} - 3 \approx -0.172$

(b)



33. $dy = 3x^2 dx$;
$\Delta y = (x + \Delta x)^3 - x^3 = x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3 = 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$

34. $dy = 8dx$; $\Delta y = [8(x + \Delta x) - 4] - [8x - 4] = 8\Delta x$

35. $dy = (2x - 2)dx$;
$\Delta y = [(x + \Delta x)^2 - 2(x + \Delta x) + 1] - [x^2 - 2x + 1]$
$= x^2 + 2x\ \Delta x + (\Delta x)^2 - 2x - 2\Delta x + 1 - x^2 + 2x - 1 = 2x\ \Delta x + (\Delta x)^2 - 2\Delta x$

36. $dy = \cos x\, dx$; $\Delta y = \sin(x + \Delta x) - \sin x$

37. **(a)** $dy = (12x^2 - 14x)dx$
(b) $dy = x\, d(\cos x) + \cos x\, dx = x(-\sin x)dx + \cos x dx = (-x \sin x + \cos x)dx$

38. **(a)** $dy = (-1/x^2)dx$ **(b)** $dy = 5 \sec^2 x\, dx$

39. **(a)** $dy = \left(\sqrt{1 - x} - \dfrac{x}{2\sqrt{1 - x}}\right) dx = \dfrac{2 - 3x}{2\sqrt{1 - x}} dx$

(b) $dy = -17(1 + x)^{-18} dx$

40. **(a)** $dy = \dfrac{(x^3 - 1)d(1) - (1)d(x^3 - 1)}{(x^3 - 1)^2} = \dfrac{(x^3 - 1)(0) - (1)3x^2 dx}{(x^3 - 1)^2} = -\dfrac{3x^2}{(x^3 - 1)^2} dx$

(b) $dy = \dfrac{(2 - x)(-3x^2)dx - (1 - x^3)(-1)dx}{(2 - x)^2} = \dfrac{2x^3 - 6x^2 + 1}{(2 - x)^2} dx$

41. $dy = \dfrac{3}{2\sqrt{3x - 2}} dx$, $x = 2$, $dx = 0.03$; $\Delta y \approx dy = \dfrac{3}{4}(0.03) = 0.0225$

42. $dy = \dfrac{x}{\sqrt{x^2 + 8}} dx$, $x = 1$, $dx = -0.03$; $\Delta y \approx dy = (1/3)(-0.03) = -0.01$

43. $dy = \dfrac{1-x^2}{(x^2+1)^2}dx,\ x=2,\ dx=-0.04;\ \Delta y \approx dy = \left(-\dfrac{3}{25}\right)(-0.04) = 0.0048$

44. $dy = \left(\dfrac{4x}{\sqrt{8x+1}} + \sqrt{8x+1}\right)dx,\ x=3,\ dx=0.05;\ \Delta y \approx dy = (37/5)(0.05) = 0.37$

45. **(a)** $A = x^2$ where x is the length of a side; $dA = 2x\,dx = 2(10)(\pm 0.1) = \pm 2$ ft^2.

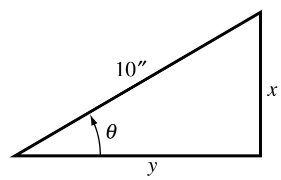
(b) relative error in x is $\approx \dfrac{dx}{x} = \dfrac{\pm 0.1}{10} = \pm 0.01$ so percentage error in x is $\approx \pm 1\%$; relative error in A is $\approx \dfrac{dA}{A} = \dfrac{2x\,dx}{x^2} = 2\dfrac{dx}{x} = 2(\pm 0.01) = \pm 0.02$ so percentage error in A is $\approx \pm 2\%$

46. **(a)** $V = x^3$ where x is the length of a side; $dV = 3x^2dx = 3(25)^2(\pm 1) = \pm 1875$ cm^3.

(b) relative error in x is $\approx \dfrac{dx}{x} = \dfrac{\pm 1}{25} = \pm 0.04$ so percentage error in x is $\approx \pm 4\%$; relative error in V is $\approx \dfrac{dV}{V} = \dfrac{3x^2dx}{x^3} = 3\dfrac{dx}{x} = 3(\pm 0.04) = \pm 0.12$ so percentage error in V is $\approx \pm 12\%$

47. **(a)** $x = 10\sin\theta,\ y = 10\cos\theta$ (see figure),

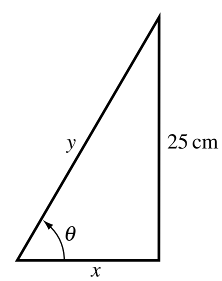
$$dx = 10\cos\theta d\theta = 10\left(\cos\frac{\pi}{6}\right)\left(\pm\frac{\pi}{180}\right) = 10\left(\frac{\sqrt{3}}{2}\right)\left(\pm\frac{\pi}{180}\right)$$
$$\approx \pm 0.151 \text{ in},$$
$$dy = -10(\sin\theta)d\theta = -10\left(\sin\frac{\pi}{6}\right)\left(\pm\frac{\pi}{180}\right) = -10\left(\frac{1}{2}\right)\left(\pm\frac{\pi}{180}\right)$$
$$\approx \pm 0.087 \text{ in}$$



(b) relative error in x is $\approx \dfrac{dx}{x} = (\cot\theta)d\theta = \left(\cot\dfrac{\pi}{6}\right)\left(\pm\dfrac{\pi}{180}\right) = \sqrt{3}\left(\pm\dfrac{\pi}{180}\right) \approx \pm 0.030$
so percentage error in x is $\approx \pm 3.0\%$;
relative error in y is $\approx \dfrac{dy}{y} = -\tan\theta d\theta = -\left(\tan\dfrac{\pi}{6}\right)\left(\pm\dfrac{\pi}{180}\right) = -\dfrac{1}{\sqrt{3}}\left(\pm\dfrac{\pi}{180}\right) \approx \pm 0.010$
so percentage error in y is $\approx \pm 1.0\%$

48. **(a)** $x = 25\cot\theta,\ y = 25\csc\theta$ (see figure);

$$dx = -25\csc^2\theta d\theta = -25\left(\csc^2\frac{\pi}{3}\right)\left(\pm\frac{\pi}{360}\right)$$
$$= -25\left(\frac{4}{3}\right)\left(\pm\frac{\pi}{360}\right) \approx \pm 0.291 \text{ cm},$$
$$dy = -25\csc\theta\cot\theta d\theta = -25\left(\csc\frac{\pi}{3}\right)\left(\cot\frac{\pi}{3}\right)\left(\pm\frac{\pi}{360}\right)$$
$$= -25\left(\frac{2}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)\left(\pm\frac{\pi}{360}\right) \approx \pm 0.145 \text{ cm}$$



(b) relative error in x is $\approx \dfrac{dx}{x} = -\dfrac{\csc^2\theta}{\cot\theta}d\theta = -\dfrac{4/3}{1/\sqrt{3}}\left(\pm\dfrac{\pi}{360}\right) \approx \pm 0.020$ so percentage error in x is $\approx \pm 2.0\%$; relative error in y is $\approx \dfrac{dy}{y} = -\cot\theta d\theta = -\dfrac{1}{\sqrt{3}}\left(\pm\dfrac{\pi}{360}\right) \approx \pm 0.005$
so percentage error in y is $\approx \pm 0.5\%$

49. $\dfrac{dR}{R} = \dfrac{(-2k/r^3)dr}{(k/r^2)} = -2\dfrac{dr}{r}$, but $\dfrac{dr}{r} \approx \pm 0.05$ so $\dfrac{dR}{R} \approx -2(\pm 0.05) = \pm 0.10$; percentage error in R is $\approx \pm 10\%$

50. $h = 12\sin\theta$ thus $dh = 12\cos\theta d\theta$ so, with $\theta = 60° = \pi/3$ radians and $d\theta = -1° = -\pi/180$ radians, $dh = 12\cos(\pi/3)(-\pi/180) = -\pi/30 \approx -0.105$ ft

51. $A = \frac{1}{4}(4)^2\sin 2\theta = 4\sin 2\theta$ thus $dA = 8\cos 2\theta d\theta$ so, with $\theta = 30° = \pi/6$ radians and $d\theta = \pm 15' = \pm 1/4° = \pm\pi/720$ radians, $dA = 8\cos(\pi/3)(\pm\pi/720) = \pm\pi/180 \approx \pm 0.017$ cm^2

52. $A = x^2$ where x is the length of a side; $\frac{dA}{A} = \frac{2x\,dx}{x^2} = 2\frac{dx}{x}$, but $\frac{dx}{x} \approx \pm 0.01$ so $\frac{dA}{A} \approx 2(\pm 0.01) = \pm 0.02$; percentage error in A is $\approx \pm 2\%$

53. $V = x^3$ where x is the length of a side; $\frac{dV}{V} = \frac{3x^2dx}{x^3} = 3\frac{dx}{x}$, but $\frac{dx}{x} \approx \pm 0.02$ so $\frac{dV}{V} \approx 3(\pm 0.02) = \pm 0.06$; percentage error in V is $\approx \pm 6\%$.

54. $\frac{dV}{V} = \frac{4\pi r^2 dr}{4\pi r^3/3} = 3\frac{dr}{r}$, but $\frac{dV}{V} \approx \pm 0.03$ so $3\frac{dr}{r} \approx \pm 0.03$, $\frac{dr}{r} \approx \pm 0.01$; maximum permissible percentage error in r is $\approx \pm 1\%$.

55. $A = \frac{1}{4}\pi D^2$ where D is the diameter of the circle; $\frac{dA}{A} = \frac{(\pi D/2)dD}{\pi D^2/4} = 2\frac{dD}{D}$, but $\frac{dA}{A} \approx \pm 0.01$ so $2\frac{dD}{D} \approx \pm 0.01$, $\frac{dD}{D} \approx \pm 0.005$; maximum permissible percentage error in D is $\approx \pm 0.5\%$.

56. $V = x^3$ where x is the length of a side; approximate ΔV by dV if $x = 1$ and $dx = \Delta x = 0.02$, $dV = 3x^2dx = 3(1)^2(0.02) = 0.06$ in^3.

57. V = volume of cylindrical rod $= \pi r^2 h = \pi r^2(15) = 15\pi r^2$; approximate ΔV by dV if $r = 2.5$ and $dr = \Delta r = 0.001$. $dV = 30\pi r\,dr = 30\pi(2.5)(0.001) \approx 0.236$ cm^3.

58. $P = \frac{2\pi}{\sqrt{g}}\sqrt{L}$, $dP = \frac{2\pi}{\sqrt{g}}\frac{1}{2\sqrt{L}}dL = \frac{\pi}{\sqrt{g}\sqrt{L}}dL$, $\frac{dP}{P} = \frac{1}{2}\frac{dL}{L}$ so the relative error in $P \approx \frac{1}{2}$ the relative error in L. Thus the percentage error in P is $\approx \frac{1}{2}$ the percentage error in L.

59. **(a)** $\alpha = \Delta L/(L\Delta T) = 0.006/(40 \times 10) = 1.5 \times 10^{-5}/°C$
(b) $\Delta L = 2.3 \times 10^{-5}(180)(25) \approx 0.1$ cm, so the pole is about 180.1 cm long.

60. $\Delta V = 7.5 \times 10^{-4}(4000)(-20) = -60$ gallons; the truck delivers $4000 - 60 = 3940$ gallons.

CHAPTER 3 SUPPLEMENTARY EXERCISES

4. **(a)**
$$\frac{dy}{dx} = \lim_{h\to 0}\frac{\sqrt{9-4(x+h)} - \sqrt{9-4x}}{h} = \lim_{h\to 0}\frac{(9-4(x+h)-(9-4x)}{h(\sqrt{9-4(x+h)}+\sqrt{9-4x})}$$
$$= \lim_{h\to 0}\frac{-4h}{h(\sqrt{9-4(x+h)}+\sqrt{9-4x})} = \frac{-4}{2\sqrt{9-4x}} = \frac{-2}{\sqrt{9-4x}}$$

(b)
$$\frac{dy}{dx} = \lim_{h\to 0}\frac{\dfrac{x+h}{x+h+1} - \dfrac{x}{x+1}}{h} = \lim_{h\to 0}\frac{(x+h)(x+1)-x(x+h+1)}{h(x+h+1)(x+1)}$$
$$= \lim_{h\to 0}\frac{h}{h(x+h+1)(x+1)} = \frac{1}{(x+1)^2}$$

5. Set $f'(x) = 0$: $f'(x) = 6(2)(2x+7)^5(x-2)^5 + 5(2x+7)^6(x-2)^4 = 0$, so $2x+7 = 0$ or $x-2 = 0$ or, factoring out $(2x+7)^5(x-2)^4$, $12(x-2) + 5(2x+7) = 0$. This reduces to $x = -7/2$, $x = 2$, or $22x + 11 = 0$, so the tangent line is horizontal at $x = -7/2, 2, -1/2$.

6. Set $f'(x) = 0$: $f'(x) = \dfrac{4(x^2+2x)(x-3)^3 - (2x+2)(x-3)^4}{(x^2+2x)^2}$, and a fraction can equal zero only if its numerator equals zero. So either $x - 3 = 0$ or, after factoring out $(x-3)^3$, $4(x^2+2x) - (2x+2)(x-3) = 0$, $2x^2 + 12x + 6 = 0$, whose roots are (by the quadratic formula) $x = \dfrac{-6 \pm \sqrt{36 - 4 \cdot 3}}{2} = -3 \pm \sqrt{6}$. So the tangent line is horizontal at $x = 3, -3 \pm \sqrt{6}$.

7. Set $f'(x) = \dfrac{3}{2\sqrt{3x+1}}(x-1)^2 + 2\sqrt{3x+1}(x-1) = 0$. If $x = 1$ then $y' = 0$. If $x \neq 1$ then divide out $x - 1$ and multiply through by $2\sqrt{3x+1}$ (at points where f is differentiable we must have $\sqrt{3x+1} \neq 0$) to obtain $3(x-1) + 4(3x+1) = 0$, or $15x + 1 = 0$. So the tangent line is horizontal at $x = 1, -1/15$.

8. $$f'(x) = 3\left(\frac{3x+1}{x^2}\right)^2 \frac{d}{dx}\frac{3x+1}{x^2} = 3\left(\frac{3x+1}{x^2}\right)^2 \frac{x^2(3) - (3x+1)(2x)}{x^4}$$
$$= -3\left(\frac{3x+1}{x^2}\right)^2 \frac{3x^2+2x}{x^4} = 0. \text{ If } f'(x) = 0$$

then $(3x+1)^2(3x^2+2x) = 0$. The tangent line is horizontal at $x = -1/3, -2/3$ ($x = 0$ is ruled out from the definition of f).

9. **(a)** $x = -2, -1, 1, 3$
(b) $(-\infty, -2)$, $(-1, 1)$, $(3, +\infty)$
(c) $(-2, -1)$, $(1, 3)$
(d) $g''(x) = f''(x)\sin x + 2f'(x)\cos x - f(x)\sin x$; $g''(0) = 2f'(0)\cos 0 = 2(2)(1) = 4$

10. **(a)** $f'(1)g(1) + f(1)g'(1) = 3(-2) + 1(-1) = -7$
(b) $\dfrac{g(1)f'(1) - f(1)g'(1)}{g(1)^2} = \dfrac{-2(3) - 1(-1)}{(-2)^2} = -\dfrac{5}{4}$
(c) $\dfrac{1}{2\sqrt{f(1)}}f'(1) = \dfrac{1}{2\sqrt{1}}3 = \dfrac{3}{2}$
(d) 0 (because $f(1)g'(1)$ is constant)

11. The equation of such a line has the form $y = mx$. The points (x_0, y_0) which lie on both the line and the parabola and for which the slopes of both curves are equal satisfy $y_0 = mx_0 = x_0^3 - 9x_0^2 - 16x_0$, so that $m = x_0^2 - 9x_0 - 16$. By differentiating, the slope is also given by $m = 3x_0^2 - 18x_0 - 16$. Equating, we have $x_0^2 - 9x_0 - 16 = 3x_0^2 - 18x_0 - 16$, or $2x_0^2 - 9x_0 = 0$. The root $x_0 = 0$ corresponds to $m = -16, y_0 = 0$ and the root $x_0 = 9/2$ corresponds to $m = -145/4, y_0 = -1305/8$. So the line $y = -16x$ is tangent to the curve at the point $(0, 0)$, and the line $y = -145x/4$ is tangent to the curve at the point $(9/2, -1305/8)$.

12. The slope of the line $x + 4y = 10$ is $m_1 = -1/4$, so we set the negative reciprocal $4 = m_2 = \dfrac{d}{dx}(2x^3 - x^2) = 6x^2 - 2x$ and obtain $6x^2 - 2x - 4 = 0$ with roots $x = \dfrac{1 \pm \sqrt{1+24}}{6} = 1, -2/3$.

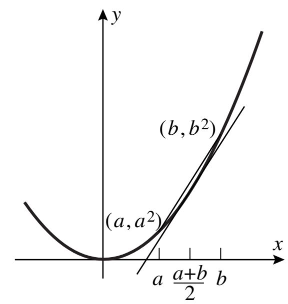
13. The line $y - x = 2$ has slope $m_1 = 1$ so we set $m_2 = \dfrac{d}{dx}(3x - \tan x) = 3 - \sec^2 x = 1$, or $\sec^2 x = 2$, $\sec x = \pm\sqrt{2}$ so $x = n\pi \pm \pi/4$ where $n = 0, \pm 1, \pm 2, \ldots$.

14. $f(x)$ is continuous and differentiable at any $x \neq 1$, so we consider $x = 1$.

(a) $\lim\limits_{x\to 1^-}(x^2 - 1) = \lim\limits_{x\to 1^+} k(x-1) = 0 = f(1)$, so any value of k gives continuity at $x = 1$.

(b) $\lim\limits_{x\to 1^-} f'(x) = \lim\limits_{x\to 1^-} 2x = 2$, and $\lim\limits_{x\to 1^+} f'(x) = \lim\limits_{x\to 1^+} k = k$, so only if $k = 2$ is $f(x)$ differentiable at $x = 1$.

15. The slope of the tangent line is the derivative $y' = 2x\Big|_{x=\frac{1}{2}(a+b)} = a + b$. The slope of the secant is $\dfrac{a^2 - b^2}{a - b} = a + b$, so they are equal.



16. To average 60 mi/h one would have to complete the trip in two hours. At 50 mi/h, 100 miles are completed after two hours. Thus time is up, and the speed for the remaining 20 miles would have to be infinite.

17. **(a)** $\Delta x = 1.5 - 2 = -0.5$; $dy = \dfrac{-1}{(x-1)^2}\Delta x = \dfrac{-1}{(2-1)^2}(-0.5) = 0.5$; and

$$\Delta y = \frac{1}{(1.5-1)} - \frac{1}{(2-1)} = 2 - 1 = 1.$$

(b) $\Delta x = 0 - (-\pi/4) = \pi/4$; $dy = \left(\sec^2(-\pi/4)\right)(\pi/4) = \pi/2$; and $\Delta y = \tan 0 - \tan(-\pi/4) = 1$.

(c) $\Delta x = 3 - 0 = 3$; $dy = \dfrac{-x}{\sqrt{25 - x^2}} = \dfrac{-0}{\sqrt{25 - (0)^2}}(3) = 0$; and

$$\Delta y = \sqrt{25 - 3^2} - \sqrt{25 - 0^2} = 4 - 5 = -1.$$

18. **(a)** $\dfrac{4^3 - 2^3}{4 - 2} = \dfrac{56}{2} = 28$ **(b)** $(dV/d\ell)|_{\ell=5} = 3\ell^2\big|_{\ell=5} = 3(5)^2 = 75$

19. **(a)** $\dfrac{dW}{dt} = 200(t - 15)$; at $t = 5$, $\dfrac{dW}{dt} = -2000$; the water is running out at the rate of 2000 gal/min.

(b) $\dfrac{W(5) - W(0)}{5 - 0} = \dfrac{10000 - 22500}{5} = -2500$; the average rate of flow out is 2500 gal/min.

20. $\cot 46^\circ = \cot\dfrac{46\pi}{180}$; let $x_0 = \dfrac{\pi}{4}$ and $x = \dfrac{46\pi}{180}$. Then

$\cot 46^\circ = \cot x \approx \cot\dfrac{\pi}{4} - \left(\csc^2\dfrac{\pi}{4}\right)\left(x - \dfrac{\pi}{4}\right) = 1 - 2\left(\dfrac{46\pi}{180} - \dfrac{\pi}{4}\right) = 0.9651$;

with a calculator, $\cot 46^\circ = 0.9657$.

21. **(a)** $h = 115\tan\phi$, $dh = 115\sec^2\phi\, d\phi$; with $\phi = 51^\circ = \dfrac{51}{180}\pi$ radians and

$d\phi = \pm 0.5^\circ = \pm 0.5\left(\dfrac{\pi}{180}\right)$ radians, $h \pm dh = 115(1.2349) \pm 2.5340 = 142.0135 \pm 2.5340$, so the height lies between 139.48 m and 144.55 m.

(b) If $|dh| \le 5$ then $|d\phi| \le \dfrac{5}{115}\cos^2 \dfrac{51}{180}\pi \approx 0.017$ radians, or $|d\phi| \le 0.98°$.

22. **(a)** $\dfrac{dT}{dL} = \dfrac{2}{\sqrt{g}}\dfrac{1}{2\sqrt{L}} = \dfrac{1}{\sqrt{gL}}$ **(b)** s/m

(c) Since $\dfrac{dT}{dL} > 0$ an increase in L gives an increase in T, which is the period. To speed up a clock, decrease the period; to decrease T, decrease L.

(d) $\dfrac{dT}{dg} = -\dfrac{\sqrt{L}}{g^{3/2}} < 0$; a decrease in g will increase T and the clock runs slower

(e) $\dfrac{dT}{dg} = 2\sqrt{L}\left(\dfrac{-1}{2}\right)g^{-3/2} = -\dfrac{\sqrt{L}}{g^{3/2}}$ **(f)** s^3/m

23. **(a)** $f'(x) = 2x, f'(1.8) = 3.6$
(b) $f'(x) = (x^2 - 4x)/(x-2)^2, f'(3.5) \approx -0.777778$

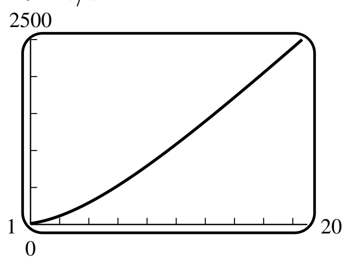
24. **(a)** $f'(x) = 3x^2 - 2x, f'(2.3) = 11.27$
(b) $f'(x) = (1 - x^2)/(x^2+1)^2, f'(-0.5) = 0.48$

25. $f'(2) \approx 2.772589; f'(2) = 4\ln 2$

26. $f'(2) = 0.312141; f'(2) = 2^{\sin 2}(\cos 2 \ln 2 + \sin 2/2)$

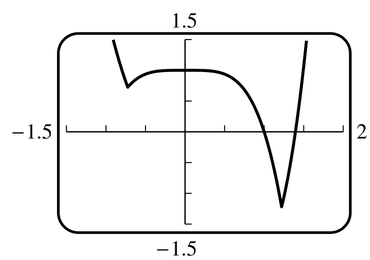
27. $v_{\text{inst}} = \lim\limits_{h\to 0}\dfrac{3(h+1)^{2.5} + 580h - 3}{10h} = 58 + \dfrac{1}{10}\dfrac{d}{dx}3x^{2.5}\bigg|_{x=1} = 58 + \dfrac{1}{10}(2.5)(3)(1)^{1.5} = 58.75$ ft/s

28. 164 ft/s



29. Solve $3x^2 - \cos x = 0$ to get $x = \pm 0.535428$.

30. When $x^4 - x - 1 > 0$, $f(x) = x^4 - 2x - 1$; when $x^4 - x - 1 < 0$, $f(x) = -x^4 + 1$, and f is differentiable in both cases. The roots of $x^4 - x - 1 = 0$ are $x_1 = -0.724492$, $x_2 = 1.220744$. So $x^4 - x - 1 > 0$ on $(-\infty, x_1)$ and $(x_2, +\infty)$, and $x^4 - x - 1 < 0$ on (x_1, x_2). Then $\lim\limits_{x\to x_1^-} f'(x) = \lim\limits_{x\to x_1^-}(4x^3 - 2) = 4x_1^3 - 2$ and $\lim\limits_{x\to x_1^+} f'(x) = \lim\limits_{x\to x_1^+} -4x^3 = -4x_1^3$ which is not equal to $4x_1^3 - 2$, so f is not differentiable at $x = x_1$; similarly f is not differentiable at $x = x_2$.



31. **(a)** $f'(x) = 5x^4$ **(b)** $f'(x) = -1/x^2$ **(c)** $f'(x) = -1/2x^{3/2}$
(d) $f'(x) = -3/(x-1)^2$ **(e)** $f'(x) = 3x/\sqrt{3x^2+5}$ **(f)** $f'(x) = 3\cos 3x$

32. $f'(x) = 2x\sin x + x^2\cos x$

33. $f'(x) = \dfrac{1 - 2\sqrt{x}\sin 2x}{2\sqrt{x}}$

34. $f'(x) = \dfrac{6x^2 + 8x - 17}{(3x+2)^2}$

35. $f'(x) = \dfrac{(1+x^2)\sec^2 x - 2x\tan x}{(1+x^2)^2}$

36. $f'(x) = \dfrac{x^2\cos\sqrt{x} - 2x^{3/2}\sin\sqrt{x}}{2x^{7/2}}$

37. $f'(x) = \dfrac{-2x^5\sin x - 2x^4\cos x + 4x^4 + 6x^2\sin x + 6x - 3x\cos x - 4x\sin x + 4\cos x - 8}{2x^2\sqrt{x^4 - 3} + 2(2 - \cos x)^2}$

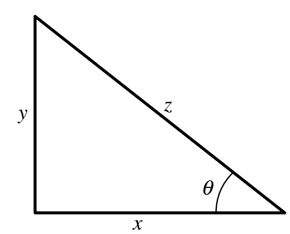
38. Differentiating, $\dfrac{2}{3}x^{-1/3} - \dfrac{2}{3}y^{-1/3}y' - y' = 0$. At $x = 1$ and $y = -1$, $y' = 2$. The tangent line is $y + 1 = 2(x - 1)$.

39. Differentiating, $(xy' + y)\cos xy = y'$. With $x = \pi/2$ and $y = 1$ this becomes $y' = 0$, so the equation of the tangent line is $y - 1 = 0(x - \pi/2)$ or $y = 1$.

40. Find $\left.\dfrac{d\theta}{dt}\right|_{\substack{x=1\\y=1}}$ given $\dfrac{dz}{dt} = a$ and $\dfrac{dy}{dt} = -b$. From the figure $\sin\theta = y/z$; when $x = y = 1$, $z = \sqrt{2}$. So $\theta = \sin^{-1}(y/z)$ and

$$\frac{d\theta}{dt} = \frac{1}{\sqrt{1 - y^2/z^2}}\left(\frac{1}{z}\frac{dy}{dt} - \frac{y}{z^2}\frac{dz}{dt}\right) = -b - \frac{a}{\sqrt{2}}$$

when $x = y = 1$.



CHAPTER 3 HORIZON MODULE

1. $x_1 = l_1\cos\theta_1, x_2 = l_2\cos(\theta_1 + \theta_2)$, so $x = x_1 + x_2 = l_1\cos\theta_1 + l_2\cos(\theta_1 + \theta_2)$ (see Figure 3 in text); similarly $y_1 = l_1\sin\theta_1 + l_2\sin(\theta_1 + \theta_2)$.

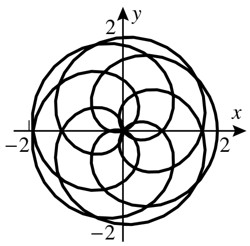
2. Fix θ_1 for the moment and let θ_2 vary; then the distance r from (x, y) to the origin (see Figure 3 in text) is at most $l_1 + l_2$ and at least $l_1 - l_2$ if $l_1 \geq l_2$ and $l_2 - l_1$ otherwise. For any fixed θ_2 let θ_1 vary and the point traces out a circle of radius r.

(a) $\{(x, y) : 0 \leq x^2 + y^2 \leq 2l_1\}$
(b) $\{(x, y) : l_1 - l_2 \leq x^2 + y^2 \leq l_1 + l_2\}$
(c) $\{(x, y) : l_2 - l_1 \leq x^2 + y^2 \leq l_1 + l_2\}$

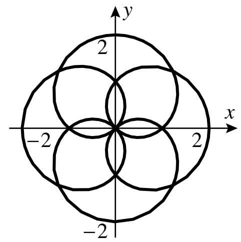
3.
$$\begin{aligned}(x, y) &= (l_1\cos\theta + l_2\cos(\theta_1 + \theta_2), l_1\sin\theta_1 + l_2\sin(\theta_1 + \theta_2)) \\ &= (\cos(\pi/4) + 3\cos(5\pi/12), \sin(\pi/4) + 3\sin(5\pi/12)) = \left(\frac{\sqrt{2} + 3\sqrt{6}}{4}, \frac{7\sqrt{2} + 3\sqrt{6}}{4}\right)\end{aligned}$$

4. $x = (1)\cos 2t + (1)\cos(2t + 3t) = \cos 2t + \cos 5t$,
$y = (1)\sin 2t + (1)\sin(2t + 3t) = \sin 2t + \sin 5t$

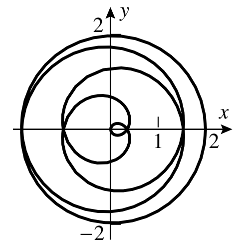
5.



$\omega_1 = 3, \omega_2 = 5$



$\omega_1 = 1, \omega_2 = 4$



$\omega_1 = 4, \omega_2 = 1$

6. $x = 2\cos t,\ y = 2\sin t$, a circle of radius 2

7. **(a)** $9 = [3\sin(\theta_1+\theta_2)]^2 + [3\cos(\theta_1+\theta_2)]^2 = [5 - 3\sin\theta_1]^2 + [3 - 3\cos\theta_1]^2$
$= 25 - 30\sin\theta_1 + 9\sin^2\theta_1 + 9 - 18\cos\theta_1 + 9\cos^2\theta_1 = 43 - 30\sin\theta_1 - 18\cos\theta_1$,
so $15\sin\theta_1 + 9\cos\theta_1 = 17$

(b) $1 = \sin^2\theta_1 + \cos^2\theta_2 = \left(\dfrac{17 - 9\cos\theta_1}{15}\right)^2 + \cos\theta_1$, or $306\cos^2\theta_1 - 306\cos\theta_1 = -64$

(c) $\cos\theta_1 = \left(153 \pm \sqrt{(153)^2 - 4(153)(32)}\right)/306 = \dfrac{1}{2} \pm \dfrac{5\sqrt{17}}{102}$

(e) If $\theta_1 = 0.792436$ rad, then $\theta_2 = 0.475882$ rad $\approx 27.2660°$;
if $\theta_1 = 1.26832$ rad, then $\theta_2 = -0.475882$ rad $\approx -27.2660°$.

8.

$$\begin{aligned}\frac{dx}{dt} &= -3\sin\theta_1\frac{d\theta_1}{dt} - (3\sin(\theta_1+\theta_2))\left(\frac{d\theta_1}{dt} + \frac{d\theta_2}{dt}\right)\\ &= -3\frac{d\theta_1}{dt}(\sin\theta_1 + \sin(\theta_1+\theta_2)) - 3\left(\sin(\theta_1+\theta_2)\right)\frac{d\theta_2}{dt}\\ &= -y\frac{d\theta_1}{dt} - 3(\sin(\theta_1+\theta_2))\frac{d\theta_2}{dt};\end{aligned}$$

similarly $\dfrac{dy}{dt} = x\dfrac{d\theta_1}{dt} + 3(\cos(\theta_1+\theta_2))\dfrac{d\theta_2}{dt}$. Now set $\dfrac{dx}{dt} = 0,\ \dfrac{dy}{dt} = 1$.

9. **(a)** $x = 3\cos(\pi/3) + 3\cos(-\pi/3) = 6\dfrac{1}{2} = 3$ and $y = 3\sin(\pi/3) - 3\sin(\pi/3) = 0$; equations (4) become $3\sin(\pi/3)\dfrac{d\theta_2}{dt} = 0$, $3\dfrac{d\theta_1}{dt} + 3\cos(\pi/3)\dfrac{d\theta_2}{dt} = 1$ with solution $d\theta_2/dt = 0$, $d\theta_1/dt = 1/3$.

(b) $x = -3,\ y = 3$, so $-3\dfrac{d\theta_1}{dt} = 0$ and $-3\dfrac{d\theta_1}{dt} - 3\dfrac{d\theta_2}{dt} = 1$, with solution $d\theta_1/dt = 0$,
$d\theta_2/dt = -1/3$.